DETERMINANT

1. INTRODUCTION :

If the equations $a_1x + b_1 = 0$, $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

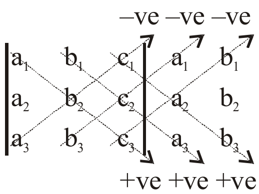
A determinant of third order consists of three rows and three columns.

2. VALUE OF A DETERMINANT :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note : Sarrus diagram to get the value of determinant of order three :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$



$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 1 : The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

(A) 213 (B) – 231 (C) 231 (D) 39

Solution :

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$$= (27 + 42) - 2\,(-36 - 12) + 3\,(28 - 6) = 231$$

Alternative : By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 3 \\ 2 & -7 \end{matrix}$$

$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$ **Ans. (C)**

3. MINORS & COFACTORS :

The minor of a given element of determinant is the determinant obtained by deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by : $C_{ij} = (-1)^{i+j}.M_{ij}$

Illustration 2 : Find the minors and cofactors of elements '–3', '5', '–1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Solution : Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$

4. EXPANSION OF A DETERMINANT IN TERMS OF THE ELEMENTS OF ANY ROW OR COLUMN:

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(i) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

D can be expressed in any of the six forms :

$a_1A_1 + b_1B_1 + c_1C_1$, $a_1A_1 + a_2A_2 + a_3A_3$,

$a_2A_2 + b_2B_2 + c_2C_2$, $b_1B_1 + b_2B_2 + b_3B_3$,

$a_3A_3 + b_3B_3 + c_3C_3$, $c_1C_1 + c_2C_2 + c_3C_3$,

where A_i, B_i & C_i (i = 1,2,3) denote cofactors of a_i, b_i & c_i respectively.

(ii) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

Hence,

$a_2A_1 + b_2B_1 + c_2C_1 = 0$,

$b_1A_1 + b_2A_2 + b_3A_3 = 0$ and so on.

where A_i, B_i & C_i (i = 1,2,3) denote cofactors of a_i, b_i & c_i respectively.

Do yourself -1 :

(i) Find minors & cofactors of elements '6', '5', '0' & '4' of the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \\ 3 & 0 & 4 \end{vmatrix}$.

(ii) Calculate the value of the determinant $\begin{vmatrix} 5 & -3 & 7 \\ -2 & 4 & -8 \\ 9 & 3 & -10 \end{vmatrix}$

(iii) The value of the determinant $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}$ is equal to -

(A) $a^3 - b^3$ (B) $a^3 + b^3$ (C) 0 (D) none of these

(iv) Find the value of 'k', if $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & k & 2 \end{vmatrix} = 4$

5. PROPERTIES OF DETERMINANTS :

(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

Illustration 3 : Prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution : $D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ (By interchanging rows & columns)

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $(C_1 \leftrightarrow C_2)$

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $(R_1 \leftrightarrow R_2)$

Illustration 4 : Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Solution : $D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$

Since all rows are same, hence value of the determinant is zero.

Do yourself -2 :

(i) Without expanding the determinant prove that $\begin{vmatrix} a & p & \ell \\ b & q & m \\ c & r & n \end{vmatrix} + \begin{vmatrix} r & n & c \\ q & m & b \\ p & \ell & a \end{vmatrix} = 0$

(ii) If $D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, then $\begin{vmatrix} 2\alpha & 2\beta \\ 2\gamma & 2\delta \end{vmatrix}$ is equal to -

(A) D (B) 2D (C) 4D (D) 16D

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note that : If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a,b,c,a_1,b_1,c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \rightarrow C_i + \alpha C_j + \beta C_k$ ($j, k \neq i$) or row (R_i) operation of the form $R_i \rightarrow R_i + \alpha R_j + \beta R_k$ ($j, k \neq i$). In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1+\alpha a_2 & b_1+\alpha b_2 & c_1+\alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3+\beta a_1 & b_3+\beta b_1 & c_3+\beta c_1 \end{vmatrix} \quad (R_1 \rightarrow R_1 + \alpha R_2; R_3 \rightarrow R_3 + \beta R_2)$$

Note :

(i) By using the operation $R_i \rightarrow xR_i + yR_j + zR_k$ ($j, k \neq i$), the value of the determinant becomes x times the original one.

(ii) While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

Illustration 5 : If $D_r = \begin{vmatrix} r & r^3 & 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$, find $\sum_{r=0}^{n} D_r$.

Solution : $$\sum_{r=0}^{n} D_r = \begin{vmatrix} \sum_{r=0}^{n} r & \sum_{r=0}^{n} r^3 & \sum_{r=0}^{n} 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = 0$$ **Ans.**

Illustration 6 : If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

(A) 2 (B) 1 (C) –1 (D) 0

Solution : Applying $(C_3 \rightarrow C_3 - C_1)$

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0 \qquad (R_3 \rightarrow R_3 - R_2;\ R_2 \rightarrow R_2 - R_1)$$

$\Rightarrow\ k - 1 = 0 \Rightarrow k = 1$ **Ans. (B)**

Do yourself - 3 :

(i) Find the value of $\begin{vmatrix} 53 & 106 & 159 \\ 52 & 65 & 91 \\ 102 & 153 & 221 \end{vmatrix}$. **(ii)** Solve for x : $\begin{vmatrix} x & 2 & 0 \\ 2+x & 5 & -1 \\ 5-x & 1 & 2 \end{vmatrix} = 0$

(iii) If $D_r = \begin{vmatrix} 2r & 1 & n \\ 1 & -2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$, then find the value of $\sum_{r=1}^{n} D_r$.

(h) **Factor theorem :** If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when x = a then (x – a) is a factor of D.

Note that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

Illustration 7 : Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Solution : Using factor theorem,

Put x = a

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes D = 0, therefore (x – a) is a factor of D.

Similarly, by putting x = b, D becomes zero, therefore (x – b) is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad ....(i)$$

To get the value of λ, put x = 0 in equation (i)

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

amb = λab ⇒ λ = m

∴ D = m(x – a)(x – b)

Do yourself - 4 :

(i) Without expanding the determinant prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) Using factor theorem, find the solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

6. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Illustration 8 : Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for n ≥ 1. Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Solution :

$$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2 = [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$D = (\alpha - \beta)^2 (\alpha + \beta - \alpha\beta - 1)^2$

∵ α & β are roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha+\beta = \frac{-b}{a} \quad \& \quad \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad |\alpha-\beta| = \frac{\sqrt{b^2-4ac}}{|a|}$$

$$D = \frac{(b^2-4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 = \frac{(b^2-4ac)(a+b+c)^2}{a^4}$$ **Ans.**

Do yourself - 5 :

(i) If the determinant $D = \begin{vmatrix} 1 & 1 & 1 \\ \alpha+\beta & \alpha^2+\beta^2 & 2\alpha\beta \\ \alpha+\beta & 2\alpha\beta & \alpha^2+\beta^2 \end{vmatrix}$ and $D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{vmatrix}$, then find the determinant D_2 such that $D_2 = \frac{D}{D_1}$.

(ii) If $D_1 = \begin{vmatrix} ab^2-ac^2 & bc^2-a^2b & a^2c-b^2c \\ ac-ab & ab-bc & bc-ac \\ c-b & a-c & b-a \end{vmatrix}$ & $D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$, then D_1D_2 is equal to -

(A) 0 (B) D_1^2 (C) D_2^2 (D) D_2^3

7. SPECIAL DETERMINANTS :

(a) Cyclic Determinant :

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

$$= -\frac{1}{2}(a+b+c) \times \{(a-b)^2+(b-c)^2+(c-a)^2\}$$

$= -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$, where ω, ω^2 are cube roots of unity

(b) Other Important Determinants :

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

Illustration 9 : Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1-\alpha^3)^2$.

Solution : This is a cyclic determinant.

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1+\alpha+\alpha^2)(1+\alpha^2+\alpha^4-\alpha-\alpha^2-\alpha^3)$$

$$= -(1+\alpha+\alpha^2)(-\alpha+1-\alpha^3+\alpha^4) = -(1+\alpha+\alpha^2)(1-\alpha)^2(1+\alpha+\alpha^2)$$

$$= -(1-\alpha)^2(1+\alpha+\alpha^2)^2 = -(1-\alpha^3)^2$$

Do yourself - 6 :

(i) The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

(A) $k(a+b)(b+c)(c+a)$ (B) $kabc(a^2+b^2+c^2)$

(C) $k(a-b)(b-c)(c-a)$ (D) $k(a+b-c)(b+c-a)(c+a-b)$

(ii) Find the value of the determinant $\begin{vmatrix} a^2+b^2 & a^2-c^2 & a^2-c^2 \\ -a^2 & 0 & c^2-a^2 \\ b^2 & -c^2 & b^2 \end{vmatrix}$.

(iii) Prove that $\begin{vmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$.

8. CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS) :

Simultaneous linear equations

- Consistent (at least one solution)
  - Exactly one solution or Unique solution
    - Trivial solution i.e. All variable zero is the only solution
  - Infinite solutions
    - Non trivial solution i.e. At least one non zero variable satisfies the system
- Inconsistent (no solution)

(a) Equations involving two variables :

(i) Consistent Equations : Definite & unique solution (Intersecting lines)

(ii) Inconsistent Equations : No solution (Parallel lines)

(iii) Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables :

Let $a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)

Then, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(iv) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations may have infinite or no solution.

Note that In case $\left.\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_1x + b_1y + c_1z = d_2 \\ a_1x + b_1y + c_1z = d_3 \end{matrix}\right\}$ (Atleast two of d_1, d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

(c) Homogeneous system of linear equations :

If x, y, z are not all zero, the condition for

$a_1x + b_1y + c_1z = 0$
$a_2x + b_2y + c_2z = 0$
$a_3x + b_3y + c_3z = 0$

to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**.

9. APPLICATION OF DETERMINANTS IN GEOMETRY :

(a) The lines : $a_1x + b_1y + c_1 = 0$ .... (i)
$a_2x + b_2y + c_2 = 0$ .... (ii)
$a_3x + b_3y + c_3 = 0$ .... (iii)

are concurrent or all three parallel if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the necessary condition for consistency of three simultaneous linear equations in 2 variables but may not be sufficient.

(b) Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(c) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2 , 3 is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If D = 0, then the three points are collinear.

(d) Equation of a straight line passing through points (x_1 , y_1) & (x_2 , y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 10 : Find the nature of solution for the given system of equations :
$x + 2y + 3z = 1;\ 2x + 3y + 4z = 3;\ 3x + 4y + 5z = 0$

Solution : $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$\because$ D = 0 but $D_1 \neq 0$
Hence no solution. **Ans.**

Illustration 11 : Find the value of λ, if the following equations are consistent :

$x + y - 3 = 0;\ (1+\lambda)x + (2+\lambda)y - 8 = 0;\ x - (1+\lambda)y + (2+\lambda) = 0$

Solution : The given equations in two unknowns are consistent, then $\Delta = 0$

$$\text{i.e. } \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (5+\lambda) - (3\lambda-5)(-2-\lambda) = 0 \quad \Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0$

$\therefore \quad \lambda = 1, -5/3$

Illustration 12 : If the system of equations $x + \lambda y + 1 = 0$, $\lambda x + y + 1 = 0$ & $x + y + \lambda = 0$. is consistent, then find the value of λ.

Solution : For consistency of the given system of equations

$$D = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3\lambda = 1 + 1 + \lambda^3$ or $\lambda^3 - 3\lambda + 2 = 0$

$\Rightarrow \quad (\lambda-1)^2 (\lambda + 2) = 0 \Rightarrow \lambda = 1$ or $\lambda = -2$ **Ans.**

Do yourself -7 :

(i) Find nature of solution for given system of equations

$2x + y + z = 3; \quad x + 2y + z = 4; \quad 3x + z = 2$

(ii) If the system of equations $x + y + z = 2$, $2x + y - z = 3$ & $3x + 2y + kz = 4$ has a unique solution, then

(A) $k \neq 0$ (B) $-1 < k < 1$ (C) $-2 < k < 1$ (D) $k = 0$

(iii) The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0 = 0$ & $-x - y + \lambda z = 0$ has a non-trivial solution, then possible values of λ are -

(A) 0 (B) 1 (C) –3 (D) $\sqrt{3}$

ANSWERS FOR DO YOURSELF

1. **(i)** minors : 4, –1, –4, 4 ; cofactors : –4, –1, 4, 4 **(ii)** –98 **(iii)** B **(iv)** 0

2. **(ii)** C

3. **(i)** 0 **(ii)** 2 **(iii)** 0

4. **(ii)** x = –1, 2

5. **(i)** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}$ **(ii)** D

6. **(i)** C **(ii)** 0

7. **(i)** infinite solutions **(ii)** A **(iii)** A

EXERCISE (O-1)

1. $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ equals-

(A) $x^2y^2z^2$ (B) $4x^2y^2z^2$ (C) xyz (D) 4xyz

DT0001

2. If $\begin{vmatrix} 1 & 3 & 4 \\ 1 & x-1 & 2x+2 \\ 2 & 5 & 9 \end{vmatrix} = 0$, then x is equal to-

(A) 2 (B) 1 (C) 4 (D) 0 **DT0002**

3. If a, b, c are in AP, then $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ equals -

(A) a + b + c (B) x + a + b + c (C) 0 (D) none of these

DT0008

4. If $px^4 + qx^3 + rx^2 + sx + t = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 2-x & x-3 \\ x-3 & x+4 & 3x \end{vmatrix}$ then t is equal to -

(A) 33 (B) 0 (C) 21 (D) none

DT0003

5. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is-

(A) 0 (B) log xyz (C) log(x + y + z) (D) logx logy logz

DT0009

6. Let a determinant is given by $A = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ and suppose det. A = 6. If $B = \begin{vmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{vmatrix}$ then

(A) det. B = 6 (B) det. B = – 6 (C) det. B = 12 (D) det. B = – 12

DT0010

7. If in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, A_1, B_1, C_1 etc. be the co-factors of a_1, b_1, c_1 etc., then which of the following relations is incorrect-

(A) $a_1A_1 + b_1B_1 + c_1C_1 = \Delta$ (B) $a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

(C) $a_3A_3 + b_3B_3 + c_3C_3 = \Delta$ (D) $a_1A_2 + b_1B_2 + c_1C_2 = \Delta$ **DT0006**

8. If $S_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$ then $\sum_{r=1}^{n} S_r$ does not depend on-

(A) x (B) y (C) n (D) all of these

DT0012

9. The value of k for which the set of equations $3x+ky-2z=0$, $x+ky+3z=0$ and $2x+3y-4z=0$ has a non-trivial solution is- **[JEE-MAIN Online 2013]**

(A) 15 (B) 16 (C) 31/2 (D) 33/2

DT0015

10. If the system of linear equations **[JEE-MAIN Online 2013]**

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then :-

(A) $a \in R - \{8\}$ and $b \in R - \{15\}$ (B) a = 8, b can be any real number

(C) a = 8, b = 15 (D) b = 15, a can be any real number

DT0016

11. Consider the system of equations : $x + ay = 0$, $y + az = 0$ and $z + ax = 0$. Then the set of all real values of 'a' for which the system has a unique solution is : **[JEE-MAIN Online 2013]**

(A) $\{1, -1\}$ (B) $R - \{-1\}$

(C) $\{1, 0, -1\}$ (D) $R - \{1\}$ **DT0017**

12. If a, b, c > 0 and x, y, z ∈ R, then the determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$ is equal to -

(A) $a^xb^yc^x$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D) zero **DT0013**

13. There are two numbers x making the value of the determinant $\begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$ equal to 86. The sum of these two numbers, is-

(A) –4 (B) 5 (C) –3 (D) 9 **DT0004**

14. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 denote the co-factors of a_1, b_1, c_1 respectively, then the value of the determinant $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is -

(A) Δ (B) Δ^2 (C) Δ^3 (D) 0 **DT0007**

EXERCISE (O-2)

1. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$, then the maximum value of f(x) , is-

(A) 2 (B) 4 (C) 6 (D) 8 **DT0019**

2. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is -

(A) 0 (B) independent of θ
(C) independent of ϕ (D) independent of θ & ϕ both **DT0024**

3. Let $D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$ then the value of $\frac{D_1}{D_2}$ where $b \neq 0$ and $ad \neq bc$, is

(A) – 2 (B) 0 (C) – 2b (D) 2b **DT0021**

4. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$ then $f(x)$ is a polynomial of degree-

(A) 0 (B) 1 (C) 2 (D) 3 **DT0022**

5. If the system of equation, $a^2x - ay = 1 - a$ & $bx + (3 - 2b)y = 3 + a$ possess a unique solution $x = 1, y = 1$ than :

(A) $a = 1; b = -1$ (B) $a = -1, b = 1$ (C) $a = 0, b = 0$ (D) none **DT0025**

6. The number of real values of x satisfying $\begin{vmatrix} x & 3x+2 & 2x-1 \\ 2x-1 & 4x & 3x+1 \\ 7x-2 & 17x+6 & 12x-1 \end{vmatrix} = 0$ is -

(A) 3 (B) 0 (C) 1 (D) infinite **DT0023**

[ONE OR MORE THAN ONE ARE CORRECT]

7. Which of the following determinant(s) vanish(es) ?

(A) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$ (B) $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

(C) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ (D) $\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_y z \\ \log_z xyz & \log_z y & 1 \end{vmatrix}$

DT0028

8. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if -

(A) a, b, c are in AP (B) a, b, c are in GP

(C) α is a root of the equation $ax^2+bx+c=0$ (D) $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

DT0029

9. System of linear equations in x,y,z

$2x + y + z = 1$

$x - 2y + z = 2$

$3x - y + 2z = 3$ have infinite solutions which

(A) can be written as $(-3\lambda - 1, \lambda, 5\lambda + 3)\ \forall\ \lambda \in R$

(B) can be written as $(3\lambda - 1, -\lambda, -5\lambda + 3)\ \forall\ \lambda \in R$

(C) are such that every solution satisfy $x - 3y + 1 = 0$

(D) are such that none of them satisfy $5x + 3z = 1$ **DT0030**

[MATRIX MATCH TYPE]

10. Consider a system of linear equations $a_ix + b_iy + c_iz = d_i$(where $a_i,b_i,c_i \neq 0$ and i = 1,2,3) & (α,β,γ) is its unique solution, then match the following conditions.

Column-I	Column-II
(A) If $a_i = k$, $d_i = k^2$, $(k \neq 0)$ and $\alpha + \beta + \gamma = 2$, then k is	(P) 1
(B) If $a_i = d_i = k \neq 0$, then $\alpha + \beta + \gamma$ is	(Q) 2
(C) If $a_i = k > 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(R) 0
(D) If $a_i = k < 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(S) 3
	(T) –1

DT0032

EXERCISE (S-1)

1. **(a)** On which one of the parameter out of a, p, d or x the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$ does not depend. **DT0042**

(b) If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different then, prove that $xyz = -1$. **DT0043**

2. Prove that :

(a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$ **DT0044**

(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = [(x-y)(y-z)(z-x)(x+y+z)]$ **DT0045**

3. (a) Let $f(x) = \begin{vmatrix} x & 1 & \frac{-3}{2} \\ 2 & 2 & 1 \\ \frac{1}{x-1} & 0 & \frac{1}{2} \end{vmatrix}$. Find the minimum value of f(x) (given x > 1). **DT0033**

(b) If $a^2 + b^2 + c^2 + ab + bc + ca \leq 0 \ \forall$ a, b, c ∈ R, then find the value of the determinant

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}.$$ **DT0034**

4. If $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ and $D' = \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$, then prove that $D' = 2D$. **DT0046**

5. If $a+b+c=0$, solve for x : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$ **DT0037**

6. Let a, b, c are the solutions of the cubic $x^3 - 5x^2 + 3x - 1 = 0$, then find the value of the determinant

$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$. **DT0040**

7. Show that, $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$ is divisible by λ^2 and find the other factor. **DT0038**

8. If $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$ and $f(x) = \sum_{j=1}^{3} \sum_{i=1}^{3} a_{ij}c_{ij}$, where a_{ij} is the element of i^{th} and j^{th} column in $\Delta(x)$ and c_{ij} is the cofactor a_{ij} $\forall$ i and j, then find the greatest value of f(x), where $x \in [-3, 18]$

DT0041

9. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2 (\beta-\gamma)^2 (\gamma-\alpha)^2$. **DT0047**

10. Solve the following sets of equations using Cramer's rule and remark about their consistency.

(a) $\begin{aligned} x + y + z - 6 &= 0 \\ 2x + y - z - 1 &= 0 \\ x + y - 2z + 3 &= 0 \end{aligned}$ **DT0049**

(b) $\begin{aligned} x + 2y + z &= 1 \\ 3x + y + z &= 6 \\ x + 2y &= 0 \end{aligned}$ **DT0050**

(c) $\begin{aligned} 7x - 7y + 5z &= 3 \\ 3x + y + 5z &= 7 \\ 2x + 3y + 5z &= 5 \end{aligned}$ **DT0051**

11. For what value of K do the following system of equations $x + Ky + 3z = 0$, $3x + Ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non trivial (i.e. not all zero) solution over the set of rationals Q. For that value of K, find all the solutions of the system. **DT0052**

12. If the equations $a(y + z) = x$, $b(z + x) = y$, $c(x + y) = z$ (where $a,b,c \neq -1$) have nontrivial solutions, then find the value of $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$. **DT0053**

13. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has alteast one solution for any real number λ. Find the set of solutions of $\lambda = -5$.
DT0054

EXERCISE (S-2)

1. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1 + a^2 + b^2)^3$. **DT0056**

2. Given $x = cy + bz$; $y = az + cx$; $z = bx + ay$, where x, y, z are not all zero, then prove that $a^2 + b^2 + c^2 + 2abc = 1$. **DT0058**

3. Investigate for what values of λ, μ the simultaneous equations $x + y + z = 6$; $x + 2y + 3z = 10$ & $x + 2y + \lambda z = \mu$ have :

(a) A unique solution. (b) An infinite number of solutions.

(c) No solution. **DT0059**

4. For what values of p, the equations : $x + y + z = 1$; $x + 2y + 4z = p$ & $x + 4y + 10z = p^2$ have a solution ? Solve them completely in each case. **DT0060**

5. Solve the equations : $Kx + 2y - 2z = 1$, $4x + 2Ky - z = 2$, $6x + 6y + Kz = 3$ considering specially the case when $K = 2$. **DT0061**

6. Find the sum of all positive integral values of a for which every solution to the system of equation $x + ay = 3$ and $ax + 4y = 6$ satisfy the inequalities $x > 1$, $y > 0$. **DT0062**

7. Given $a = \frac{x}{y-z}$; $b = \frac{y}{z-x}$; $c = \frac{z}{x-y}$, where x, y, z are not all zero, prove that : $1 + ab + bc + ca = 0$.

DT0063

8. Solve the system of equations : $\left.\begin{matrix} z + ay + a^2x + a^3 = 0 \\ z + by + b^2x + b^3 = 0 \\ z + cy + c^2x + c^3 = 0 \end{matrix}\right]$ where $a \neq b \neq c$. **DT0064**

EXERCISE (JM)

1. The number of values of k for which the linear equations

$4x + ky + 2z = 0$, $kx + 4y + z = 0$, $2x + 2y + z = 0$ possess a non-zero solution is : **[AIEEE - 2011]**

(1) 1 (2) zero (3) 3 (4) 2 **DT0067**

2. If the trivial solution is the only solution of the system of equations

$x - ky + z = 0$, $kx + 3y - kz = 0$, $3x + y - z = 0$ Then the set of all values of k is: **[AIEEE - 2011]**

(1) $\{2, -3\}$ (2) $R - \{2, -3\}$ (3) $R - \{2\}$ (4) $R - \{-3\}$

DT0068

3. The number of values of k, for which the system of equations : **[JEE(Main)-2013]**

$(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution, is -

(1) infinite (2) 1 (3) 2 (4) 3 **DT0069**

4. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, then

K is equal to : **[JEE(Main)-2014]**

(1) $\alpha\beta$ (2) $\frac{1}{\alpha\beta}$ (3) 1 (4) -1 **DT0070**

5. The set of all values of λ for which the system of linear equations :

$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$

has a non-trivial solution **[JEE(Main)-2015]**

(1) contains two elements (2) contains more than two elements

(3) is an empty set (4) is a singleton **DT0071**

6. The system of linear equations $x + \lambda y - z = 0$, $\lambda x - y - z = 0$, $x + y - \lambda z = 0$ has a non-trivial solution for : **[JEE(Main)-2016]**

(1) exactly three values of λ. (2) infinitely many values of λ.

(3) exactly one value of λ. (4) exactly two values of λ. **DT0072**

7. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x + y + z = 1$$
$$x + ay + z = 1$$
$$ax + by + z = 0$$

has no solution, then S is : **[JEE(Main)-2017]**

(1) a singleton (2) an empty set

(3) an infinite set (4) a finite set containing two or more elements

DT0073

8. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to :

[JEE(Main)-2018]

(1) (–4, 3) (2) (–4, 5) (3) (4, 5) (4) (–4, –5)

DT0074

9. If the system of linear equations

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to : **[JEE(Main)-2018]**

(1) 10 (2) – 30 (3) 30 (4) –10

DT0075

10. The system of linear equations

$$x + y + z = 2$$
$$2x + 3y + 2z = 5$$
$$2x + 3y + (a^2 - 1)z = a + 1$$

[JEE(Main)-2019]

(1) has infinitely many solutions for a = 4 (2) is inconsistent when $|a| = \sqrt{3}$

(3) is inconsistent when a = 4 (4) has a unique solution for $|a| = \sqrt{3}$

DT0088

11. If the system of linear equations

$$x - 4y + 7z = g$$
$$3y - 5z = h$$
$$-2x + 5y - 9z = k$$

is consistent, then : **[JEE(Main)-2019]**

(1) g + h + k = 0 (2) 2g + h + k = 0 (3) g + h + 2k = 0 (4) g + 2h + k = 0

DT0076

12. If the system of equations

$$x+y+z = 5$$
$$x+2y+3z = 9$$
$$x+3y+\alpha z = \beta$$

has infinitely many solutions, then $\beta-\alpha$ equals: **[JEE(Main)-2019]**

(1) 5 (2) 18 (3) 21 (4) 8 **DT0089**

13. Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is : **[JEE(Main)-2019]**

(1) –7 (2) $2(\sqrt{2}+2)$ (3) –5 (4) $2(\sqrt{2}+1)$

DT0077

node06\B0B0-BA\Kota\JEE(Advanced)\Leader\Maths\Sheet\Determinant & Matrix\Eng\02 Ex.p65

14. Let a_1,a_2,a_3, ..., a_{10} be in G.P. with $a_i > 0$ for $i = 1,2,....,10$ and S be the set of pairs (r,k), r k∈N (the set of natural numbers) for which $\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$

Then the number of elements in S, is : **[JEE(Main)-2019]**

(1) Infinitely many (2) 4 (3) 10 (4) 2 **DT0078**

15. The number of values of $\theta \in (0,\pi)$ for which the system of linear equations

$$x + 3y + 7z = 0$$
$$-x + 4y + 7z = 0$$
$$(\sin 3\theta)x + (\cos 2\theta)y + 2z = 0$$

has a non-trivial solution, is : **[JEE(Main)-2019]**

(1) One (2) Three (3) Four (4) Two

DT0090

16. If the system of linear equations

$$2x + 2y + 3z = a$$
$$3x - y + 5z = b$$
$$x - 3y + 2z = c$$

where a, b, c are non-zero real numbers, has more then one solution, then : **[JEE(Main)-2019]**

(1) $b - c - a = 0$ (2) $a + b + c = 0$ (3) $b + c - a = 0$ (4) $b - c + a = 0$

DT0091

17. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)(x + a + b + c)^2$, $x \neq 0$ and $a + b + c \neq 0$, then x is equal to :- **[JEE(Main)-2019]**

(1) $-(a + b + c)$ (2) $2(a + b + c)$ (3) abc (4) $-2(a + b + c)$

DT0092

18. An ordered pair(α,β) for which the system of linear equations

$$(1+\alpha)x + \beta y + z = 2$$
$$\alpha x + (1+\beta)y + z = 3$$
$$\alpha x + \beta y + 2z = 2$$

has a unique solution is **[JEE(Main)-2019]**

(1) (1,–3) (2) (–3,1) (3) (2, 4) (4) (–4, 2)

DT0093

19. The set of all values of λ for which the system of linear equations.

$$x - 2y - 2z = \lambda x$$
$$x + 2y + z = \lambda y$$
$$-x - y = \lambda z$$

has a non-trivial solution. **[JEE(Main)-2019]**

(1) contains more than two elements (2) is a singleton

(3) is an empty set (4) contains exactly two elements

DT0079

20. If $A=\begin{bmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{bmatrix}$; then for all $\theta\in\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$, det(A) lies in the interval :

[JEE(Main)-2019]

(1) $\left[\frac{5}{2},4\right)$ (2) $\left(\frac{3}{2},3\right]$ (3) $\left(0,\frac{3}{2}\right]$ (4) $\left(1,\frac{5}{2}\right]$

DT0094

EXERCISE (JA)

1. Which of the following values of α satisfy the equation $\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2\\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2\end{vmatrix}=-648\alpha$?

[JEE(Advanced)-2015, 4M, –2M]

(A) –4 (B) 9 (C) –9 (D) 4 **DT0085**

2. The total number of distinct $x \in R$ for which $\begin{vmatrix}x & x^2 & 1+x^3\\ 2x & 4x^2 & 1+8x^3\\ 3x & 9x^2 & 1+27x^3\end{vmatrix}=10$ is

[JEE(Advanced)-2016, 3(0)]

DT0086

3. Let $a,\lambda,m \in \mathbb{R}$. Consider the system of linear equations

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

Which of the following statement(s) is(are) correct ?

(A) If a = –3, then the system has infinitely many solutions for all values of λ and μ

(B) If a ≠ –3, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for a = –3

(D) If $\lambda + \mu \neq 0$, then the system has no solution for a = –3 **[JEE(Advanced)-2016, 4(–2)]**

DT0087

ANSWER KEY

EXERCISE (O-1)

1. D **2.** A **3.** C **4.** C **5.** A **6.** C **7.** D
8. D **9.** D **10.** C **11.** B **12.** D **13.** A **14.** B

EXERCISE (O-2)

1. C **2.** B **3.** A **4.** C **5.** A **6.** D **7.** A,B,C,D
8. B,D **9.** A,B,D **10.** (A)→(Q); (B)→(P); (C)→(Q,S); (D)→(R,T)

EXERCISE (S-1)

1. **(a)** p **3.** **(a)** 4, **(b)** 65 **5.** x = 0 or $x = \pm\sqrt{\frac{3}{2}\left(a^2+b^2+c^2\right)}$ **6.** 80

7. $\lambda^2(a^2+b^2+c^2+\lambda)$ **8.** 0

10. **(a)** x = 1, y = 2, z = 3; consistent **(b)** x = 2, y = –1, z = 1; consistent **(c)** inconsistent

11. $K = \frac{33}{2}, x:y:z = -\frac{15}{2}:1:-3$ **12.** 2

13. If $\lambda \neq -5$ then $x = \frac{4}{7}; y = -\frac{9}{7}$ & z = 0; If $\lambda = -5$ then $x = \frac{4-5K}{7}; y = \frac{13K-9}{7}$ and z = K, where $K \in R$

EXERCISE (S-2)

3. (a) $\lambda \neq 3$ (b) $\lambda = 3, \mu = 10$ (c) $\lambda = 3, \mu \neq 10$

4. x = 1 + 2k, y = –3K, z = K, when p = 1; x = 2K, y = 1 – 3K, z = K, when p = 2; where $K \in R$

5. If $K \neq 2, \frac{x}{2(K+6)} = \frac{y}{2K+3} = \frac{z}{6(K-2)} = \frac{1}{2\left(K^2+2K+15\right)}$

If K = 2, then x = λ, $y = \frac{1-2\lambda}{2}$ and z = 0 where $\lambda \in R$

6. 4 **8.** $x = -(a+b+c), y = ab+bc+ca, z = -abc$

EXERCISE (JM)

1. 4 **2.** 2 **3.** 2 **4.** 3 **5.** 1 **6.** 1 **7.** 1
8. 2 **9.** 1 **10.** 2 **11.** 2 **12.** 4 **13.** 3 **14.** 1
15. 4 **16.** 1 **17.** 4 **18.** 3 **19.** 2 **20.** 2

EXERCISE (JA)

1. B,C **2.** 2 **3.** B,C,D

MATRIX

1. INTRODUCTION :

A rectangular array of mn numbers (which may be **real or complex**) in the form of 'm' horizontal lines (called **rows**) and 'n' vertical lines (called **columns**), is called a matrix of order m by n, written as $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ) or‖ An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A=[a_{ij}]_{m\times n}$. The number a_{11}, a_{12},.....etc are known as the elements of the matrix A, a_{ij} belongs to the i^{th} row and j^{th} column and is called the **(i, j)th** element of the matrix $A = [a_{ij}]$.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements :

$a_{11} = 1, a_{12} = 2, a_{13} = 3, a_{21} = 0, a_{22} = -1, a_{23} = 9.$

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \ldots\ldots\ldots\ldots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{12} \\ : \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : ($A = \mathbf{O}_{m\times n}$) An $m \times n$ matrix whose all entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $\mathbf{n > m}$ e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

(e) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $\mathbf{m > n}$ e.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : If number of rows = number of columns $\Rightarrow$ matrix is a square matrix. If number of rows = number of columns = n then, matrix is of the **order 'n'**.

Note : The pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

3. TRACE OF MATRIX :

The sum of the elements of a **square matrix** A lying along the principal diagonal is called the trace of A i.e. (tr(A)). Thus, if $A = [a_{ij}]_{n \times n}$, then $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ...a_{nn}$

Properties of trace of a matrix :

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar then

(i) $tr(\lambda A) = \lambda tr(A)$ (ii) $tr(A + B) = tr(A) + tr(B)$ (iii) $tr(AB) = tr(BA)$

4. SQUARE MATRICES

- Triangular Matrix
  - $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ Upper Triangular $a_{ij} = 0 \ \forall \ i > j$
  - ; $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{bmatrix}$ Lower Triangular $a_{ij} = 0 \ \forall \ i < j$
- Diagonal Matrix denoted as $A = diag\ (a_{11}, a_{22} ...., a_{nn})$ where $a_{ij} = 0$ for $i \neq j$
  - Scalar Matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ If $a_{11} = a_{22} = a_{33} = a$
  - Unit or Identity Matrix $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ If $a_{11} = a_{22} = a_{33} = 1$

Note :

(i) Minimum number of zeros in triangular matrix of order $n = n(n-1)/2$.

(ii) Minimum number of zero in a diagonal matrix of order $n = n(n-1)$.

5. EQUALITY OF MATRICES :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if

(a) both have the same order. (b) $a_{ij} = b_{ij}$ for each pair of i & j .

Illustration 1 : Find the value of x, y, z and w which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$$

Solution : As the given matrices are equal so their corresponding elements are equal.

$x + 3 = -x - 1 \Rightarrow 2x = -4$

$\therefore x = -2$ ....(i)

$2y + x = 0 \Rightarrow 2y - 2 = 0$ [from (i)]

$\Rightarrow y = 1$ ....(ii)

$z - 1 = 3 \Rightarrow z = 4$ ....(iii)

$4w - 8 = 2w \Rightarrow 2w = 8$

$\therefore w = 4$ ....(iv) **Ans.**

Do yourself -1 :

(i) Find 2 × 3 matrix $[a_{ij}]_{2\times 3}$, where $a_{ij} = i + 2j$

(ii) Find the minimum number of zeroes in a triangular matrix of order 4.

(iii) Find minimum number of zeros in a diagonal matrix of order 6.

(iv) If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the values of x,y,a and b.

6. ALGEBRA OF MATRICES :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) **Addition of matrices is commutative :** i.e. A + B = B + A

(b) **Matrix addition is associative :** (A + B) + C = A + (B + C)

(c) **Additive inverse :** If A + B = **O** = B + A, then B is called **additive inverse** of A.

(d) **Existence of additive identity :** Let $A = [a_{ij}]$ be an m × n matrix and **O** be an m × n zero matrix, then A + **O** = **O** + A = A. In other words, **O** is the **additive identity** for matrix addition.

(e) **Cancellation laws** hold good in case of addition of matrices. If A,B,C are matrices of the same order, then A + B = A + C ⇒ B = C (**left cancellation law**) and B + A = C + A ⇒ B = C (**right-cancellation law**)

Note : The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Illustration 2 : If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and A + B – D = **O** (zero matrix), then D matrix will be-

(A) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Solution : Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore \quad A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

⇒ –a = 0 ⇒ a = 0, 1 – b = 0 ⇒ b = 1,

3 – c = 0 ⇒ c = 3, 7 – d = 0 ⇒ d = 7,

5 – e = 0 ⇒ e = 5, 6 – f = 0 ⇒ f = 6

$$\therefore \quad D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

Ans. (C)

Do yourself-2 :

(i) If $A = \begin{bmatrix} 2 & 3 & 9 \\ 8 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 & 2 \\ 6 & 4 & 8 \end{bmatrix}$, then find a matrix C such that A – B + C = **O** and also find the order of the matrix C.

(ii) If $A = \begin{bmatrix} 8 & 9 \\ 7/2 & 8 \\ 1 & -1 \end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive inverse will be the matrix itself.

7. MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Properties of scalar multiplication :

(a) If A and B are two matrices of the same order and 'k' be a scalar then k(A + B) = kA + kB.

(b) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.

(c) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

8. MULTIPLICATION OF MATRICES (Row by Column) :

Let A be a matrix of order m × n and B be a matrix of order p × q, then the matrix multiplication AB is possible if and only if n = p and matrices are said to be **conformable** for multiplication.

In the product AB, A is called pre-factor and B is called post factor.

⇒ AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is m × p & $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3\times 4}$

Then order of AB is 2 × 4.

$$(AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{r=1}^{3} a_{1r}b_{r1}$$

$$(AB)_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{r=1}^{3} a_{2r}b_{r3}$$

In general $(ab)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{r=1}^{3} a_{ir}b_{rj}$

Illustration 3 : If $[1 \ x \ 2]\begin{bmatrix}2 & 3 & 1\\0 & 4 & 2\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\1\\-1\end{bmatrix} = \mathbf{O}$, then the value of x is :-

(A) –1 (B) 0 (C) 1 (D) 2

Solution : The LHS of the equation

$$= [2 \quad 4x+9 \quad 2x+5]\begin{bmatrix}x\\1\\-1\end{bmatrix} = [2x + 4x + 9 - 2x - 5] = 4x + 4$$

Thus $4x + 4 = 0 \Rightarrow x = -1$ **Ans. (A)**

Illustration 4 : If A, B are two matrices such that $A + B = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$, $A - B = \begin{bmatrix}3 & 2\\-2 & 0\end{bmatrix}$, then find AB.

Solution : Given $A + B = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ ....(i) & $A - B = \begin{bmatrix}3 & 2\\-2 & 0\end{bmatrix}$ ....(ii)

Adding (i) & (ii)

$$2A = \begin{bmatrix}4 & 4\\0 & 4\end{bmatrix} \Rightarrow A = \begin{bmatrix}2 & 2\\0 & 2\end{bmatrix}$$

Subtracting (ii) from (i)

$$2B = \begin{bmatrix}-2 & 0\\4 & 4\end{bmatrix} \Rightarrow B = \begin{bmatrix}-1 & 0\\2 & 2\end{bmatrix}$$

Now $AB = \begin{bmatrix}2 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 2\end{bmatrix} = \begin{bmatrix}2 & 4\\4 & 4\end{bmatrix}$ **Ans.**

9. PROPERTIES OF MATRIX MULTIPLICATION :

(a) **Matrix multiplication is not commutative :** i.e. **AB ≠ BA**

Here both AB & BA exist and also they are of the same type but **AB ≠ BA.**

Example :

Let $A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ & $B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$; then $AB = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$; $BA = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

$\Rightarrow$ AB ≠ BA (in general)

(b) $AB = \mathbf{O} \not\Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$ (in general)

Let $A = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$ & $B = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$, then $AB = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Note :

If A and B are two non - zero matrices such that AB = **O** then A and B are called the divisors of zero. If A and B are two matrices such that

(i) AB = BA then A and B are said to commute

(ii) AB = –BA then A and B are said to anticommute

(c) Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then (AB) C = A(BC)

(d) Distributivity :

$$\left.\begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned}\right] \text{Provided A,B \& C are conformable for respective products}$$

Illustration 5 : Let $A=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, B=\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \& C=\begin{bmatrix} 1 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$ be the matrices then, prove that in matrix multiplication cancellation law does not hold.

Solution : We have to show that AB = AC; though B is not equal to C.

We have $AB=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}=\begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times4}$

Now, $AC=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}=\begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times4}$

Here, AB = AC though B is not equal to C. Thus cancellation law does not hold in general.

Do yourself - 3 :

(i) If $A=\begin{bmatrix} 2 & 9 \\ -4 & 5 \end{bmatrix}, B=\begin{bmatrix} 1 & 5 \\ 6 & 8 \end{bmatrix}$ and $C=\begin{bmatrix} 9 & -7 \\ -2 & 4 \end{bmatrix}$, then show that A(B + C) = AB + AC.

(ii) If $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, then prove that $(A-B)^2 \neq A^2 - 2AB + B^2$.

(iii) Find the value of x : $2\begin{bmatrix} 3 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix}+x\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \end{bmatrix}=\begin{bmatrix} 4 & 2 & -2 \\ -8 & -14 & -2 \end{bmatrix}$

10. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

For a square matrix A, $A^n = \underbrace{A.A.A..............A}_{\text{upto n times}}$, where $n \in N$

Note :

(i) $A^m.A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn}$, where $m,n \in N$

(iii) If A and B are square matrices of same order and AB = BA then

$(A+B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ... + {}^nC_nB^n$

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.

Do yourself -4 :

(i) If $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, then prove that $A^n=\begin{bmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{bmatrix}$ where n is positive integer.

(ii) If $A=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}$, where $i=\sqrt{-1}$ and $x \in N$, then A^{4x} equals -

(A) $\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ (B) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ (C) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (D) $\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}$

11. SPECIAL SQUARE MATRICES :

(a) Idempotent Matrix : A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following :

(i) $A^n = A \ \forall\ n \geq 2, n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1

(b) Periodic Matrix : A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) Nilpotent Matrix : A square matrix of the order 'n' is said to be nilpotent matrix of order m, m $\in N$, if $A^m = \mathbf{O}$ & $A^{m-1} \neq \mathbf{O}$.

(d) Involutary Matrix : If $A^2 = I$, the matrix is said to be an involutary matrix. i.e. square roots of identity matrix is involutary matrix.

Note : The determinant value of involutary matrix is 1 or –1.

Illustration 6 : Show that the matrix $A=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ is idempotent.

Solution :

$$A^2 = A.A = \begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}\times\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}2.2+(-2).(-1)+(-4).1 & 2(-2)+(-2).3+(-4).(-2) & 2.(-4)+(-2).4+(-4).(-3)\\ (-1).2+3.(-1)+4.1 & (-1).(-2)+3.3+4.(-2) & (-1).(-4)+3.4+4.(-3)\\ 1.2+(-2).(-1)+(-3).1 & 1.(-2)+(-2).3+(-3).(-2) & 1.(-4)+(-2).4+(-3).(-3)\end{bmatrix}$$

$$=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix} = A$$

Hence the matrix A is idempotent.

Illustration 7 : Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

$\therefore \quad A^3 = \mathbf{O} \qquad$ i.e., $\qquad A^k = \mathbf{O}$

Here $\qquad k = 3$

Hence A is nilpotent of order 3.

Illustration 8 : Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Solution : $$A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence the given matrix A is involutory.

Illustration 9: Show that a square matrix A is involutory, iff (I – A) (I + A) = **O**

Solution : Let A be involutory

Then $A^2 = I$

$(I - A)(I + A) = I^2 + IA - AI - A^2 = I + A - A - A^2 = I - A^2 = \mathbf{O}$

Conversly, let $(I - A)(I + A) = \mathbf{O}$

$\Rightarrow I^2 + IA - AI - A^2 = \mathbf{O}$ $\Rightarrow$ $I + A - A - A^2 = \mathbf{O}$

$\Rightarrow I - A^2 = \mathbf{O}$ $\Rightarrow$ A is involutory

Do yourself - 5 :

(i) The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutary matrix

(C) nilpotent matrix (D) periodic matrix

(ii) If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x

12. THE TRANSPOSE OF A MATRIX : (Changing rows & columns)

Let A be any matrix of order m × n. Then A^T or $A' = [a_{ji}]$ for $1 \le i \le m$ & $1 \le j \le n$ of order n × m

Properties of transpose :

If A^T & B^T denote the transpose of A and B ,

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

Note : In general : $(A_1 . A_2, ... A_n)^T = A_n^T . ... A_2^T . A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

Illustration 10 : If A and B are matrices of order m × n and n × m respectively, then order of matrix $B^T(A^T)^T$ is -

(A) m × n (B) m × m (C) n × n (D) Not defined

Solution : Order of B is n × m so order of B^T will be m × n

Now $(A^T)^T = A$ & its order is m × n. For the multiplication $B^T(A^T)^T$

Number of columns in prefactor ≠ Number of rows in post factor.

Hence this multiplication is not defined. **Ans. (D)**

13. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $A A^T = I$

Note :

(i) The determinant value of orthogonal matrix is either 1 or –1.

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

If $AA^T = I$, then

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \text{ and } \sum_{i=1}^{3} a_i b_i = \sum_{i=1}^{3} b_i c_i = \sum_{i=1}^{3} c_i a_i = 0$$

Illustration 11 : Determine the values of α, β, γ, when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution : Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

$\therefore \quad A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

But given A is orthogonal.

$\therefore \quad AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$4\beta^2 + \gamma^2 = 1$ ....(i)

$2\beta^2 - \gamma^2 = 0$ ....(ii)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ ....(iii)

From (i) and (ii), $6\beta^2 = 1 \quad \therefore \quad \beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm\frac{1}{\sqrt{2}}$, $\beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$ **Ans.**

Do yourself - 6 :

(i) If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$, then show that $(AB)^T = B^T.A^T$.

(ii) If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

(iii) If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & ½ & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

(iv) Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is an orthogonal matrix.

14. SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}\ \forall$ i & j (conjugate elements are equal).
Hence for symmetric matrix $\mathbf{A = A^T}$.

Note : Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) Skew symmetric matrix :

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}\ \forall$ i & j (the pair of conjugate elements are additive inverse of each other). For a skew symmetric matrix $\mathbf{A = -A^T}$.

Note :

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0\ \forall\ i$. Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

(c) Properties of symmetric & skew symmetric matrix :

(i) A is symmetric if $A^T = A$ & A is skew symmetric if $A^T = -A$

(ii) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(iii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iv) If A & B are symmetric matrices then,

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(v) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}} \text{ and } A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Illustration 12 : If A is symmetric as well as skew symmetric matrix, then A is -

(A) diagonal matrix (B) null matrix (C) triangular matrix (D) none of these

Solution : Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

for i = j, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

for $i \neq j$, $a_{ij} = -a_{ji}$ [$\because$ A is skew symmetric] & $a_{ij} = a_{ji}$ [$\because$ A is symmetric]

$\therefore$ $a_{ij} = 0$ for all $i \neq j$

so, $a_{ij} = 0$ for all 'i' and 'j' i.e. A is null matrix. **Ans. (B)**

Do yourself - 7 :

(i) If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x.

(ii) Express matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.

15. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adjA, is defined as the transpose of the cofactor matrix.

Then, $\text{adj}A = [C_{ij}]^T \Rightarrow \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Theorem : A (adj. A) = (adj. A) .A = |A| I_n.

Proof : $A.(\text{adj } A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A.(\text{Adj. } A) = |A| I$$

(whatever may be the value only |A| will come out as a common element)

If $|A| \neq 0$, then $\dfrac{\mathbf{A.(adj.A)}}{\mathbf{|A|}} = \mathbf{I}$ = unit matrix of the same order as that of A

Properties of adjoint matrix :

If **A** be a square matrix of order **n**, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where $|A| \neq 0$

(iii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, where $|A| \neq 0$

(iv) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(v) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

(vi) $\text{adj } A^T = (\text{adj } A)^T$

Method to find adjoint of a 2 × 2 square matrix, directly :

Let A be a 2 × 2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

e.g. If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Illustration 13 : If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj A is equal to -

(A) $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

(B) $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$

(D) none of these

Solution : $\text{adj. } A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ **Ans. (A)**

Illustration 14 : If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then adj (adj A) is equal to -

(A) $8\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (B) $16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (C) $64\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (D) none of these

Solution : $|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now $\text{adj}(\text{adj } A) = |A|^{3-2} A$

$= 8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans. (B)**

Do yourself - 8 :

(i) For any 2×2 matrix, if $A(AdjA) = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$, then $|A|$ is equal -

(A) 20 (B) 625 (C) 15 (D) 0

(ii) Which of the following is/are incorrect ?

(A) Adjoint of a symmetric matrix is skew symmetric matrix.

(B) Adjoint of a diagonal matrix is a diagonal matrix.

(C) $A(AdjA) = (AdjA)A = |A|I$

(D) Adjoint of a unit matrix is a diagonal matrix

(iii) If A be a square matrix of the order 5 and B = Adj(A) then find Adj (5A).

(iv) If A be a square matrix of order 4 and $|A| = 3$ then find adj(adjA).

16. INVERSE OF A MATRIX (Reciprocal Matrix) :

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, AB = I = BA.

B is called the **inverse** (reciprocal) of A and is denoted by $\mathbf{A^{-1}}$. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A| \, I_n$

$A^{-1}. \, A(\text{adj } A) = A^{-1} \, I_n \, |A|$

$I_n(\text{adj } A) = A^{-1} \, |A| \, I_n$

$\therefore \quad \mathbf{A^{-1} = \dfrac{(adj\ A)}{|A|}}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $\mathbf{|A| \neq 0}$

Properties of inverse :

(i) If A & B are invertible matrices of the same order, then $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$.

Note: If $A_1, A_2, \ldots\ldots A_n$ are all invertible square matrices of order n then $(A_1A_2\ldots\ldots A_n)^{-1} = A_n^{-1} \, A_{n-1}^{-1} \ldots\ldots A_2^{-1} \, A_1^{-1}$

(ii) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(iii) If A is invertible, (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}; \ k \in N$

(iv) If A is non-singular matrix, then $|A^{-1}| = |A|^{-1}$

(v) If idempotent matrix is invertible then its inverse will be identity matrix.

(vi) A nilpotent matrix will not be invertible because its determinant value is zero.

(vii) Orthogonal matrix A is always invertible and $A^{-1} = A^T$.

(viii) $A = A^{-1}$ for an involutary matrix.

Cancellation law : Let A,B,C be square matrices of the same order 'n'.

If A is a non-singular matrix, then

(a) $AB = AC \Rightarrow B = C$ (Left cancellation law)

(b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

Note that these cancellation laws hold only if the matrix 'A' is **non-singular** (i.e.$|A| \neq 0$).

Illustration 15 : Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution : $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$\Rightarrow (A^{-1})^T = A^{-1}$ Hence proved

Illustration 16 : $\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$ is equal to -

(A) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (D) none of these

Solution : $\begin{bmatrix} 1 & \tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}^{-1} = \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$\therefore$ Product $= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta/2-\sin^2\theta/2 & -2\sin\theta/2.\cos\theta/2 \\ 2\sin\theta/2.\cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ **Ans. (C)**

Illustration 17: If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and M = AB, then M^{-1} is equal to-

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$ (C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Solution : $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, adj $M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore \quad M^{-1} = \dfrac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ **Ans. (C)**

Do yourself -9 :

(i) If 'A' is a square matrix such that $A^2 = I$ then A^{-1} is equal to -

(A) A + I (B) A (C) 0 (D) 2A

(ii) If 'A' is an orthogonal matrix, then A^{-1} equals -

(A) A (B) A^T (C) A^2 (D) none of these

(iii) If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to -

(A) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (B) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ (C) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (D) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

17. MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_nx^0$, then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI^n$.

where A is the given square matrix. If $f(A)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

18. CHARACTERISTIC EQUATION :

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation of A. After solving the characterstic polynomial the values of 'x' are said to be characterstic roots of the polynomial.

Note : (i) Sum of the roots of the characteristic equation is equal to trace of the matrix.

(ii) Product of the roots of the characteristic equation is equal to the determinant value.

(iii) The degree of characteristic equation is same as the order of the matrix.

Illustration 18 : If $f(x) = x^2 - 3x + 3$ and $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ be a square matrix then prove that $f(A) = \mathbf{O}$. Hence find A^4.

Solution : $A^2 = A.A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix}$

Hence $A^2 - 3A + 3I = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

Aliter : $\because |A - XI| = 0 \Rightarrow \begin{vmatrix} 2-x & 1 \\ -1 & 1-x \end{vmatrix} = 0$

$\Rightarrow (2 - x)(1 - x) + 1 = 0 \Rightarrow x^2 - 3x + 3 = 0$ (characterstic polynomial)

by Cayley-Hamilton Theorem $A^2 - 3A + 3I = \mathbf{O}$. Hence proved.

Now $A^2 = 3A - 3I$

squaring on both the sides

$A^4 = 9(A^2 - 2A + I)$

$$= 9\left(\begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 9\begin{bmatrix} 3-4+1 & 3-2 \\ -3+2 & -2+1 \end{bmatrix}$$

$$= 9\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix}$$

19. CAYLEY - HAMILTON THEOREM :

Every square matrix A satisfy its characteristic equation

i.e. $a_0x^n + a_1x^{n-1} + \dots\dots + a_{n-1}x + a_n = 0$ is the characteristic equation of A, then

$a_0A^n + a_1A^{n-1} + \dots\dots + a_{n-1}A + a_nI = \mathbf{O}$

Note : This theorem is helpful to find the inverse of any non-singular square matrix.

i.e. $a_0A^n + a_1A^{n-1} + \dots\dots + a_{n-1}A + a_nI = \mathbf{O}$

On multiplying by A^{-1} on both the sides of above equation, we get

$$A^{-1} = -\frac{1}{a_n}\left(a_0A^{n-1} + a_1A^{n-2} + \dots\dots a_{n-1}I\right)$$

Illustration 19 : If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, show that $5A^{-1} = A^2 + A - 5I$

Solution : We have the characteristic equation of A.

$|A - xI| = 0$

i.e. $\begin{vmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = 0$

i.e. $x^3 + x^2 - 5x - 5 = 0$

Using Cayley – Hamilton theorem

$A^3 + A^2 - 5A - 5I = \mathbf{O} \quad \Rightarrow \quad 5I = A^3 + A^2 - 5A$

Multiplying by A^{-1}, we get $5A^{-1} = A^2 + A - 5I$

Do yourself -10 :

(i) Determine the characteristic roots of the matrix A. Hence find the trace and determinant value of A.

Where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and also prove that $A^3 - 18A^2 + 45A = \mathbf{O}$.

20. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow$ A X = B ....(i)

Multiplying adjA on both the sides of (i)

$\Rightarrow$ (adjA) AX = (adjA)B $\Rightarrow$ |A|X = (adjA) B

$$|A|X = (\text{adj}A)B$$

- If $|A| \neq 0$: $X = \dfrac{(\text{adj}A)B}{|A|}$ or $X = A^{-1}B$ (unique solution)
- If $|A| = 0$:
  - $(\text{adj}A)B$ = null matrix $\rightarrow$ Infinite solutions or no solution
  - If $(\text{adj }A)B \neq O$ $\rightarrow$ Inconsistent (no solution)

Illustration 20: Solve the system $\begin{matrix} x+y+z=16 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ using matrix method.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

Then the system is AX = B.

$|A| = 6$, hence A is non singular,

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ x = 1, y = 2, z = 3 **Ans.**

Do yourself -11 :

(i) The system of equations x + 2y – 3z = 1, x – y + 4z = 0, 2x + y + z = 1 has -

(A) only two solutions (B) only one solution

(C) no solution (D) infinitely many solutions

(ii) The system of equations x + y + z = 8, x – y + 2z = 6, 3x + 5y – 7z = 14 has-

(A) Unique solution (B) infinite number of solutions

(C) no solution (D) none of these

ANSWERS FOR DO YOURSELF

1 : **(i)** $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ **(ii)** 6 **(iii)** 30 **(iv)** x = 2, y = –1, a = 1, b = –3

2 : **(i)** $\begin{bmatrix} -7 & -10 & -7 \\ -2 & 6 & 3 \end{bmatrix}$ & 2×3 **(ii)** $\begin{bmatrix} -8 & -9 \\ -7/2 & -8 \\ -1 & 1 \end{bmatrix}$

3 : **(iii)** x = –2

4 : **(ii)** C

5 : **(i)** C **(ii)** x = 0

6 : **(ii)** $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

7 : **(i)** –4 **(ii)** $\begin{bmatrix} 2 & 7 & 4 \\ 7 & -7 & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & \frac{3}{2} \\ -3 & -\frac{3}{2} & 0 \end{bmatrix}$

8 : **(i)** C **(ii)** A **(iii)** 625 B **(iv)** 9A

9 : **(i)** B **(ii)** B **(iii)** A

10 : **(i)** λ = 0,3 and 15 tr(A) = 18, |A| = 0

11 : **(i)** D **(ii)** A

EXERCISE (O-1)

1 Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$, then Tr (A) – Tr (B) has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

MT0001

2. If $\begin{bmatrix}x & 3x-y\\ zx+z & 3y-w\end{bmatrix}=\begin{bmatrix}3 & 2\\ 4 & 7\end{bmatrix}$, then

(A) $x = 3$, $y = 7$, $z = 1$, $w = 14$ (B) $x = 3$, $y = -5$, $x = -1$, $w = -4$

(C) $x = 3$, $y = 6$, $z = 2$, $w = 7$ (D) None of these **MT0002**

3. The matrix $A^2+4A-5I$, where I is identity matrix and $A=\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}$ equals :**[JEE-MAIN Online 2013]**

(A) $32\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ (B) $4\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}$ (C) $4\begin{bmatrix}0 & -1\\ 2 & 2\end{bmatrix}$ (D) $32\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}$

MT0003

4. If $M=\begin{bmatrix}0 & 2\\ 5 & 0\end{bmatrix}$ and $N=\begin{bmatrix}0 & 5\\ 2 & 0\end{bmatrix}$, then M^{2011} is -

(A) $10^{1005}M$ (B) $10^{1005}N$ (C) $10^{2010}M$ (D) $10^{2011}M$

MT0008

5. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $A^2-kA-I_2=0$, then value of k is-

(A) 4 (B) 2 (C) 1 (D) –4 **MT0004**

6. Let three matrices are $A=\begin{bmatrix}2 & 1\\ 4 & 1\end{bmatrix}$; $B=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}$ and $C=\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}$, then

$$t_r(A)+t_r\left(\frac{ABC}{2}\right)+t_r\left(\frac{A(BC)^2}{4}\right)+t_r\left(\frac{A(BC)^3}{8}\right)+.......+\infty \text{ is equal to-}$$

(A) 6 (B) 9 (C) 12 (D) none **MT0009**

7. For a matrix A = $\begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$, the value of $\prod_{r=1}^{50}\begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$ is equal to -

(A) $\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 4950 \\ 0 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 5050 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2500 \\ 0 & 1 \end{bmatrix}$

MT0010

8. A and B are two given matrices such that the order of A is 3×4 , if A′ B and BA′ are both defined then

(A) order of B′ is 3 × 4
(B) order of B′A is 4 × 4
(C) order of B′A is 3 × 3
(D) B′A is undefined **MT0005**

9. If the product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}......\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is equal to the matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$ then the value of n is equal to -

(A) 26 (B) 27 (C) 377 (D) 378 **MT0011**

10. Consider a matrix A (θ) = $\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ then

(A) A(θ) is symmetric
(B) A(θ) is skew symmetric
(C) $A^{-1}(\theta) = A(\pi - \theta)$
(D) $A^2(\theta) = A\left(\frac{\pi}{2} - 2\theta\right)$ **MT0012**

11. If p, q, r are 3 real number satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ 0\ 1]$, then 2p + q – r equals :- **[JEE-MAIN Online 2013]**

(A) –1 (B) 4 (C) –3 (D) 2 **MT0006**

12. If A, B and C are $n \times n$ matrices and det(A) = 2, det(B) = 3 and det(C) = 5, then the value of the $\det(A^2BC^{-1})$ is equal to

(A) $\frac{6}{5}$ (B) $\frac{12}{5}$ (C) $\frac{18}{5}$ (D) $\frac{24}{5}$ **MT0007**

13. Which of the following is an orthogonal matrix -

(A) $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$
(B) $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

(C) $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$
(D) $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$ **MT0013**

14. Matrix A $= \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, if xyz = 60 and 8x + 4y + 3z = 20, then A (adj A) is equal to -

(A) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ (B) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$ (C) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ (D) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

MT0017

15. The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is a

(A) non-singular (B) Idempotent (C) Nilpotent (D) Orthogonal

MT0014

16. If A and B are symmetric matrices, then ABA is -

(A) symmetric matrix (B) skew symmetric matrix

(C) diagonal matrix (D) scalar matrix **MT0016**

17. Number of real values of λ for which the matrix $A = \begin{bmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ \lambda+3 & \lambda-2 & \lambda+7 \end{bmatrix}$ has no inverse

(A) 0 (B) 1 (C) 2 (D) infinite

MT0018

18. If $A = [a_{ij}]_{2\times2}$ where $a_{ij} = \begin{cases} i+j & i \neq j \\ i^2 - 2j & i = j \end{cases}$, then A^{-1} is equal to -

(A) $\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ (B) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ 3 & -1 \end{bmatrix}$ (C) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$ (D) $\frac{1}{3}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

MT0020

EXERCISE (O-2)

1. Let A, other than I or –I, be a 2 × 2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A. **[JEE-MAIN Online 2013]**

Statement–1 : Tr (A) = 0

Statement–2 : det (A) = –1

(A) Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation for Statement-1

(B) Statement-1 and Statement-2 are true and Statement-2 is a correct explanation for Statement-1.

(C) Statement-1 is true and Statement-2 is false.

(D) Statement-1 is false and Statement-2 is true. **MT0021**

2. Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22} \right\}$. Then the number of non-singular matrices in the set S is : **[JEE-MAIN Online 2013]**

(A) 24 (B) 10 (C) 20 (D) 27 **MT0025**

3. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix}$ where x,y,z ∈ R. If $B^T AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ then the number of ordered triplet (x,y,z) is-

(A) 2 (B) 6 (C) 8 (D) 9 **MT0027**

4. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of matrix A, then α is -

(A) –2 (B) –1 (C) 2 (D) 5 **MT0026**

[ONE OR MORE THAN ONE ARE CORRECT]

5. Let det(adj(adjA)) = 14^4 where $A = \begin{bmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $x \neq -\frac{11}{3}$, then

(A) x = 1 (B) det(2A) = 112 (C) x = 2 (D) det(2A) = 256

MT0038

6. Let $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, then -

(A) $7|A| = \frac{1}{2}$ (B) $|\text{adj } A| = \frac{1}{196}$

(C) trace (adjA) = $-\frac{1}{7}$ (D) Matrix A is a symmetric matrix

MT0039

7. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, then which of the following is(are) true ?

(trace of A denotes sum of principal diagonal elements of A)

(A) A is invertible (B) trace(adj(adj(A))) = 144

(C) trace(adj(adj(A))) = 8 (D) |adj A| is less than 400 **MT0040**

8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$, then-

(A) $k = 1, \alpha = 14$ (B) $\alpha = 14, \beta = 22$ (C) $k = -1, \beta = 22$ (D) $\alpha = -14, \beta = -22$

MT0029

9. If A and B are 3×3 matrices and $|A| \neq 0$, then which of the following are true?

(A) $|AB| = 0 \Rightarrow |B| = 0$ (B) $|AB| = 0 \Rightarrow B = 0$

(C) $|A^{-1}| = |A|^{-1}$ (D) $|A + A| = 2|A|$ **MT0030**

10. If D_1 and D_2 are two 3×3 diagonal matrices where none of the diagonal element is zero, then -

(A) D_1D_2 is a diagonal matrix (B) $D_1D_2 = D_2D_1$

(C) $D_1^2 + D_2^2$ is a diagonal matrix (D) none of these **MT0031**

11. Let $A = a_{ij}$ be a matrix of order 3 where $a_{ij} = \begin{bmatrix} x & \text{if } i = j, \ x \in R \\ 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{bmatrix}$, then which of the following hold(s) good ?

(A) for x = 2, A is a diagonal matrix.

(B) A is a symmetric matrix

(C) for x = 2, det A has the value equal to 6

(D) Let f (x) = det A, then the function f (x) has both the maxima and minima. **MT0033**

12. If A & B are square matrices of order 2 such that $A + \text{adj}(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $A^T - \text{adj}(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then-

(A) B is symmetric matrix (B) $A^n = A \ \forall \ n \in N$

(C) $|A + A^2 + A^3 + A^4 + A^5| = 0$ (D) $|B + B^2 + B^3 + B^4 + B^5| = 0$ **MT0034**

13. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ & $A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, (where $n \geq 2$ & $n \in N$), then -

(A) a = d (B) b = c

(C) b = a + 1 if n is odd (D) b = a – 1 if n is even **MT0036**

14. If A and B are two orthogonal matrices of order 3, then -

(A) A and B both will be invertible matrices (B) matrix ABA will also be orthogonal

(C) matrix A^2B^2 will also be orthogonal (D) maximum value of $\det\left(\frac{A}{2}\text{adj}(2B)\right)$ is 8.

MT0035

15. If A & B are two non singular matrices of order 3×3 such that $A^T + B = I$ & $BA^T = -B$, then which is/are always true (where X^T denotes transpose of X and I denotes unit matrix)-

(A) $|B| = 2$ (B) $|B| = 8$ (C) $|A| = -1$ (D) $|A| = 1$ **MT0037**

(where |X| denotes determinant value of X)

Paragraph for Question 16 to 17

Consider the system AX = B, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

16. Sum of elements of (adjA) B is-
(A) –1 (B) 2 (C) –2 (D) –4 **MT0041**

17. Value of $tr(XB^T)$ is (where tr(A) denotes trace of matrix A)-
(A) 0 (B) 1 (C) 2 (D) 3 **MT0041**

EXERCISE (S-1)

1. Find the number of 2 × 2 matrix satisfying following conditions :
(i) a_{ij} is 1 or –1 ; (ii) $a_{11}a_{21} + a_{12}a_{22} = 0$ **MT0043**

2. Find the value of x and y that satisfy the equations $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$ **MT0044**

3. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the matrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

MT0047

4. If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\frac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

MT0049

5. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when bc = 1/4.

Hence otherwise evaluate a. **MT0056**

6. If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A) \cdot \frac{1}{2}(I - A) = \mathbf{O}$. **MT0057**

7. Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent matrix.

Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$. **MT0058**

8. $A=\begin{pmatrix}3 & a & -1\\ 2 & 5 & c\\ b & 8 & 2\end{pmatrix}$ is Symmetric and $B=\begin{pmatrix}d & 3 & a\\ b-a & e & -2b-c\\ -2 & 6 & -f\end{pmatrix}$ is Skew Symmetric, then find AB.

Is AB a symmetric, Skew Symmetric or neither of them. Justify your answer. **MT0050**

9. Express the matrix $\begin{bmatrix}1 & 2 & 5\\ 2 & 3 & -6\\ -1 & 0 & 4\end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also express the matrix as a sum of a symmetric and a skew symmetric matrix. **MT0051**

10. (a) A is a square matrix of order n.
ℓ = maximum number of distinct entries if A is a triangular matrix
m = maximum number of distinct entries if A is a diagonal matrix
p = minimum number of zeroes if A is a triangular matrix.
If $\ell + 5 = p + 2m$, find the order of the matrix. **MT0052**

(b) Let A be the set of all 3×3 skew symmetric matrices whose entries are either –1, 0 or 1. If there are exactly three 0's, three 1's and three (–1)'s, then find the number of such matrices. **MT0053**

11. If A is an idempotent non-zero matrix and I is an identity matrix of the same order, find the value of n, $n \in N$, such that $(A + I)^n = I + 127\,A$. **MT0054**

12. Let $A=\begin{bmatrix}2 & 0 & 7\\ 0 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-x & 14x & 7x\\ 0 & 1 & 0\\ x & -4x & -2x\end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and $AB \neq I$ (where I is an identity matrix of order 3×3).

Find the value of Tr. $\left(AB + (AB)^2 + (AB)^3 + ... + (AB)^{100}\right)$,

where Tr. (A) denotes the trace of matrix A. **MT0059**

13. If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then prove that value of f and g satisfying the matrix equation $A^2 + fA + gI = \mathbf{O}$ are equal to $-t_r(A)$ and determinant of A respectively. Given a, b, c, d are non zero reals and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. **MT0118**

14. Let $A=\begin{bmatrix}3x^2\\ 1\\ 6x\end{bmatrix}$, $B=[a\ b\ c]$ and $C=\begin{bmatrix}(x+2)^2 & 5x^2 & 2x\\ 5x^2 & 2x & (x+2)^2\\ 2x & (x+2)^2 & 5x^2\end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$. Given that $tr(AB) = tr(C)\ \forall\ x \in R$, where tr(A) denotes trace of A. Find the value of (a + b + c) **MT0055**

EXERCISE (S-2)

1. For the matrix $A=\begin{bmatrix}4 & -4 & 5\\ -2 & 3 & -3\\ 3 & -3 & 4\end{bmatrix}$ find A^{-2}. **MT0068**

2. (a) Given $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 4 & 1\\ 2 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$. Find P such that $BPA=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$. **MT0062**

(b) Find the matrix A satisfying the matrix equation, $\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}.A.\begin{bmatrix}3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}2 & 4\\ 3 & -1\end{bmatrix}$.

MT0063

3. Let A be a 3 × 3 matrix such that $a_{11}=a_{33}=2$ and all the other $a_{ij}=1$. Let $A^{-1}=xA^2+yA+zI$, then find the value of $(x+y+z)$ where I is a unit matrix of order 3. **MT0064**

4. Given that $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 2 & 3\\ 1 & -1 & 3\end{bmatrix}$, $C=\begin{bmatrix}2 & 1 & 1\\ 2 & 2 & 1\\ 1 & 1 & 1\end{bmatrix}$, $D=\begin{bmatrix}10\\ 13\\ 9\end{bmatrix}$ and that Cb = D.

Solve the matrix equation Ax = b. **MT0065**

5. Let $A=\begin{bmatrix}1 & \frac{3}{2}\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}4 & -3\\ -2 & 2\end{bmatrix}$ and $C_r=\begin{bmatrix}r.3^r & 2^r\\ 0 & (r-1)3^r\end{bmatrix}$ be 3 given matrices.

Compute the value of $\sum_{r=1}^{50} tr.\left((AB)^r C_r\right)$. (where tr.(A) denotes trace of matrix A) **MT0067**

6. Given $A=\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}$; $B=\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(a) AX = A (b) XA = I (c) XB = **O** but BX ≠ **O**. **MT0069**

7. Find the product of two matrices A & B, where $A=\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$ & $B=\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$ and use it to solve the following system of linear equations, x + y + 2z = 1; 3x + 2y + z = 7; 2x + y + 3z = 2.

MT0074

8. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(a) has a unique solution ; (b) has no solution and (c) has infinitely many solutions **MT0073**

9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then solve the following matrix equation.

(a) $AX = B - I$ (b) $(B - I)X = IC$ (c) $CX = A$ **MT0072**

10. $A_{3\times 3}$ is a matrix such that $|A|=a$, $B = (\text{adj } A)$ such that $|B| = b$. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots\ldots.$ up to ∞, and $a = 3$. **MT0071**

11. If A and B are square matrices of order 3, where $|A| = -2$ and $|B| = 1$, then find $\left|\left(A^{-1}\right)\text{adj}\left(B^{-1}\right)\text{adj}\left(2A^{-1}\right)\right|$. **MT0060**

EXERCISE (JM)

1. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012]**

(1) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (4) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ **MT0081**

2. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

[JEE(Main) - 2013]

(1) 4 (2) 11 (3) 5 (4) 0 **MT0082**

3. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, the BB' equals :

[JEE(Main) - 2014]

(1) $I + B$ (2) I (3) B^{-1} (4) $(B^{-1})'$ **MT0083**

4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to : **[JEE(Main)-2015]**

(1) (2, 1) (2) (–2, –1) (3) (2, –1) (4) (–2, 1) **MT0084**

5. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = A A^T$, then $5a + b$ is equal to : **[JEE(Main)-2016]**

(1) 13 (2) –1 (3) 5 (4) 4 **MT0085**

6. If $A=\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to :- **[JEE(Main)-2017]**

(1) $\begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$ (2) $\begin{bmatrix}72 & -84\\ -63 & 51\end{bmatrix}$ (3) $\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$ (4) $\begin{bmatrix}51 & 84\\ 63 & 72\end{bmatrix}$

MT0086

7. If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then the matrix A^{-50} when $\theta=\frac{\pi}{12}$, is equal to : **[JEE(Main)-2019]**

(1) $\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$ (2) $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ (3) $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ (4) $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

MT0119

8. If $A=\begin{bmatrix}e^t & e^{-t}\cos t & e^{-t}\sin t\\ e^t & -e^{-t}\cos t-e^{-t}\sin t & -e^{-t}\sin t+e^{-t}\cos t\\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t\end{bmatrix}$. Then A is - **[JEE(Main)-2019]**

(1) Invertible only if $t=\frac{\pi}{2}$ (2) not invertible for any tεR

(3) invertible for all tεR (4) invertible only if t=π **MT0120**

9. Let $A=\begin{bmatrix}2 & b & 1\\ b & b^2+1 & b\\ 1 & b & 2\end{bmatrix}$ where $b > 0$. Then the minimum value of $\frac{\det(A)}{b}$ is : **[JEE(Main)-2019]**

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $-2\sqrt{3}$ (4) $2\sqrt{3}$ **MT0121**

10. Let $A=\begin{pmatrix}0 & 2q & r\\ p & q & -r\\ p & -q & r\end{pmatrix}$. It $AA^T = I_3$, then |p| is : **[JEE(Main)-2019]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{1}{\sqrt{6}}$ (4) $\frac{1}{\sqrt{3}}$ **MT0122**

11. Let A and B be two invertible matrices of order 3 × 3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to :- **[JEE(Main)-2019]**

(1) 16 (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1 **MT0087**

12. Let $P=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 9 & 3 & 1\end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q-P^5 = I_3$. Then $\frac{q_{21}+q_{31}}{q_{32}}$ is equal to :

[JEE(Main)-2019]

(1) 15 (2) 9 (3) 135 (4) 10 **MT0088**

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EXERCISE (JA)

1. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **[JEE 2010, 3]**

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2 **MT0098**

2. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then [k] is equal to

[**Note :** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k]. **[JEE 2010, 3]**

MT0099

3. Let ω≠1 be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

where each of a,b and c is either ω or ω^2. Then the number of distinct matrices in the set S is-

(A) 2 (B) 6 (C) 4 (D) 8

[JEE 2011, 3, (–1)]

MT0102

4. Let M be 3 × 3 matrix satisfying $M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

Then the sum of the diagonal entries of M is **[JEE 2011, 4]**

MT0103

5. Let P =$[a_{ij}]$ be a 3 × 3 matrix and let Q = $[b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is - **[JEE 2012, 3M, –1M]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13} **MT0104**

6. If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[JEE 2012, 3M, –1M]**

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D) PX = –X

MT0105

7. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) - **[JEE 2012, 4M]**

(A) –2 (B) –1 (C) 1 (D) 2 **MT0106**

8. For 3 × 3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

(A) $N^TM N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj (M N) for all invertible matrices M and N

[JEE-Advanced 2013, 4, (–1)]

MT0107

9. Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with nonzero entries in the main diagonal

(D) the product of entries in the main diagonal of M is not the square of an integer

[JEE(Advanced)-2014, 3]

MT0108

10. Let M and N be two 3 × 3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3 × 3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

[JEE(Advanced)-2014, 3]

MT0109

11. Let X and Y be two arbitrary, 3 × 3, non-zero, skew-symmetric matrices and Z be an arbitrary 3 × 3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced)-2015, 4M, –2M]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

MT0110

12. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then-

(A) $\alpha = 0,\ k = 8$ (B) $4\alpha - k + 8 = 0$

(C) $\det(P\operatorname{adj}(Q)) = 2^9$ (D) $\det(Q\operatorname{adj}(P)) = 2^{13}$

[JEE(Advanced)-2016, 4(–2)]

MT0111

13. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE(Advanced)-2016, 3(–1)]**

(A) 52 (B) 103 (C) 201 (D) 205 **MT0112**

14. Which of the following is(are) NOT the square of a 3×3 matrix with real entries ?

[JEE(Advanced)-2017, 4(–2)]

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

MT0113

15. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced)-2017, 3(–1)]**

(A) 198 (B) 126 (C) 135 (D) 162 **MT0114**

16. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced)-2017, 3]**

MT0115

17. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced)-2018, 4(–2)]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$ **MT0116**

18. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ . **[JEE(Advanced)-2018, 3(0)]**
MT0117

19. Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If
α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and
β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,
then the value of $\alpha^* + \beta^*$ is **[JEE(Advanced)-2019, 3(–1)]**

(1) $-\frac{37}{16}$ (2) $-\frac{29}{16}$ (3) $-\frac{31}{16}$ (4) $-\frac{17}{16}$ **MT0123**

20. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $a + b = 3$
(2) $\det(\text{adj}M^2) = 81$
(3) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$
(4) If $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

MT0124

21. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_K^T$

where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct?

[JEE(Advanced)-2019, 4(–1)]

(1) $X - 30I$ is an invertible matrix

(2) The sum of diagonal entries of X is 18

(3) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$

(4) X is a symmetric matrix **MT0125**

22. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) For x = 1, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(2) There exists a real number x such that PQ = QP

(3) $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in \mathbb{R}$

(4) For x = 0, if $R \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then a + b = 5 **MT0126**

ANSWER KEY

EXERCISE (O-1)

1 C	**2.** A	**3.** B	**4.** A	**5.** A	**6.** A	**7.** D
8. B	**9.** B	**10.** C	**11.** C	**12.** B	**13.** A	**14.** C
15. B	**16.** A	**17.** D	**18.** A			

EXERCISE (O-2)

1. A	**2.** C	**3.** C	**4.** D	**5.** A,B	**6.** B,C,D	**7.** A,B,D
8. B,C	**9.** A,C	**10.** A,B,C	**11.** B,D	**12.** A,C	**13.** A,B,C,D	**14.** A,B,C,D
15. B,C	**16.** C	**17.** A				

EXERCISE (S-1)

1. 8 **2.** $x = \frac{3}{2}, y = 2$ **4.** 1 **5.** f(a) = 1/4, a = 1/2 **7.** $\begin{bmatrix} 1 & 0 \\ 4014 & 1 \end{bmatrix}$

8. AB is neither symmetric nor skew symmetric

9. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$ **10.** (a) 4, (b) 8

11. n = 7 **12.** 100 **13.** f = – (a + d); g = ad – bc **14.** 7

EXERCISE (S-2)

1. $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$ **2.** (a) $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$; (b) $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

3. 1 **4.** $x_1 = 1,\ x_2 = -1,\ x_3 = 1$ **5.** $3(49.3^{50} + 1)$

6. (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in R$; (ii) X does not exist;

(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ $a, c \in R$ and $3a + c \neq 0$; $3b + d \neq 0$

7. x = 2, y = 1, z = –1 **8.** (i) $a \neq -3, b \in R$; (ii) $a = -3$ and $b \neq 1/3$; (iii) $a = -3,\ b = 1/3$

9. (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$, (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (c) no solution **10.** 225 **11.** – 8

EXERCISE (JM)

1. 1	**2.** 2	**3.** 2	**4.** 2	**5.** 3	**6.** 3	**7.** 1
8. 3	**9.** 4	**10.** 1	**11.** 2	**12.** 4		

EXERCISE (JA)

1. A,	**2.** 4	**3.** A	**4.** 9	**5.** D	**6.** D	**7.** A,D
8. C,D	**9.** C,D	**10.** A,B	**11.** C,D	**12.** B,C	**13.** B	**14.** A,B
15. A	**16.** 1	**17.** A,D	**18.** 4	**19.** 2	**20.** 1,3,4	**21.** 2,3,4
22. 3,4						

Important Notes