VECTORS

1. INTRODUCTION :

Vectors constitute one of the several Mathematical systems which can be usefully employed to provide mathematical handling for certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics.

Vectors facilitate mathematical study of such physical quantities as possess Direction in addition to Magnitude. Velocity of a particle, for example, is one such quantity.

2. Physical quantities are broadly divided in two categories viz **(a) Vector Quantities** & **(b) Scalar quantities.**

(a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note :

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

For example, the rotations of a rigid body through finite angles have both magnitude & direction but do not satisfy the law of vector addition therefore not a vector.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

3. MATHEMATICAL DESCRIPTION OF VECTOR & SCALAR :

To understand vectors mathematically we will first understand directed line segment.

Directed line segment :

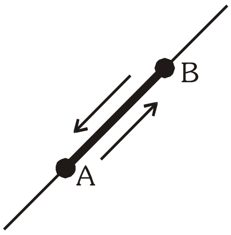
Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$. The two end points of a directed line segment are not interchangeable and the directed line segments.

$\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector :

A directed line segment is called vector. Every directed line segment have three essential characteristics.



(i) Length : The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

(ii) Support : The line of unlimited length of which a directed line segment is a part is called its line of support or simply the Support.

(iii) Sense : The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ from B to A so that the sense of a directed line segment is from its initial to the terminal point.

(b) Scalar :

Any real number is a scalar.

4. EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(a) the same length,

(b) the same or parallel supports and

(c) the same sense.

Note : Components of two equal vectors taken in any arbitrary direction are equal. i.e. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, where $\hat{i},\ \hat{j}$ & $\hat{k}$ are the unit vectors taken along co-ordinate axes, then $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.

Illustration 1 : Let $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k},\ \vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{c} = -2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{r} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$, then find $\lambda + \mu + \nu$.

Solution :

$$3\hat{i} + 2\hat{j} - 5\hat{k} = \lambda(2\hat{i} - \hat{j} + \hat{k}) + \mu(\hat{i} + 3\hat{j} - 2\hat{k}) + \nu(-2\hat{i} + \hat{j} - 3\hat{k})$$

$$= (2\lambda + \mu - 2\nu)\hat{i} + (-\lambda + 3\mu + \nu)\hat{j} + (\lambda - 2\mu - 3\nu)\hat{k}$$

Equating components of equal vectors

$2\lambda + \mu - 2\nu = 3$ ....(i)

$-\lambda + 3\mu + \nu = 2$ ....(ii)

$\lambda - 2\mu - 3\nu = -5$ ....(iii)

on solving (i), (ii) & (iii)

we get $\lambda = 3,\ \mu = 1,\ \nu = 2$

So $\lambda + \mu + \nu = 6$. **Ans.**

Do yourself - 1 :

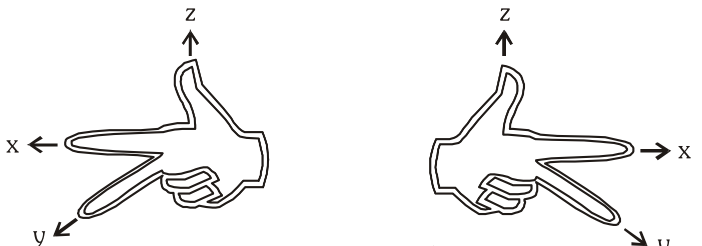
(i) If $\vec{a} = 2\hat{i} + \mu\hat{j} - 7\hat{k}$ and $\vec{b} = \lambda\hat{i} + \sqrt{3}\,\hat{j} - 7\hat{k}$ are two equal vectors, then find $\lambda^2 + \mu^2$.

(ii) If $\vec{a}$, $\vec{b}$ are two vectors then which of the following statements is/are correct -

(A) $\vec{a} = -\vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$ (B) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$

(C) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$ (D) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm 2\vec{b}$

5. LEFT AND RIGHT - HANDED ORIENTATION (CONFIGURATIONS) :



Right - Handed Triplet Left - Handed Triplet

For each hand take the directions Ox, Oy and Oz as shown in the figure. Thus we get two rectangular coordinate systems. Can they be made congruent ? They cannot be, because the two hands have different orientations. Therefore these two systems are different.

A rectangular coordinate system which can be made congruent with the system formed with the help of right hand (or left hand) is called a right handed (or left handed) rectangular coordinates system.

Thus we have the following condition to identify these two systems using sense of rotation :

(a) If the rotation from Ox to Oy is in the anticlockwise direction and Oz is directed upwards (see right hand), then the system is right handed.

(b) If the rotation from Ox to Oy is clockwise and Oz is directed upward (see left hand) then the system is left handed.

Here after we shall use the right-handed rectangular Cartesian coordinate system (or Ortho-normal system).

6. ALGEBRA OF VECTORS :

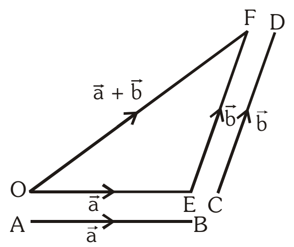
It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(a) Addition of two vectors :

The vectors have magnitude as well as direction, therefore their addition is different than addition of real numbers.

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, which are represented by $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Their addition can be performed in the following two ways :

(i) **Triangle law of addition of vectors :** If two vectors can be represented in magnitude and direction by the two sides of a triangle, taken in order, then their sum will be represented by the third side in reverse order.



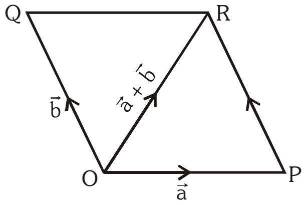
Let O be the fixed point in the plane of vectors. Draw a line segment $\overrightarrow{OE}$ from O, equal and parallel to $\overrightarrow{AB}$, which represents the vector $\vec{a}$. Now from E, draw a line segment $\overrightarrow{EF}$, equal and parallel to $\overrightarrow{CD}$, which represents the vector $\vec{b}$. Line segment $\overrightarrow{OF}$ obtained by joining O and F represents the sum of vectors $\vec{a}$ and $\vec{b}$.

i.e. $\overrightarrow{OE}+\overrightarrow{EF}=\overrightarrow{OF}$

or $\vec{a}+\vec{b}=\overrightarrow{OF}$

This method of addition of two vectors is called **Triangle law of addition of vectors.**

(ii) Parallelogram law of addition of vectors : If two vectors be represented in magnitude and direction by the two adjacent sides of a parallelogram then their sum will be represented by the diagonal through the co-initial point.



Let $\vec{a}$ and $\vec{b}$ be vectors drawn from point O denoted by line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Now complete the parallelogram OPRQ. Then the vector represented by the diagonal OR will represent the sum of the vectors $\vec{a}$ and $\vec{b}$.

i.e. $\overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OR}$

or $\vec{a}+\vec{b}=\overrightarrow{OR}$

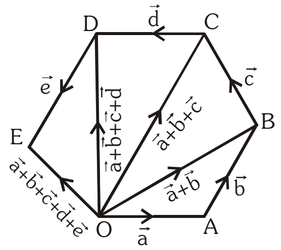
This method of addition of two vectors is called **Parallelogram law of addition of vectors.**

(iii) Properties of vector addition :

(1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative) (2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

(3) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ (additive identity) (4) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ (additive inverse)

(b) Polygon law of vector Addition (Addition of more than two vectors):


D
$\vec{d}$
C
$\vec{e}$
$\vec{c}$
E
B
$\vec{a}+\vec{b}+\vec{c}+\vec{d}$
$\vec{a}+\vec{b}+\vec{c}$
$\vec{a}+\vec{b}$
$\vec{b}$
$\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$
O
$\vec{a}$
A

Addition of more than two vectors is found to be by repetition of triangle law. Suppose we have to find the sum of five vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ and $\vec{e}$. If these vectors be represented by line segment $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DE}$ respectively, then their sum will be denoted by $\overrightarrow{OE}$. This is the vector represented by rest (last) side of the polygon OABCDE in reverse order. We can also make it clear this way :

By triangle's law

$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$ or $\vec{a}+\vec{b}=\overrightarrow{OB}$

$\overrightarrow{OB}+\vec{c}=\overrightarrow{OC}$ or $(\vec{a}+\vec{b})+\vec{c}=\overrightarrow{OC}$

$\overrightarrow{OC}+\vec{d}=\overrightarrow{OD}$ or $(\vec{a}+\vec{b}+\vec{c})+\vec{d}=\overrightarrow{OD}$

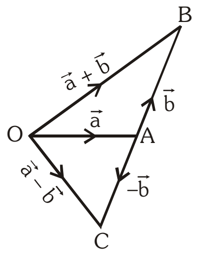
$\overrightarrow{OD}+\vec{e}=\overrightarrow{OE}$ or $(\vec{a}+\vec{b}+\vec{c}+\vec{d})+\vec{e}=\overrightarrow{OE}$

Here, we see that $\overrightarrow{OE}$ is represented by the line segment joining the initial point O of the first vector $\vec{a}$ and the final point of the last vector $\vec{e}$.

In order to find the sum of more that two vectors by this method, a polygon is formed. Therefore this method is known as the **polygon law of addition.**

Note : If the initial point of the first vector and the final point of the last vector are the same, then the sum of the vectors will be a null vector.

(c) Subtraction of Vectors :



Vector $-\vec{b}$ has length equals to vector $\vec{b}$ but its direction is opposite. Subtraction of vector $\vec{a}$ and $\vec{b}$ is defined as addition of $\vec{a}$ and $(-\vec{b})$. It is written as follows :

$$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$$

Geometrical representation :

In the given diagram, $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$. We extend the line AB in opposite direction upto C, where AB = AC. The line segment $\overrightarrow{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overrightarrow{OC}$ is $\vec{a}+(-\vec{b})$. i.e. denotes the vector $\vec{a}-\vec{b}$.

Note :

(i) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

(ii) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(iii) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtraction of vectors does not obey the associative law.

(d) Multiplication of vector by scalars :

If $\vec{a}$ is a vector & m is a scalar, then m($\vec{a}$) is a vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

(i) $m(\vec{a})=(\vec{a})m=m\vec{a}$

(ii) $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

(iii) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

(iv) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

Illustration 2 : ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ equals :-

(A) $\overrightarrow{OP}$ (B) $2\overrightarrow{OP}$ (C) $3\overrightarrow{OP}$ (D) $4\overrightarrow{OP}$

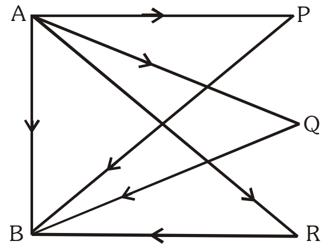
Solution : Since, P bisects both the diagonal AC and BD, so

$\therefore\ \overrightarrow{OA}+\overrightarrow{OC}=2\overrightarrow{OP}$ and $\overrightarrow{OB}+\overrightarrow{OD}=2\overrightarrow{OP} \Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$ **Ans. [D]**

Illustration 3 : A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP},\overrightarrow{AQ},\overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB},\overrightarrow{QB},\overrightarrow{RB}$ acts on point B then resultant is :-

(A) $3\overrightarrow{AB}$ (B) $3\overrightarrow{BA}$ (C) $3\overrightarrow{PQ}$ (D) $3\overrightarrow{PR}$

Solution : From figure



$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$

$\overrightarrow{AQ}+\overrightarrow{QB}=\overrightarrow{AB}$

$\overrightarrow{AR}+\overrightarrow{RB}=\overrightarrow{AB}$

So $(\overrightarrow{AP}+\overrightarrow{AQ}+\overrightarrow{AR})+(\overrightarrow{PB}+\overrightarrow{QB}+\overrightarrow{RB})=3\overrightarrow{AB}$

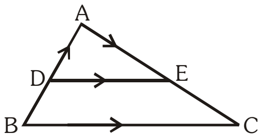
so required resultant = $3\overrightarrow{AB}$. **Ans. [A]**

Illustration 4 : Prove that the line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

Solution : Let the middle points of side AB and AC of a ΔABC be D and E respectively.

$\overrightarrow{BA}=2\overrightarrow{DA}$ and $\overrightarrow{AC}=2\overrightarrow{AE}$

Now in ΔABC, by triangle law of addition



$\overrightarrow{BA}+\overrightarrow{AC}=\overrightarrow{BC}$

$2\overrightarrow{DA}+2\overrightarrow{AE}=\overrightarrow{BC} \Rightarrow \overrightarrow{DA}+\overrightarrow{AE}=\frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{DE}=\frac{1}{2}\overrightarrow{BC}$

Hence, line DE is parallel to third side BC of triangle and half of it.

Do yourself - 2 :

(i) If $\vec{a}, \vec{b}, \vec{c}$ be the vectors represented by the sides of a triangle taken in order, then prove that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

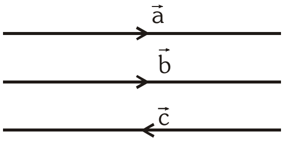
(ii) If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, then prove that the points P, Q and R are collinear.

(iii) For any two vectors $\vec{a}$ and $\vec{b}$ prove that

(a) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ (b) $|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$ (c) $|\vec{a}+\vec{b}|\ge|\vec{a}|-|\vec{b}|$

Note : In general for any non-zero vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ one may note that although $\vec{\mathbf{a}}+\vec{\mathbf{b}}+\vec{\mathbf{c}}=\vec{\mathbf{0}}$ but it will not always represent the three sides of a triangle.

7. COLLINEAR VECTORS :



Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Note :

(i) Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a}=K\vec{b}$, where $K\in R$

(ii) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are two collinear vectors then $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$.

(iii) If $\vec{\mathbf{a}}$ & $\vec{\mathbf{b}}$ are two non-zero, non-collinear vectors such that $x\vec{a}+y\vec{b}=\vec{0}\ \Rightarrow\ x=y=0$

8. CO-INITIAL VECTORS :

Vectors having same initial point are called Co-initial Vectors.

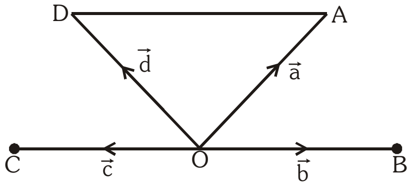


Illustration 5 : If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then find the value of x for which vectors :

$\vec{\alpha}=(x-2)\vec{a}+\vec{b}$ and $\vec{\beta}=(3+2x)\vec{a}-2\vec{b}$ are collinear.

Solution : Since the vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore$ there exist scalar λ such that $\vec{\alpha}=\lambda\vec{\beta}$

$\Rightarrow\ (x-2)\vec{a}+\vec{b}=\lambda\,\{(3+2x)\vec{a}-2\vec{b}\}\ \Rightarrow\ (x-2-\lambda\,(3+2x))\vec{a}+(1+2\lambda)\vec{b}=\vec{0}$

$\Rightarrow\ x-2-\lambda\,(3+2x)=0$ and $1+2\lambda=0$

$x-2-\lambda(3+2x)=0$ and $\lambda=-\frac{1}{2}$

$\Rightarrow\ x-2+\frac{1}{2}(3+2x)=0\ \Rightarrow\ 4x-1=0\ \Rightarrow\ x=\frac{1}{4}$. **Ans.**

Illustration 6 : If $A\equiv(2\hat{i}+3\hat{j})$, $B\equiv(p\hat{i}+9\hat{j})$ and $C\equiv(\hat{i}-\hat{j})$ are collinear, then the value of p is :-

(A) 1/2 (B) 3/2 (C) 7/2 (D) 5/2

Solution : $\overrightarrow{AB}=(p-2)\hat{i}+6\hat{j}$, $\overrightarrow{AC}=-\hat{i}-4\hat{j}$

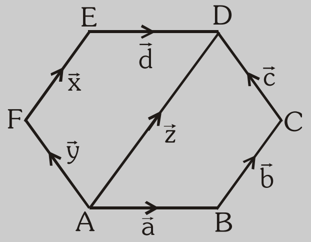
Now A, B, C are collinear $\Leftrightarrow\ \overrightarrow{AB}\,||\,\overrightarrow{AC}\ \Leftrightarrow\ \frac{p-2}{-1}=\frac{6}{-4}\ \Leftrightarrow\ p=7/2$ **Ans. [C]**

Illustration 7 : The value of λ when $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=8\hat{i}+\lambda\hat{j}+4\hat{k}$ are parallel is :-

(A) 4 (B) – 6 (C) – 12 (D) 1

Solution : Since $\vec{a}$ & $\vec{b}$ are parallel $\Rightarrow\ \frac{2}{8}=-\frac{3}{\lambda}=\frac{1}{4}\ \Rightarrow\ \lambda=-12$ **Ans. [C]**

Do yourself - 3 :



(i) In the given figure of regular hexagon, which vectors are (provided vertices A, B, C, D, E, F are all fixed) :

(a) Parallel (b) Equal

(c) Coinitial (d) Parallel but not equal.

(ii) If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 8\hat{i} - 12\hat{j} + 16\hat{k}$ such that $\vec{a} = \lambda\vec{b}$, then λ equals to

(iii) If $3\vec{a} + 2\vec{b} = 5\vec{c}$ and $8\vec{a} - 7\vec{b} = 4\vec{c}$, then which statement is/are true :

(A) $|\vec{a}| > |\vec{b}|$ (B) $|\vec{c}| > |\vec{b}|$

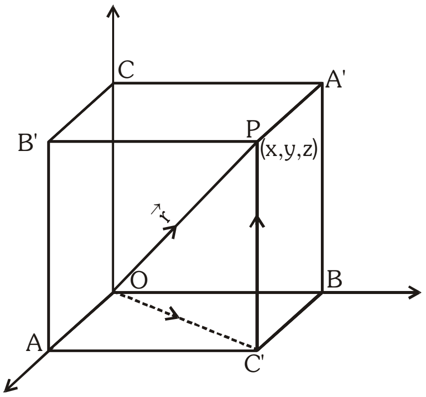
(C) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors. (D) $|\vec{a}| = |\vec{b}|$

9. COPLANAR VECTORS :

A given number of vectors are called coplanar if their supports are all parallel to the same plane. Note that “TWO VECTORS ARE ALWAYS COPLANAR ”.

Note : Coplanar vectors may have any directions or magnitude.

10. REPRESENTATION OF A VECTOR IN SPACE IN TERMS OF 3 ORTHONORMAL TRIAD OF UNIT VECTORS :



Let P(x, y, z) be a point in space with reference to OX, OY and OZ as the coordinate axes, then OA = x, OB = y and OC = z

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively, then

$$\overrightarrow{OA} = x\hat{i},\ \overrightarrow{OB} = y\hat{j},\ \overrightarrow{OC} = z\hat{k}$$

$$\overrightarrow{OP} = \overrightarrow{OC'} + \overrightarrow{C'P} = \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC} \qquad [\because\ \overrightarrow{C'P} = \overrightarrow{OC}]$$

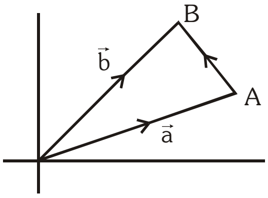
$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{i} + y\hat{j} + z\hat{k}$$

If $\overrightarrow{OP} = \vec{r}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \overline{OP} = \sqrt{x^2 + y^2 + z^2}$$

11. POSITION VECTOR :



Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a} =$ pv of B – pv of A.

12. ZERO VECTOR OR NULL VECTOR :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a zero vector. It is denoted by $\vec{0}$. It can have any arbitrary direction and any line as its line of support.

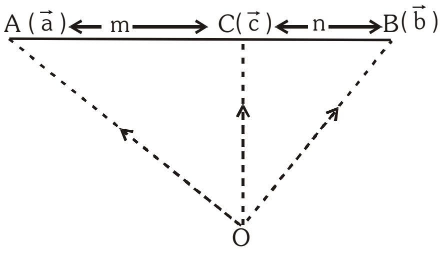
13. UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ (provided $|\vec{a}| \neq 0$)

14. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point $C(\vec{r})$ which divides AB in the ratio m : n is given by :

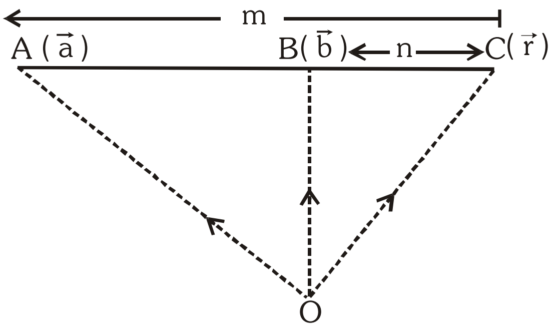
(a) Internal Division :



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Note : Position vector of mid point of **AB** = $\dfrac{\vec{a}+\vec{b}}{2}$

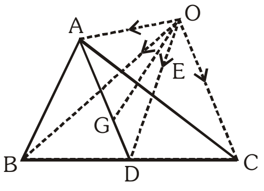
(b) External division :



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Illustration 8 : Prove that the medians of a triangle are concurrent.

Solution : Let ABC be a triangle and position vectors of three vertices A, B and C with respect to the origin O be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



$\therefore \quad \overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$

Again, let D be the middle point of the side BC,

so the position vector of point D is $\overrightarrow{OD} = \dfrac{\vec{b}+\vec{c}}{2}$

Now take a point G, which divides the median AD in the ratio 2 : 1.

Position vector of point G is $\overrightarrow{OG} = \dfrac{1.\overrightarrow{OA} + 2.\overrightarrow{OD}}{1+2} = \dfrac{1.\vec{a} + 2.\frac{1}{2}\left(\vec{b}+\vec{c}\right)}{1+2} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Similarly, the position vector of the middle points of the other two medians, which divide the medians in the ratio 2 : 1 will comes out to the same $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$, which is the position vector of G.

Hence, the medians of the triangles meet in G i.e. are concurrent.

Illustration 9 : If the middle points of sides BC, CA & AB of triangle ABC are respectively D, E, F then position vector of centroid of triangle DEF, when position vector of A,B,C are respectively $\hat{i}+\hat{j},\ \hat{j}+\hat{k},\ \hat{k}+\hat{i}$ is -

(A) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $(\hat{i}+\hat{j}+\hat{k})$ (C) $2(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

Solution : The position vector of points D, E, F are respectively $\frac{\hat{i}+\hat{j}}{2}+\hat{k},\ \hat{i}+\frac{\hat{k}+\hat{j}}{2}$ and $\frac{\hat{i}+\hat{k}}{2}+\hat{j}$

So, position vector of centroid of ΔDEF $=\frac{1}{3}\left[\frac{\hat{i}+\hat{j}}{2}+\hat{k}+\hat{i}+\frac{\hat{k}+\hat{j}}{2}+\frac{\hat{i}+\hat{k}}{2}+\hat{j}\right]=\frac{2}{3}[\hat{i}+\hat{j}+\hat{k}]$.

Ans. [D]

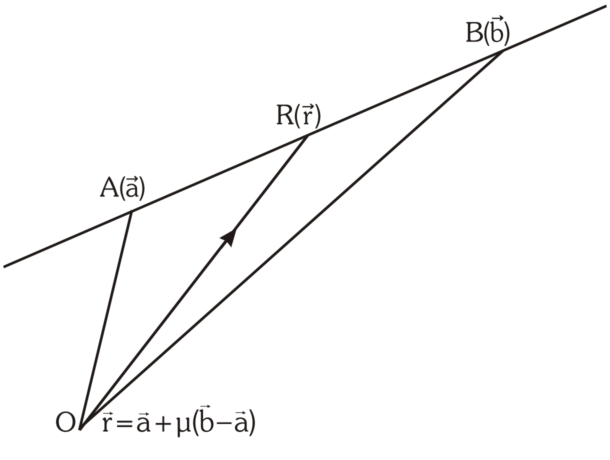
Do yourself - 4 :

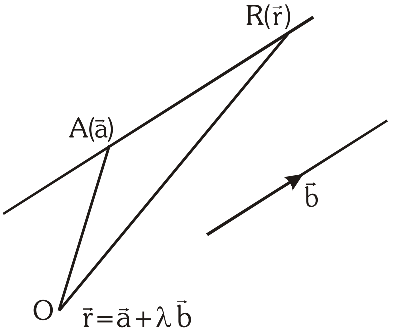
(i) Find the position vectors of the points which divide the join of the points $2\vec{a}-3\vec{b}$ and $3\vec{a}-2\vec{b}$ internally and externally in the ratio 2 : 3,

(ii) ABCD is a parallelogram and P is the point of intersection of its diagonals. If O is the origin of reference, show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$

(iii) Find the unit vector in the direction of $3\hat{i}-6\hat{j}+2\hat{k}$.

15. VECTOR EQUATION OF A LINE :





Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{\mathbf{r}}=\vec{\mathbf{a}}+\mathbf{t}(\vec{\mathbf{b}}-\vec{\mathbf{a}})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{\mathbf{r}}=\vec{\mathbf{a}}+\mathbf{t}\vec{\mathbf{b}}$.

Note :

(i) Equations of the bisectors of the angles between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is, $\vec{\mathbf{r}}=\vec{\mathbf{a}}+\mathbf{t}(\hat{\mathbf{b}}+\hat{\mathbf{c}})$ & $\vec{\mathbf{r}}=\vec{\mathbf{a}}+\mathbf{p}(\hat{\mathbf{c}}-\hat{\mathbf{b}})$.

(ii) In a plane, two lines are either intersecting or parallel.

(iii) Two non parallel nor intersecting lines are called **skew lines**.

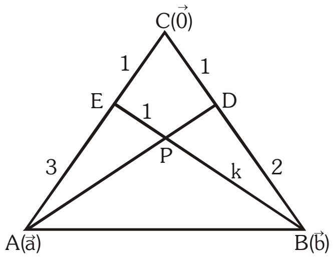
Illustration 10 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods. **(JEE-1993)**

Solution : Let the position vectors of A and B be a and b respectively. Equations of AD and BE are

$$\vec{r} = \vec{a} + t(\vec{b}/3 - \vec{a}) \qquad \text{........ (i)}$$

$$\vec{r} = \vec{b} + s(\vec{a}/4 - \vec{b}) \qquad \text{........ (ii)}$$

If they intersect at P we must have identical values of r.



Comparing the coefficients of a and b in (i) and (ii), we get

$$1 - t = \frac{s}{4}, \frac{t}{3} = 1 - s$$

solving we get $t = \frac{9}{11},\ s = \frac{8}{11}$.

Putting for t or s in (i) or (ii), we get the point P as $\frac{2\vec{a} + 3\vec{b}}{11}$.

Let P divide BE in the ratio k : 1, then P is $\frac{k.\frac{\vec{a}}{4} + \vec{b}}{k+1} = \frac{2\vec{a} + 3\vec{b}}{11}$.

Comparing $\vec{a}$ and $\vec{b}$, we get $11k = 8(k + 1)$ and $11 = 3(k + 1)$ $\quad \therefore \quad k = \frac{8}{3}$

and this satisfies the 2nd relation also. Hence the required ratio is 8 : 3. **Ans.**

Illustration 11 : Find whether the given lines are coplanar or not

$$\vec{r} = \hat{i} - \hat{j} - 10\hat{k} + \lambda(2\hat{i} - 3\hat{j} + 8\hat{k})\ ;\ \vec{r} = 4\hat{i} - 3\hat{j} - \hat{k} + \mu(\hat{i} - 4\hat{j} + 7\hat{k})$$

Solution : $L_1 : \vec{r} = (2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k}$

$L_2 : \vec{r} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$

The given lines are not parallel. For coplanarity, the lines must intersect.

$$\therefore \quad (2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$$

$$2\lambda + 1 = 4 + \mu \qquad \text{.........(i)}$$

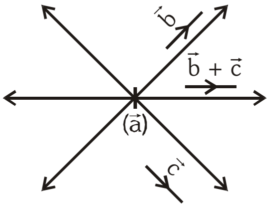
$$1 + 3\lambda = 4\mu + 3 \qquad \text{.........(ii)}$$

$$8\lambda - 10 = 7\mu - 1 \qquad \text{.........(iii)}$$

Solving (i) & (ii), $\lambda = 2$, $\mu = 1$ and $\lambda = 2$, $\mu = 1$ satisfies equation (iii)

Given lines are intersecting & hence coplanar. **Ans.**

16. TEST OF COLLINEARITY OF THREE POINTS :



(a) 3 points A B C will be collinear if $\overrightarrow{AB}=\lambda\overrightarrow{BC}$, where $\lambda\in R$

(b) Three points A, B, C with position vectors $\vec{\mathbf a},\vec{\mathbf b},\vec{\mathbf c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $\mathbf{x}\vec{\mathbf a}+\mathbf{y}\vec{\mathbf b}+\mathbf{z}\vec{\mathbf c}=\vec{\mathbf 0}$, where $\mathbf{x+y+z=0}$

(c) Collinearly can also be checked by first finding the equation of line through two points and satisfying the third point.

Illustration 12 : Prove that the points with position vectors $\vec a=\hat i-2\hat j+3\hat k,\ \vec b=2\hat i+3\hat j-4\hat k\ \&\ -7\hat j+10\hat k$ are collinear.

Solution : If we find, three scalars ℓ, m & n such that $\ell\vec a+m\vec b+n\vec c=\vec 0$, where $\ell+m+n=0$ then points are collinear.

$$\ell(\hat i-2\hat j+3\hat k)+m(2\hat i+3\hat j-4\hat k)+n(-7\hat j+10\hat k)=\vec 0$$

$$\Rightarrow\quad (\ell+2m)\hat i+(-2\ell+3m-7n)\hat j+(3\ell-4m+10n)\hat k=\vec 0$$

$$\Rightarrow\quad \ell+2m=0,\ -2\ell+3m-7n=0,\ 3\ell-4m+10n=0$$

Solving, we get $\ell=2,\ m=-1,\ n=-1$

since $\ell+m+n=0$

Hence, the points are collinear.

Aliter :

$$\overrightarrow{AB}=\vec b-\vec a=\left(2\hat i+3\hat j-4\hat k\right)-\left(\hat i-2\hat j+3\hat k\right)=\hat i+5\hat j-7\hat k$$

$$\overrightarrow{BC}=\vec c-\vec b=\left(-7\hat j+10\hat k\right)-\left(2\hat i+3\hat j-4\hat k\right)=-2\hat i-10\hat j+14\hat k=-2\left(\hat i+5\hat j-7\hat k\right)$$

$$\therefore\quad \overrightarrow{AB}=-2\overrightarrow{BC}$$

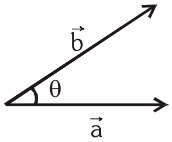
Hence $\vec a,\vec b\ \&\ \vec c$ are collinear.

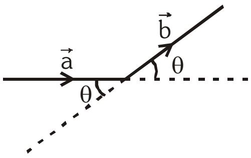
Do yourself - 5 :

(i) The position vectors of the points P, Q, R are $\hat i+2\hat j+3\hat k$, $-2\hat i+3\hat j+5\hat k$ and $7\hat i-\hat k$ respectively. Prove that P, Q and R are collinear.

17. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT) :

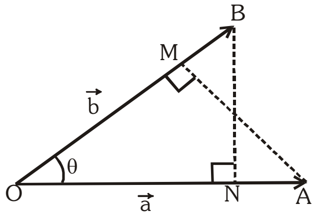
Definition : Let $\vec a$ and $\vec b$ be two non zero vectors inclined at an angle θ. Then the scalar product of $\vec a$ with $\vec b$ is denoted by $\vec a\ .\ \vec b$ and is defined as $\vec a\,.\vec b=|\vec a||\vec b|\cos\theta;\ 0\le\theta\le\pi$.





Geometrical Interpretation of Scalar product :

$|\overrightarrow{OA}| = |\vec{a}|,\ |\overrightarrow{OB}| = |\vec{b}|$



Now $\quad \vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|\,(\text{OB}\cos\theta)$

$=$ (magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$)

Again, $\quad \vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|\,(|\vec{a}|\cos\theta)$

$=$ (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$)

(a) $\mathbf{\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)}$

Note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

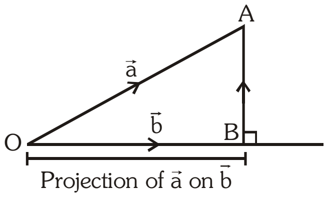
(b) (i) $\mathbf{\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2}$ (ii) $\mathbf{\vec{a}.\vec{b} = \vec{b}.\vec{a}}$ (commutative)

(c) $\mathbf{\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}}$ (distributive)

(d) $\mathbf{\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}\,;\ (\vec{a}, \vec{b} \ne \vec{0})}$

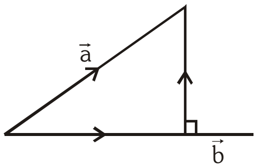
(e) $\mathbf{\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1;\ \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0}$

(f) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. (Provided $|\vec{b}| \ne 0$)



Note :

(i) The vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ [by triangle law of vector Addition]



(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \quad 0 \le \phi \le \pi$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}},\ |\vec{b}| = \sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}$

(iv) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

(v) Minimum values of $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

(vi) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

(g) Vector equation of angle bisector :

A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^{+}$.

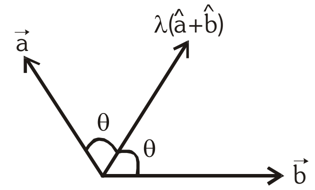


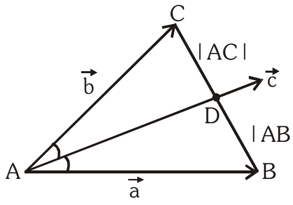
Illustration 13 : The vector $\vec{c}$, directed along the bisector of the angle between the vector $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$ is -

(A) $\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) none of these

Solution : Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$

and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

Angle bisector of A divides the BC in the ratio of $\left|\overrightarrow{AB}\right|:\left|\overrightarrow{AC}\right|$,



$\left|\overrightarrow{AB}\right|=9$, $\left|\overrightarrow{AC}\right|=3$

$$\overrightarrow{AD}=\left(\frac{9(-2\hat{i}-\hat{j}+2\hat{k})+3(7\hat{i}-4\hat{j}-4\hat{k})}{9+3}\right)=\frac{\hat{i}-7\hat{j}+2\hat{k}}{4}$$

$$\vec{c}=\left(\frac{\overrightarrow{AD}}{\left|\overrightarrow{AD}\right|}\right)5\sqrt{6}=\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$$ **Ans.[A]**

Illustration 14 : If pth, qth, rth terms of a G.P. are the positive numbers a, b, c then angle between the vectors $\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $(q-r)\hat{i}+(r-p)\hat{j}+(p-q)\hat{k}$ is :-

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{a^2+b^2+c^2}}\right)$ (D) none of these

Solution : Let x_0 be first term and x the common ratio of the G.P.

$\therefore$ $a=x_0\,x^{p-1}$, $b=x_0\,x^{q-1}$, $c=x_0\,x^{r-1}$ $\Rightarrow$ $\log a=\log x_0+(p-1)\log x$;

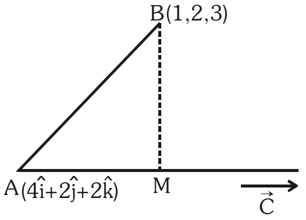
$\log b=\log x_0+(q-1)\log x$; $\log c=\log x_0+(r-1)\log x$

If $\vec{a}=\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $\vec{b}=(q-r)\,\hat{i}+(r-p)\,\hat{j}+(p-q)\,\hat{k}$

$\therefore$ $\vec{a}.\vec{b}=\Sigma 2\,(\log a)\,(q-r)=2\sum(\log x_0+(p-1)\log x)(q-r)=0 \Rightarrow \vec{a}\wedge\vec{b}=\frac{\pi}{2}$. **Ans.**

Illustration 15: Find the distance of the point B($\hat{i}+2\hat{j}+3\hat{k}$) from the line which is passing through A($4\hat{i}+2\hat{j}+2\hat{k}$) and which is parallel to the vector $\vec{C}=2\hat{i}+3\hat{j}+6\hat{k}$. **(Roorkee 1993)**

Solution :



$AB=\sqrt{3^2+1^2}=\sqrt{10}$

$$AM=\overrightarrow{AB}.\hat{c}=(-3\hat{i}+\hat{k}).\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

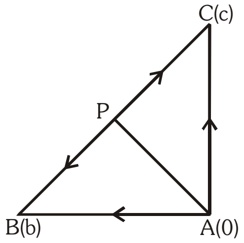
$=-6+6=0$

$BM^2=AB^2-AM^2$

So, $BM=AB=\sqrt{10}$ **Ans.**

Illustration 16 : Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

Solution : The triangle being isosceles, we have



AB = AC .... (i)

Now $\overrightarrow{AP}=\frac{\vec{b}+\vec{c}}{2}$ where P is mid-point of BC.

Also $\overrightarrow{BC}=\vec{c}-\vec{b}$

$$\therefore \quad \overrightarrow{AP}.\overrightarrow{BC}=\frac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b})=\frac{1}{2}(c^2-b^2)$$

$$=\frac{1}{2}(AC^2-AB^2)=0 \qquad \{\text{by (i)}\}$$

$\therefore$ Median AP is perpendicular to base BC.

Do yourself - 6 :

(i) Find the angle between two vectors $\vec{a}$ & $\vec{b}$ with magnitude 2 and 1 respectively and such that $\vec{a}.\vec{b}=\sqrt{3}$.

(ii) Find the value of $(\vec{a}+3\vec{b}).(2\vec{a}-\vec{b})$ if $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}+2\hat{j}-\hat{k}$.

(iii) The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1, find λ.

(iv) Find the projection of the vector $\vec{a}=4\hat{i}-2\hat{j}+\hat{k}$ on the vector $\vec{b}=3\hat{i}+6\hat{j}+2\hat{k}$. Also find vector component of $\vec{a}$ along $\vec{b}$ and perpendicular to $\vec{b}$.

(v) Find the unit vectors along the angle bisectors between the vectors $\vec{a}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{b}=-3\hat{i}+6\hat{j}+2\hat{k}$.

18. LINEAR COMBINATIONS :

Given a finite set of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$,........... then the vector $\vec{\mathbf{r}}=\mathbf{x}\vec{\mathbf{a}}+\mathbf{y}\vec{\mathbf{b}}+\mathbf{z}\vec{\mathbf{c}}+$ **...........** is called a **linear combination** of $\vec{a}$, $\vec{b}$, $\vec{c}$,........ for any x, y, z......... $\in$ R.

FUNDAMENTAL THEOREM IN PLANE :

Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x,\ y \in R$ such that $\mathbf{x\vec{a} + y\vec{b} = \vec{r}}$.

Illustration 17 : Find a vector $\vec{c}$ in the plane of $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}.(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

Solution : Any vector in the plane of $\vec{a}$ & $\vec{b}$ can be written as $x\vec{a} + y\vec{b}$

let $\vec{c} = x\vec{a} + y\vec{b}$ [by fundamental theorem in plane]

Now, given that

$\vec{c}.\vec{b} = 0 \Rightarrow (x\vec{a} + y\vec{b}).\vec{b} = 0$

$x\vec{a}.\vec{b} + y\vec{b}^2 = 0$

$\Rightarrow \quad x(-2 + 1 - 1) + y(3) = 0$

$-2x + 3y = 0$ ....(i)

Also $(x\vec{a} + y\vec{b}).(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow \quad x\vec{a}.(-2\hat{i} + 3\hat{j} - \hat{k}) + y\vec{b}.(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow \quad x(-4 + 3 + 1) + y(2 + 3 - 1) = -1$

$$y = -\frac{1}{4}$$

$$x = \frac{3y}{2} = -\frac{3}{8}$$

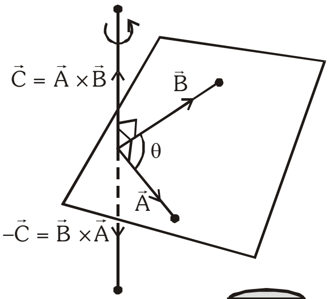
Hence the required vector $\vec{c} = -\frac{3}{8}(2\hat{i} + \hat{j} - \hat{k}) - \frac{1}{4}(-\hat{i} + \hat{j} + \hat{k})$

$= \frac{1}{8}[-6\hat{i} - 3\hat{j} + 3\hat{k} + 2\hat{i} - 2\hat{j} - 2\hat{k}] = \frac{1}{8}[-4\hat{i} - 5\hat{j} + \hat{k}]$ **Ans.**

Do yourself - 7 :

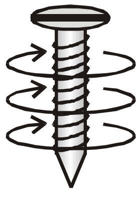
(i) Find a vector $\vec{r}$ in the plane of $\vec{p} = -\hat{i} + \hat{j}$ and $\vec{q} = -\hat{j} + \hat{k}$ such that $\vec{r}$ is perpendicular to $\vec{p}$ and $\vec{r}.\vec{q} = -2$.

19. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :


$\vec{C} = \vec{A} \times \vec{B}$
$\vec{B}$
θ
$\vec{A}$
$-\vec{C} = \vec{B} \times \vec{A}$

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\hat{n}$ forms a right handed screw system.

Sign convention :



Right handed screw system : $\vec{a}, \vec{b}$ and $\hat{\mathbf{n}}$ form a right handed system it means that if we rotate vector $\vec{a}$ towards the direction of $\vec{b}$ through the angle θ, then $\hat{n}$ advances in the same direction as a right handed screw would, if turned in the same way.

(b) **Lagranges Identity :** For any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}.\vec{b})^2}$ (ii) $\vec{c}.\vec{a} = 0; \vec{c}.\vec{b} = 0$ and (iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system

(d) (i) $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \vec{\mathbf{0}} \Leftrightarrow \vec{\mathbf{a}}$ **&** $\vec{\mathbf{b}}$ are parallel (collinear) $(\vec{\mathbf{a}} \neq \vec{\mathbf{0}},\ \vec{\mathbf{b}} \neq \vec{\mathbf{0}})$ i.e. $\vec{\mathbf{a}} = \mathbf{K}\vec{\mathbf{b}}$, where K is a scalar

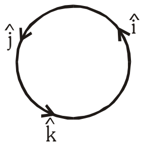
(ii) $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \neq \vec{\mathbf{b}} \times \vec{\mathbf{a}}$ **(not commutative)**

(iii) $(\mathbf{m}\vec{\mathbf{a}})\times\vec{\mathbf{b}} = \vec{\mathbf{a}}\times(\mathbf{m}\vec{\mathbf{b}}) = \mathbf{m}(\vec{\mathbf{a}}\times\vec{\mathbf{b}})$ where m is a scalar.

(iv) $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}+\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\times\vec{\mathbf{b}}) + (\vec{\mathbf{a}}\times\vec{\mathbf{c}})$ **(distributive over addition)**

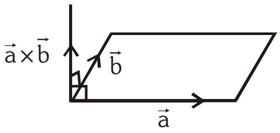
(v) $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \vec{\mathbf{0}}$

(vi) $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}},\ \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$



(e) If $\vec{\mathbf{a}} = \mathbf{a_1}\hat{\mathbf{i}} + \mathbf{a_2}\hat{\mathbf{j}} + \mathbf{a_3}\hat{\mathbf{k}}$ & $\vec{\mathbf{b}} = \mathbf{b_1}\hat{\mathbf{i}} + \mathbf{b_2}\hat{\mathbf{j}} + \mathbf{b_3}\hat{\mathbf{k}}$, then $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix}$.

(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) (i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{\mathbf{n}} = \pm \dfrac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}}{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|}$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{\mathbf{r}(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}{|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$, then $\mathbf{\sin\theta} = \dfrac{|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|}$

(h) Vector area :

(i) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle $\mathbf{ABC} = \dfrac{\mathbf{1}}{\mathbf{2}}\left[\vec{\mathbf{a}}\times\vec{\mathbf{b}} + \vec{\mathbf{b}}\times\vec{\mathbf{c}} + \vec{\mathbf{c}}\times\vec{\mathbf{a}}\right]$.

(ii) The points A, B & C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$

(iii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{\mathbf{1}}{\mathbf{2}}\left|\vec{\mathbf{d}}_{\mathbf{1}}\times\vec{\mathbf{d}}_{\mathbf{2}}\right|$.

Illustration 18 : Find the vectors of magnitude 5 which are perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution : Unit vectors perpendicular to $\vec{a}$ & $\vec{b} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$\because \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore \quad \text{Unit vectors} = \pm\frac{(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$$

Hence the required vectors are $\pm\dfrac{5\sqrt{3}}{3}(\hat{i} + \hat{j} + \hat{k})$ **Ans.**

Illustration 19 : If $\vec{a}, \vec{b}, \vec{c}$ are three non zero vectors such that $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$.

Solution : $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{a} = \vec{b}\times\vec{c}$

$\Rightarrow \quad \vec{c} \perp \vec{a},\ \vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b},\ \vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a} \perp \vec{b},\ \vec{b} \perp \vec{c}$ and $\vec{c} \perp \vec{a}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

Again, $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{b}\times\vec{c} = \vec{a}$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = |\vec{c}|$ and $|\vec{b}\times\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\dfrac{\pi}{2} = |\vec{c}|$ and $|\vec{b}||\vec{c}|\sin\dfrac{\pi}{2} = |\vec{a}|$ $\quad\left(\because \vec{a} \perp \vec{b} \text{ and } \vec{b} \perp \vec{c}\right)$

$\Rightarrow \quad |\vec{a}||\vec{b}| = |\vec{c}|$ and $|\vec{b}||\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{b}|^2\,|\vec{c}| = |\vec{c}|$

$\Rightarrow \quad |\vec{b}|^2 = 1$

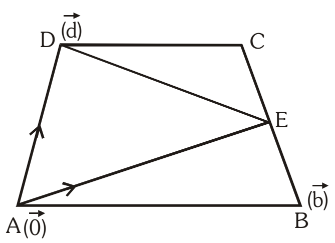
$\Rightarrow \quad |\vec{b}| = 1$

putting in $|\vec{a}||\vec{b}| = |\vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\vec{c}|$

Illustration 20 : Show that the area of the triangle formed by joining the extremities of an oblique side of a trapezium to the midpoint of opposite side is half that of the trapezium.

Solution : Let ABCD be the trapezium and E be the midpoint of BC. Let A be the initial point and let $\vec{b}$ be the position vector of B and $\vec{d}$ that of D. Since DC is parallel to AB, $t\vec{b}$ is a vector along DC, so that the position vector of c is $\vec{d} + t\vec{b}$.



$\Rightarrow$ the position vector of E is $\dfrac{\vec{b}+\vec{d}+t\vec{b}}{2}=\dfrac{\vec{d}+(1+t)\vec{b}}{2}$

Area of $\Delta AED=\dfrac{1}{2}\left|\dfrac{\vec{d}+(1+t)\vec{b}}{2}\times\vec{d}\right|=\dfrac{1}{4}(1+t)\,|\vec{b}\times\vec{d}\,|$

Area of the trapezium $=$ Area (ΔACD) $+$ Area (ΔABC).

$$=\frac{1}{2}|\vec{b}\times(\vec{d}+t\vec{b})|+\frac{1}{2}|(\vec{d}+t\vec{b})\times\vec{d}\,|$$

$$=\frac{1}{2}|\vec{b}\times\vec{d}\,|+\frac{t}{2}|\vec{b}\times\vec{d}\,|=\frac{1}{2}(1+t)\,|\vec{b}\times\vec{d}\,|=2\Delta AED$$

Illustration 21 : Let $\vec{a}$ & $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-\left(\vec{a}.\vec{b}\right)\vec{b}$ & $\vec{v}=\left(\vec{a}\times\vec{b}\right)$, then $|\vec{v}|$ is - **[JEE 99]**

(A) $|\vec{u}|$ (B) $|\vec{u}|+|\vec{u}.\vec{a}|$ (C) $|\vec{u}|+|\vec{u}.\vec{b}|$ (D) $\vec{u}+\vec{u}.(\vec{a}+\vec{b})$

Solution : $\vec{u}.\vec{a}=\vec{a}.\vec{a}-(\vec{a}.\vec{b})(\vec{a}.\vec{b})$

$=1-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$=1-\cos^2\theta=\sin^2\theta$

$|\vec{v}|=|\vec{a}\times\vec{b}|=\sin\theta$

$|\vec{u}|=\sqrt{\vec{u}.\vec{u}}$

$=\sqrt{\vec{a}.\vec{a}-2(\vec{a}.\vec{b})^2+(\vec{a}.\vec{b})^2|\vec{b}|^2}=\sqrt{1-(\vec{a}.\vec{b})^2}=\sin\theta$

$\therefore$ $|\vec{v}|=|\vec{u}|$ also $\vec{u}.\vec{b}=0$

Hence, $|\vec{v}|=|\vec{u}|=|\vec{u}|+|\vec{u}.\vec{b}|$ **Ans. (A, C)**

Do yourself - 8 :

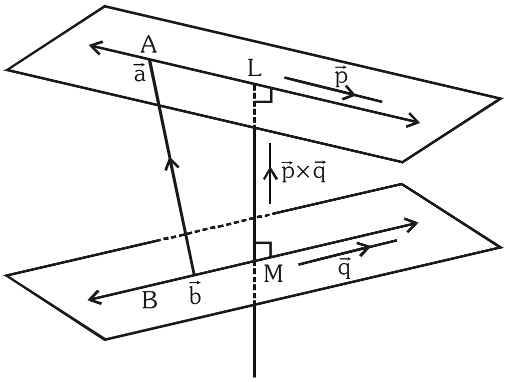
(i) If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, then show that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$ when $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

(ii) Find $\vec{a}\times\vec{b}$, if $\vec{a}=2\hat{i}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

(iii) For any two vectors $\vec{u}$ & $\vec{v}$, prove that **[JEE 98]**

(a) $(\vec{u}.\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2$ (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)=(1-\vec{u}.\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

20. SHORTEST DISTANCE BETWEEN TWO LINES :



If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines**. **In other words the lines which are not coplanar are skew lines**. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p}\times\vec{q}$

i.e. $\overrightarrow{LM} = |\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}| = |\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p}\times\vec{q}|$

$$= \left|\frac{\overrightarrow{\mathbf{AB}}.(\vec{\mathbf{p}}\times\vec{\mathbf{q}})}{\vec{\mathbf{p}}\times\vec{\mathbf{q}}}\right| = \left|\frac{(\vec{\mathbf{b}}-\vec{\mathbf{a}}).(\vec{\mathbf{p}}\times\vec{\mathbf{q}})}{|\vec{\mathbf{p}}\times\vec{\mathbf{q}}|}\right|$$

(a) **The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e.** $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q} \Rightarrow \left[(\vec{\mathbf{b}}-\vec{\mathbf{a}})\ \ \vec{\mathbf{p}}\ \ \vec{\mathbf{q}}\right] = \mathbf{0}$

(b) If two lines are given by $\vec{\mathbf{r}}_1 = \vec{\mathbf{a}}_1 + \mathbf{K_1}\vec{\mathbf{b}}$ & $\vec{\mathbf{r}}_2 = \vec{\mathbf{a}}_2 + \mathbf{K_2}\vec{\mathbf{b}}$ i.e. they are parallel then, $d = \left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

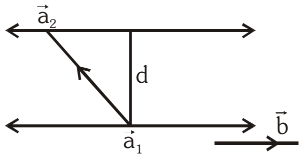


Illustration 22 : Find the shortest distance between the lines

$$\vec{r} = (4\hat{i}-\hat{j}) + \lambda(\hat{i}+2\hat{j}-3\hat{k}) \text{ and } \vec{r} = (\hat{i}-\hat{j}+2\hat{k}) + \mu(2\hat{i}+4\hat{j}-5\hat{k})$$

Solution : We known, the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Comparing the given equation with the equations $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $r = \vec{a}_2 + \lambda\vec{b}_2$ respectively,

we have $\vec{a}_1 = 4\hat{i}-\hat{j}, \vec{a}_2 = \hat{i}-\hat{j}+2\hat{k}$, $\vec{b}_1 = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}_2 = 2\hat{i}+4\hat{j}-5\hat{k}$

$$\text{Now, } \vec{a}_2-\vec{a}_1 = -3\hat{i}+0\hat{j}+2\hat{k} \text{ and } \vec{b}_1\times\vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i}-\hat{j}+0\hat{k}$$

$$\therefore\ (\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2) = (-3\hat{i}+0\hat{j}+2\hat{k}).(2\hat{i}-\hat{j}+0\hat{k}) = -6 \text{ and } |\vec{b}_1\times\vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$$

$$\therefore\quad \text{Shortest distance } d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}.$$

Ans.

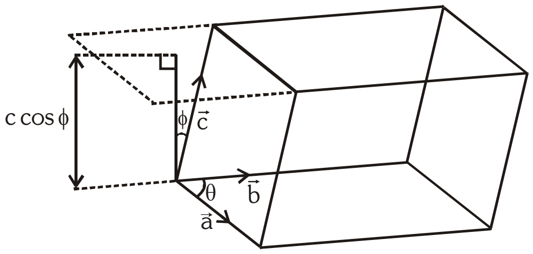
Do yourself - 9 :

(i) Find the shortest distance between the lines :

$$\vec{r}_1 = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(2\hat{i}+3\hat{j}+4\hat{k}) \ \&\ \vec{r}_2 = (2\hat{i}+4\hat{j}+5\hat{k}) + \mu(3\hat{i}+4\hat{j}+5\hat{k}).$$

21. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as : $(\vec{\mathbf{a}}\times\vec{\mathbf{b}}).\vec{\mathbf{c}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}||\vec{\mathbf{c}}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ & $\vec{\mathbf{c}}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\,\vec{b}\,\vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general, if $\vec{a} = a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b} = b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ & $\vec{c} = c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$; where $\vec{l}, \vec{m}$ & $\vec{n}$ are non coplanar vectors.

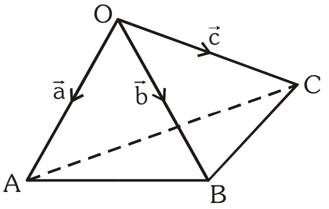
(v) **If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.**

(vi) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Note : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non- coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(vii) $[\hat{i}\,\hat{j}\,\hat{k}] = 1$ **(viii)** $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ **(ix)** $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}]+[\vec{b}\,\vec{c}\,\vec{d}]$

(viii) The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}$, $\vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$



The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

(ix) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ & $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Illustration 23 : For any three vectors $\vec{a},\vec{b},\vec{c}$ prove that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

Solution : We have $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= \{(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\}.(\vec{c}+\vec{a}) = \{\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}\}.(\vec{c}+\vec{a})$ $\{\because \vec{b}\times\vec{b} = 0\}$

$= \{\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}\}.(\vec{c}+\vec{a}) = (\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{c}).\vec{c}+(\vec{b}\times\vec{c}).\vec{c}+(\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{c}).\vec{a}+(\vec{b}\times\vec{c}).\vec{a}$

$= [\vec{a}\,\vec{b}\,\vec{c}]+0+0+0+0+[\vec{b}\,\vec{c}\,\vec{a}]$ $\{\because [\vec{a}\,\vec{c}\,\vec{c}] = 0, [\vec{b}\,\vec{c}\,\vec{c}] = 0, [\vec{a}\,\vec{b}\,\vec{a}] = 0, [\vec{a}\,\vec{c}\,\vec{a}] = 0\}$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{c}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$. **Ans.**

Illustration 24 : If $\vec{a}, \vec{b}$ are non-zero and non-collinear vectors then show $\vec{a} \times \vec{b} = [\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} + [\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$

Solution : Let $\vec{a} \times \vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$

$$(\vec{a} \times \vec{b}).\hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}).\hat{i}$$

$$(\vec{a} \times \vec{b}).\hat{i} = x$$

also $(\vec{a} \times \vec{b}).\hat{j} = y$ & $(\vec{a} \times \vec{b}).\hat{k} = z$

$\therefore \quad \vec{a} \times \vec{b} = [\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} + [\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$ **Ans.**

Do yourself - 10 :

(i) If $\vec{a}, \vec{b}, \vec{c}$ are three non coplanar mutually perpendicular unit vectors then find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(ii) If $\vec{r}$ be a vector perpendicular to $\vec{a} + \vec{b} + \vec{c}$, where $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = z$ and $\vec{r} = \ell(\vec{b} \times \vec{c}) + m(\vec{c} \times \vec{a}) + n(\vec{a} \times \vec{b})$, then find $l + m + n$.

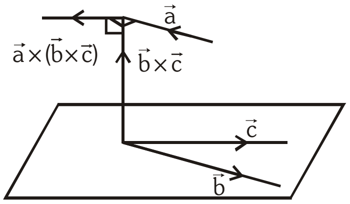
(iii) Find the volume of the parallelepiped whose coterminous edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

(iv) Examine whether the vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ form a left handed or right handed system.

22. VECTOR TRIPLE PRODUCT :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ is a vector & is called a **vector triple product**.

Geometrical interpretation of $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$



Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b} \times \vec{c})$ but $(\vec{b} \times \vec{c})$ is a vector perpendicular to the plane $\vec{b}$ & $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is vector lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \mathbf{x}\vec{\mathbf{b}} + \mathbf{y}\vec{\mathbf{c}}$ where x & y are scalars.

(a) $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}}.\vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}.\vec{\mathbf{b}})\vec{\mathbf{c}}$

(b) $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = (\vec{\mathbf{a}}.\vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{b}}.\vec{\mathbf{c}})\vec{\mathbf{a}}$

(c) $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} \neq \vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$

Illustration 25 : Prove that $\left[\vec a\times\vec b\ \ \vec b\times\vec c\ \ \vec c\times\vec a\right]=\left[\vec a\vec b\vec c\right]^2$

Solution : We have, $\left[\vec a\times\vec b\ \ \vec b\times\vec c\ \ \vec c\times\vec a\right]=\left\{\left(\vec a\times\vec b\right)\times\left(\vec b\times\vec c\right)\right\}.\left(\vec c\times\vec a\right)$

$$=\left\{\vec d\times\left(\vec b\times\vec c\right)\right\}.\left(\vec c\times\vec a\right) \qquad \text{(where, } \vec d=(\vec a\times\vec b)\text{)}$$

$$=\left\{\left(\vec d.\vec c\right)\vec b-\left(\vec d.\vec b\right)\vec c\right\}.\left(\vec c\times\vec a\right)=\left\{\left(\left(\vec a\times\vec b\right).\vec c\right)\vec b-\left(\left(\vec a\times\vec b\right).\vec b\right)\vec c\right\}.\left(\vec c\times\vec a\right)$$

$$=\left\{\left[\vec a\vec b\vec c\right]\vec b-0\right\}.\left(\vec c\times\vec a\right) \qquad (\because\ [\vec a\vec b\vec b]=0)$$

$$=\left[\vec a\vec b\vec c\right]\left\{\vec b.\left(\vec c\times\vec a\right)\right\}=\left[\vec a\vec b\vec c\right]\left[\vec b\vec c\vec a\right]=\left[\vec a\vec b\vec c\right]\left[\vec a\vec b\vec c\right]=\left[\vec a\vec b\vec c\right]^2$$

Illustration 26 : Show that $(\vec b\times\vec c).(\vec a\times\vec d)+(\vec c\times\vec a).(\vec b\times\vec d)+(\vec a\times\vec b).(\vec c\times\vec d)=0$

Solution : Let $\vec b\times\vec c=\vec u,\vec c\times\vec a=\vec v,\vec c\times\vec d=\vec w$

$\therefore$ L.H.S $=\vec u.(\vec a\times\vec d)+\vec v.(\vec b\times\vec d)+(\vec a\times\vec b).\vec w=(\vec u\times\vec a).\vec d+(\vec v\times\vec b).\vec d+\vec a.(\vec b\times\vec w)$

$$=[(\vec b\times\vec c)\times\vec a].\vec d+[(\vec c\times\vec a)\times\vec b].\vec d+\vec a.[\vec b\times(\vec c\times\vec d)]$$

$$=[(\vec b.\vec a)\vec c-(\vec c.\vec a)\vec b].\vec d+[(\vec c.\vec b)\vec a-(\vec a.\vec b)\vec c].\vec d+\vec a.[(\vec b.\vec d)\vec c-(\vec b.\vec c)\vec d]$$

$$=\{(\vec a.\vec b)(\vec c.\vec d)\}-\{(\vec c.\vec a)(\vec b.\vec d)\}+\{(\vec c.\vec b)(\vec a.\vec d)\}-\{(\vec a.\vec b)(\vec c.\vec d)\}+\{(\vec a.\vec c)(\vec b.\vec d)\}-\{(\vec a.\vec d)(\vec b.\vec c)\}=0$$

$=$ R.H.S.

Do yourself - 11 :

(i) If $\vec a=2\hat i-4\hat j+7\hat k$, $\vec b=3\hat i+5\hat j-9\hat k$ and $\vec c=\hat i+\hat j+\hat k$, then find $[\vec a\ \vec b\ \vec c]$ and also $\vec a\times(\vec b\times\vec c)$.

23. LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

(a) If $\vec x_1, \vec x_2, ...\vec x_n$ are n non zero vectors, & $k_1, k_2, ...\ k_n$ are n scalars & if the linear combination $\mathbf{k_1\vec x_1+k_2\vec x_2+....k_n\vec x_n=\vec 0\Rightarrow k_1=0,\ k_2=0....k_n=0}$, then we say that vectors $\vec x_1, \vec x_2, ...\vec x_n$ are **linearly independent vectors.**

(b) If $\vec x_1, \vec x_2, ...\vec x_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**. i.e. if $\mathbf{k_1\vec x_1+k_2\vec x_2+.......k_n\vec x_n=\vec 0}$ & if there exists at least one $\mathbf{k_r\neq 0}$ then $\mathbf{\vec x_1, \vec x_2, ...\vec x_n}$ are said to be **linearly dependent.**

FUNDAMENTAL THEOREM IN SPACE :

Let $\vec a, \vec b, \vec c$ be non-zero, non-coplanar vectors in space. Then any vector $\vec r$, can be uniquely expressed as a linear combination of $\vec a, \vec b, \vec c$ i.e. There exist some unique x, y, z $\in$ R such that $\mathbf{\vec r=x\vec a+y\vec b+z\vec c}$.

Note :

(i) If $\vec{a} = 3\hat{i}+2\hat{j}+5\hat{k}$ then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also, $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors then $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0} \Rightarrow x=y=z=0$

(iii) $\hat{i}, \hat{j}, \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=\vec{0} \Rightarrow K_1 = 0 = K_2 = K_3$

(iv) Two vectors $\vec{\mathbf{a}}$ **&** $\vec{\mathbf{b}}$ are **linearly dependent** $\Rightarrow \vec{a}$ is a parallel to $\vec{b}$ i.e. $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}} \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a}\times\vec{b}\neq\vec{0}$, then $\vec{a}$ & $\vec{b}$ are **linearly independent**.

(v) If three vectors $\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent**, then they are **coplanar** i.e. $[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]=\mathbf{0}$ conversely, if $[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]\neq\mathbf{0}$, then the vectors are linearly independent.

Illustration 27 : Show that points with position vectors $\vec{a}-2\vec{b}+3\vec{c}, -2\vec{a}+3\vec{b}-\vec{c}, 4\vec{a}-7\vec{b}+7\vec{c}$ are collinear. It is given that vectors $\vec{a},\vec{b},\vec{c}$ are non-coplanar.

Solution : The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$$\lambda_1(\vec{a}-2\vec{b}+3\vec{c})+\lambda_2(-2\vec{a}+3\vec{b}-\vec{c})+\lambda_3(4\vec{a}-7\vec{b}+7\vec{c})=\vec{0}$$

with $\lambda_1+\lambda_2+\lambda_3=0$

equating the coefficients $\vec{a}, \vec{b}$ and $\vec{c}$ separately to zero, we get

$\lambda_1-2\lambda_2+4\lambda_3=0, -2\lambda_1+3\lambda_2-7\lambda_3=0$ and $3\lambda_1-\lambda_2+7\lambda_3=0$

on solving we get, $\lambda_1=-2, \lambda_2=1, \lambda_3=1$

So that $\lambda_1+\lambda_2+\lambda_3=0$

Hence the given vectors are collinear.

24. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=\vec{0}$, where $x+y+z+w=0$

25. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$, then the two systems are called Reciprocal System of vectors.

Note : $a'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $b'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $c'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

26. PROPERTIES OF RECIPROCAL SYSTEM OF VECTORS :

(a) $\vec a.\vec b' = \vec a.\vec c' = \vec b.\vec c' = \vec c.\vec a' = \vec c.\vec b' = 0$

(b) The scalar triple product $[\vec a\ \ \vec b\ \ \vec c]$ formed by three non-coplanar vectors $\vec a,\ \vec b,\ \vec c$ is the reciprocal of the scalar triple product formed from reciprocal system.

Illustration 28 : Find a set of vectors reciprocal to the vectors $\vec a, \vec b$ and $\vec a \times \vec b$.

Solution : Let the given vectors be denoted by $\vec a, \vec b$ and $\vec c$ where $\vec c = \vec a \times \vec b$

$$\therefore \quad [\vec a\,\vec b\,\vec c] = (\vec a \times \vec b).\vec c = (\vec a \times \vec b).(\vec a \times \vec b) = (\vec a \times \vec b)^2$$

and let the reciprocal system of vectors be $\vec a'\vec b'$and$\vec c'$

$$\therefore \quad \vec a' = \frac{\vec b \times \vec c}{[\vec a\,\vec b\,\vec c]} = \frac{\vec b \times (\vec a \times \vec b)}{|\vec a \times \vec b|^2};\ \vec b' = \frac{\vec c \times \vec a}{[\vec a\,\vec b\,\vec c]} = \frac{(\vec a \times \vec b) \times \vec a}{|\vec a \times \vec b|^2};\ \vec c' = \frac{\vec a \times \vec b}{[\vec a\,\vec b\,\vec c]} = \frac{\vec a \times \vec b}{|\vec a \times \vec b|^2}$$

$\therefore$ $\vec a', \vec b', \vec c'$ are required reciprocal system of vectors for $\vec a, \vec b$ and $\vec a \times \vec b$. **Ans.**

Illustration 29 : If $\vec a' = \dfrac{\vec b \times \vec c}{[\vec a\,\vec b\,\vec c]}, \vec b' = \dfrac{\vec c \times \vec a}{[\vec a\,\vec b\,\vec c]}, \vec c' = \dfrac{\vec a \times \vec b}{[\vec a\,\vec b\,\vec c]}$, then shown that ; $\vec a \times \vec a' + \vec b \times \vec b' + \vec c \times \vec c' = \vec 0$

where $\vec a, \vec b, \vec c$ are non-coplanar vectors.

Solution : Here

$$\vec a \times \vec a' = \frac{\vec a \times (\vec b \times \vec c)}{[\vec a\,\vec b\,\vec c]}$$

$$\vec a \times \vec a' = \frac{(\vec a.\vec c)\vec b - (\vec a.\vec b)\vec c}{[\vec a\,\vec b\,\vec c]}$$

Similarly $\vec b \times \vec b' = \dfrac{(\vec b.\vec a)\vec c - (\vec b.\vec c)\vec a}{[\vec a\,\vec b\,\vec c]}$ & $\vec c \times \vec c' = \dfrac{(\vec c.\vec b)\vec a - (\vec c.\vec a)\vec b}{[\vec a\,\vec b\,\vec c]}$

$$\vec a \times \vec a' + \vec b \times \vec b' + \vec c \times \vec c' = \frac{(\vec a.\vec c)\vec b - (\vec a.\vec b)\vec c + (\vec b.\vec a)\vec c - (\vec b.\vec c)\vec a + (\vec c.\vec b)\vec a - (\vec c.\vec a)\vec b}{[\vec a\,\vec b\,\vec c]}$$

$[\because \vec a.\vec b = \vec b.\vec a$ etc.]

$= \vec 0$. **Ans.**

Do yourself - 12 :

(i) If $\vec p = \dfrac{\vec b \times \vec c}{[\vec b\ \vec c\ \vec a]}, \vec q = \dfrac{\vec c \times \vec a}{[\vec c\ \vec a\ \vec b]}, \vec r = \dfrac{\vec a \times \vec b}{[\vec a\ \vec b\ \vec c]}$,

then find the value of $(\vec a + \vec b).\vec p + (\vec b + \vec c).\vec q + (\vec c + \vec a).\vec r$.

(ii) If $\vec a$, $\vec b$ and $\vec c$ are non zero, non coplanar vectors determine whether the vectors $\vec r_1 = 2\vec a - 3\vec b + \vec c$, $\vec r_2 = 3\vec a - 5\vec b + 2\vec c$ and $\vec r_3 = 4\vec a - 5\vec b + \vec c$ are linearly independent or dependent.

Miscellaneous Illustrations :

Illustration 30 : Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$, then prove that $|(\vec{u}\times\vec{v})\,.\,\vec{w}|\le\frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

[JEE 99]

Solution : $\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$ ......(i)

$\Rightarrow\quad \vec{w}\times\vec{u}=\vec{v}-\vec{w}\ \Rightarrow\ (\vec{w}\times\vec{u})^2=v^2+w^2-2\vec{v}.\vec{w}$

$\Rightarrow\quad 2\vec{v}.\vec{w}=1+w^2-(\vec{u}\times\vec{w})^2$ ......(ii)

also taking dot product of (i) with $\vec{v}$, we get

$\vec{w}.\vec{v}+(\vec{w}\times\vec{u}).\vec{v}=\vec{v}.\vec{v}$

$\Rightarrow\quad \vec{v}.(\vec{w}\times\vec{u})=1-\vec{w}.\vec{v}$ ....(iii) $\{\because \vec{v}.\vec{v}=|\vec{v}|^2=1\}$

Now ; $\vec{v}.(\vec{w}\times\vec{u})=1-\frac{1}{2}(1+w^2-(\vec{u}\times\vec{w})^2)$ (using (ii) and (iii))

$=\frac{1}{2}-\frac{w^2}{2}+\frac{(\vec{u}\times\vec{w})^2}{2}$ $(\therefore\ 0\le\cos^2\theta\le1)$

$=\frac{1}{2}(1-w^2+w^2\sin^2\theta)$ ......(iv)

as we know ; $0\le w^2\cos^2\theta\le w^2$

$\therefore\quad \frac{1}{2}\ge\frac{1-w^2\cos^2\theta}{2}\ge\frac{1-w^2}{2}$

$\Rightarrow\quad \frac{1-w^2\cos^2\theta}{2}\le\frac{1}{2}$ ......(v)

from (iv) and (v)

$|\vec{v}.(\vec{w}\times\vec{u})|\le\frac{1}{2}$

Equality holds only when $\cos^2\theta=0\ \Rightarrow\ \theta=\frac{\pi}{2}$

i.e., $\vec{u}\perp\vec{w}\Rightarrow\vec{u}.\vec{w}=0\quad\Rightarrow\quad\vec{w}+(\vec{w}\times\vec{u})=\vec{v}$

$\Rightarrow\quad \vec{u}\cdot\vec{w}+\vec{u}.(\vec{w}\times\vec{u})=\vec{u}.\vec{v}$ (taking dot with $\vec{u}$)

$\Rightarrow\quad 0+0=\vec{u}\cdot\vec{v}\ \Rightarrow\ \vec{u}\cdot\vec{v}=0\ \Rightarrow\ \vec{u}\perp\vec{v}$

Illustration 31 : A point $A(x_1, y_1)$ with abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with ordinate $y_2 = 11$ are given in a rectangular cartesian system of co-ordinates OXY on the part of the curve $y = x^2 - 2x + 3$ which lies in the first quadrant. Find the scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution : Since (x_1, y_1) and (x_2, y_2) lies on $y = x^2 - 2x + 3$.

$\therefore\quad y_1 = x_1^2 - 2x_1 + 3$

$y_1 = 1^2 - 2(1) + 3$ (as $x_1 = 1$)

$y_1 = 2$

so the co-ordinates of A(1, 2)

Also, $y_2 = x_2^2 - 2x_2 + 3$

$11 = x_2^2 - 2x_2 + 3 \Rightarrow x_2 = 4,\ x_2 \neq -2$ (as B lie in 1st quadrant)

$\therefore$ co-ordinates of B (4, 11).

Hence, $\overrightarrow{OA} = \hat{i} + 2\hat{j}$ and $\overrightarrow{OB} = 4\hat{i} + 11\hat{j}$

$\Rightarrow \quad \overrightarrow{OA}\,.\,\overrightarrow{OB} = 4 + 22 = 26.$

Illustration 32 : If 'a' is real constant and A, B, C are variable angles

and $\sqrt{a^2-4}\tan A + a\tan B + \sqrt{a^2+4}\tan C = 6a$,

then find the least value of $\tan^2 A + \tan^2 B + \tan^2 C$

Solution : The given relation can be re-written as :

$$(\sqrt{a^2-4}\,\hat{i} + a\hat{j} + \sqrt{a^2+4}\,\hat{k}).(\tan A\,\hat{i} + \tan B\,\hat{j} + \tan C\,\hat{k}) = 6a$$

$$\Rightarrow \quad \sqrt{(a^2-4)+a^2+(a^2+4)}.\sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\ .\cos\theta = 6a$$

(as $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$)

$$\Rightarrow \quad \sqrt{3}\,a\,.\sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\,\cos\theta = 6a$$

$\Rightarrow \quad \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta$ ......(i)

also, $12\sec^2\theta \geq 12$ (as, $\sec^2\theta \geq 1$) ......(ii)

from (i) and (ii),

$\tan^2 A + \tan^2 B + \tan^2 C \geq 12$

$\therefore$ least value of $\tan^2 A + \tan^2 B + \tan^2 C = 12$.

Illustration 33 : $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar unit vectors such that angle between any two is α. If $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} = \ell\vec{a} + m\vec{b} + n\vec{c}$, then determine ℓ, m, n in terms of α. **(JEE-1997)**

Solution : $a^2 = b^2 = c^2 = 1$, $[abc] \neq 0$

$\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{c} = \vec{c}\,.\,\vec{a} = \cos\alpha$ .... (i)

Multiply both sides of given relation scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$0 + [\vec{a}\ \vec{b}\ \vec{c}] = \ell.1 + (m+n)\cos\alpha$ .... (ii)

$0 = m + (n+\ell)\cos\alpha$ .... (iii)

$[\vec{a}\ \vec{b}\ \vec{c}] + 0 = (\ell+m)\cos\alpha + n$ .... (iv)

Adding, we get

$2[\vec{a}\ \vec{b}\ \vec{c}] = (\ell+m+n) + 2(\ell+m+n)\cos\alpha$

or $\quad 2[\vec{a}\ \vec{b}\ \vec{c}] = (\ell+m+n)(1+2\cos\alpha)$ .... (v)

From (ii), $(m+n) = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]-\ell}{\cos\alpha}$

Putting in (v), we get $2[\vec{a}\,\vec{b}\,\vec{c}] = \left\{\ell + \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]-\ell}{\cos\alpha}\right\}(1+2\cos\alpha)$

or $\quad [\vec{a}\,\vec{b}\,\vec{c}]\left\{2-\dfrac{1+2\cos\alpha}{\cos\alpha}\right\}=\ell\left(1-\dfrac{1}{\cos\alpha}\right)(1+2\cos\alpha)$

$\therefore \quad \ell=\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{(1+2\cos\alpha)(1-\cos\alpha)}=n$ $\qquad$ {as above}

and $\quad m=-(n+\ell)\cos\alpha=\dfrac{-2[\vec{a}\,\vec{b}\,\vec{c}]\cos\alpha}{(1+2\cos\alpha)(1-\cos\alpha)}$

Thus the values of ℓ, m, n depend on $[\vec{a}\ \vec{b}\ \vec{c}]$

Hence we now find the value of scalar $[\vec{a}\ \vec{b}\ \vec{c}]$ in terms of α.

$$\text{Now } [\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c}\end{vmatrix}=\begin{vmatrix}1 & \cos\alpha & \cos\alpha\\ \cos\alpha & 1 & \cos\alpha\\ \cos\alpha & \cos\alpha & 1\end{vmatrix} \qquad (\text{Apply } C_1+C_2+C_3)$$

$$=(1+2\cos\alpha)\begin{vmatrix}1 & \cos\alpha & \cos\alpha\\ 1 & 1 & \cos\alpha\\ 1 & \cos\alpha & 1\end{vmatrix} \qquad (\text{Apply } R_2-R_1 \text{ and } R_3-R_1)$$

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]^2=(1+2\cos\alpha)(1-\cos\alpha)^2$

$\therefore \quad \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{1-\cos\alpha}=\sqrt{1+2\cos\alpha}$

Putting in the value of ℓ, m, n we have $\ell=\dfrac{1}{\sqrt{(1+2\cos\alpha)}}=n, m=\dfrac{-2\cos\alpha}{\sqrt{(1+2\cos\alpha)}}$ **Ans.**

ANSWERS FOR DO YOURSELF

1 : **(i)** 7 **(ii)** A

3 : **(i)** **(a)** $\vec{a},\vec{d};\vec{b},\vec{x},\ \vec{z};\vec{c},\ \vec{y}$ **(b)** $\vec{b},\vec{x};\vec{a},\vec{d};\vec{c},\ \vec{y}$ **(c)** $\vec{a},\vec{y},\vec{z}$ **(d)** $\vec{b},\vec{z};\ \vec{x},\vec{z}$

(ii) $\dfrac{1}{4}$ **(iii)** A,B,C

4 : **(i)** $\dfrac{12\vec{a}-13\vec{b}}{5},-5\vec{b}$ **(iii)** $\dfrac{3}{7}\hat{i}-\dfrac{6}{7}\hat{j}+\dfrac{2}{7}\hat{k}$

6 : **(i)** $\dfrac{\pi}{6}$ **(ii)** -15 **(iii)** 1 **(iv)** $\dfrac{2}{7},\dfrac{2}{49}(3\hat{i}+6\hat{j}+2\hat{k})$ and $\dfrac{190\hat{i}-110\hat{j}+45\hat{k}}{49}$

(v) $-\dfrac{2}{21}\hat{i}+\dfrac{32}{21}\hat{j}-\dfrac{8}{21}\hat{k}$

7 : **(i)** $\vec{r}=\dfrac{2}{3}\left(\hat{i}+\hat{j}-2\hat{k}\right)$ **8 :** **(ii)** $-\hat{i}-\hat{j}+2\hat{k}$ **9 :** **(i)** $\dfrac{1}{\sqrt{6}}$

10 : **(i)** ±1 **(ii)** 0 **(iii)** 7 **(iv)** Right handed system

11 : **(i)** 62, $92\hat{i}+102\hat{j}+32\hat{k}$ **12 :** **(i)** 3 **(ii)** linearly dependent.

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. A (1, –1, –3), B (2, 1, –2) & C (–5, 2, –6) are the position vectors of the vertices of a triangle ABC. The length of the bisector of its internal angle at A is : **VT0001**

(A) $\sqrt{10}/4$ (B) $3\sqrt{10}/4$ (C) $\sqrt{10}$ (D) none

2. Let $\vec{p}$ is the p.v. of the orthocentre & $\vec{g}$ is the p.v. of the centroid of the triangle ABC where circumcentre is the origin. If $\vec{p} = K\vec{g}$, then K = **VT0002**

(A) 3 (B) 2 (C) 1/3 (D) 2/3

3. Four points A(+1, –1, 1) ; B(1, 3, 1) ; C(4, 3, 1) and D(4, – 1, 1) taken in order are the vertices of

(A) a parallelogram which is neither a rectangle nor a rhombus

(B) rhombus

(C) an isosceles trapezium

(D) a cyclic quadrilateral. **VT0005**

4. Let α, β & γ be distinct real numbers. The points whose position vector's are $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$; $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$ and $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

(A) are collinear (B) form an equilateral triangle

(C) form a scalene triangle (D) form a right angled triangle **VT0006**

5. If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$ & $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ constitute the sides of a ΔABC, then the length of the median bisecting the vector $\vec{c}$ is

(A) $\sqrt{2}$ (B) $\sqrt{14}$ (C) $\sqrt{74}$ (D) $\sqrt{6}$ **VT0007**

6. $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of which are collinear and the vector $\vec{a} + \vec{b}$ is collinear with $\vec{c}$, $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + \vec{b} + \vec{c}$ is equal to - **VT0009**

(A) $\vec{a}$ (B) $\vec{b}$ (C) $\vec{c}$ (D) none of these

7. Consider the following 3 lines in space

$$L_1 : \vec{r} = 3\hat{i} - \hat{j} + 2\hat{k} + \lambda(2\hat{i} + 4\hat{j} - \hat{k})$$

$$L_2 : \vec{r} = \hat{i} + \hat{j} - 3\hat{k} + \mu(4\hat{i} + 2\hat{j} + 4\hat{k})$$

$$L_3 : \vec{r} = 3\hat{i} + 2\hat{j} - 2\hat{k} + t(2\hat{i} + \hat{j} + 2\hat{k})$$

Then which one of the following pair(s) are in the same plane. **VT0011**

(A) only L_1L_2 (B) only L_2L_3 (C) only L_3L_1 (D) L_1L_2 and L_2L_3

8. The vectors $3\hat{i} - 2\hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} + 5\hat{k}$ & $2\hat{i} + \hat{j} - 4\hat{k}$ form the sides of a triangle. Then triangle is

(A) an acute angled triangle (B) an obtuse angled triangle

(C) an equilateral triangle (D) a right angled triangle **VT0013**

9. If the vectors $3\bar{p}+\bar{q}\,;5\bar{p}-3\bar{q}$ and $2\bar{p}+\bar{q}\,;4\bar{p}-2\bar{q}$ are pairs of mutually perpendicular vectors then $\sin\left(\widehat{\bar{p}\,\bar{q}}\right)$ is **VT0014**

(A) $\sqrt{55}/4$ (B) $\sqrt{55}/8$ (C) 3/16 (D) $\sqrt{247}/16$

10. The set of values of c for which the angle between the vectors $cx\hat{i}-6\hat{j}+3\hat{k}$ & $x\hat{i}-2\hat{j}+2cx\hat{k}$ is acute for every $x \in R$ is

(A) (0, 4/3) (B) [0, 4/3] (C) (11/9, 4/3) (D) [0, 4/3) **VT0016**

11. Let $\vec{u}=\hat{i}+\hat{j}, \vec{v}=\hat{i}-\hat{j}$ and $\vec{w}=\hat{i}+2\hat{j}+3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n}=0$ and $\vec{v}\cdot\hat{n}=0$, then $|\vec{w}\cdot\hat{n}|$ is equal to

(A) 1 (B) 2 (C) 3 (D) 0 **VT0017**

12. If the vector $6\hat{i}-3\hat{j}-6\hat{k}$ is decomposed into vectors parallel and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ then the vectors are :

(A) $-\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $7\hat{i}-2\hat{j}-5\hat{k}$ (B) $-2\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $8\hat{i}-\hat{j}-4\hat{k}$

(C) $+2\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $4\hat{i}-5\hat{j}-8\hat{k}$ (D) none **VT0018**

13. Let $\vec{r}=\vec{a}+\lambda\vec{l}$ and $\vec{r}=\vec{b}+\mu\vec{m}$ be two lines in space where $\vec{a}=5\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=-\hat{i}+7\hat{j}+8\hat{k}$, $\vec{l}=-4\hat{i}+\hat{j}-\hat{k}$ and $\vec{m}=2\hat{i}-5\hat{j}-7\hat{k}$ then the p.v. of a point which lies on both of these lines, is

(A) $\hat{i}+2\hat{j}+\hat{k}$ (B) $2\hat{i}+\hat{j}+\hat{k}$ **VT0019**

(C) $\hat{i}+\hat{j}+2\hat{k}$ (D) non existent as the lines are skew

14. Let A(1, 2, 3), B(0, 0, 1), C(–1, 1, 1) are the vertices of a ΔABC.

(i) The equation of internal angle bisector through A to side BC is

(A) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\mu(3\hat{i}+2\hat{j}+3\hat{k})$ (B) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\mu(3\hat{i}+4\hat{j}+3\hat{k})$

(C) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\mu(3\hat{i}+3\hat{j}+2\hat{k})$ (D) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\mu(3\hat{i}+3\hat{j}+4\hat{k})$

(ii) The equation of median through C to side AB is

(A) $\vec{r}=-\hat{i}+\hat{j}+\hat{k}+p(3\hat{i}-2\hat{k})$ (B) $\vec{r}=-\hat{i}+\hat{j}+\hat{k}+p(3\hat{i}+2\hat{k})$

(C) $\vec{r}=-\hat{i}+\hat{j}+\hat{k}+p(-3\hat{i}+2\hat{k})$ (D) $\vec{r}=-\hat{i}+\hat{j}+\hat{k}+p(3\hat{i}+2\hat{j})$

(iii) The area (ΔABC) is equal to

(A) $\frac{9}{2}$ (B) $\frac{\sqrt{17}}{2}$ (C) $\frac{17}{2}$ (D) $\frac{7}{2}$ **VT0020**

15. If $\vec{a}+\vec{b}+\vec{c}=0$, $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$, then the angle between $\vec{a}$ & $\vec{b}$ is :

(A) π/6 (B) 2π/3 (C) 5π/3 (D) π/3 **VT0021**

16. A line passes through the point $A\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and is parallel to the vector $\vec{V}\left(\hat{i}+\hat{j}+\hat{k}\right)$. The shortest distance from the origin, of the line is - **VT0022**

(A) $\sqrt{2}$ (B) $\sqrt{4}$ (C) $\sqrt{5}$ (D) $\sqrt{6}$

17. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ to $\vec{c}+\vec{a}$ & $\vec{c}$ to $\vec{a}+\vec{b}$. Then $\left|\vec{a}+\vec{b}+\vec{c}\right|$ is :

(A) $2\sqrt{5}$ (B) $2\sqrt{2}$ (C) $10\sqrt{5}$ (D) $5\sqrt{2}$ **VT0023**

18. If a vector $\vec{a}$ of magnitude 50 is collinear with vector $\vec{b}=6\hat{i}-8\hat{j}-\frac{15}{2}\hat{k}$ and makes an acute angle with positive z-axis then :

(A) $\vec{a}=4\vec{b}$ (B) $\vec{a}=-4\vec{b}$ (C) $\vec{b}=4\vec{a}$ (D) none **VT0025**

19. A, B, C & D are four points in a plane with pv's $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ respectively such that $\left(\vec{a}-\vec{d}\right)\cdot\left(\vec{b}-\vec{c}\right)=\left(\vec{b}-\vec{d}\right)\cdot\left(\vec{c}-\vec{a}\right)=0$. Then for the triangle ABC, D is its

(A) incentre (B) circumcentre (C) orthocentre (D) centroid **VT0026**

20. $\vec{a}$ and $\vec{b}$ are unit vectors inclined to each other at an angle α, $\alpha \in (0, \pi)$ and $\left|\vec{a}+\vec{b}\right| < 1$. Then $\alpha \in$

(A) $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ (B) $\left(\frac{2\pi}{3}, \pi\right)$ (C) $\left(0, \frac{\pi}{3}\right)$ (D) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ **VT0027**

21. Image of the point P with position vector $7\hat{i}-\hat{j}+2\hat{k}$ in the line whose vector equation is, $\vec{r}=9\hat{i}+5\hat{j}+5\hat{k}+\lambda(\hat{i}+3\hat{j}+5\hat{k})$ has the position vector

(A) $(-9, 5, 2)$ (B) $(9, 5, -2)$ (C) $(9, -5, -2)$ (D) none **VT0028**

22. Let $\hat{a}$, $\hat{b}$, $\hat{c}$ are three unit vectors such that $\hat{a}+\hat{b}+\hat{c}$ is also a unit vector. If pairwise angles between $\hat{a}$, $\hat{b}$, $\hat{c}$ are θ_1, θ_2 and θ_3 respectively then $\cos\theta_1+\cos\theta_2+\cos\theta_3$ equals

(A) 3 (B) – 3 (C) 1 (D) – 1 **VT0029**

23. Cosine of an angle between the vectors $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$ if $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}\,^\wedge\,\vec{b}=60°$ is

(A) $\sqrt{3/7}$ (B) $9/\sqrt{21}$ (C) $3/\sqrt{7}$ (D) none **VT0031**

24. Given three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ each two of which are non collinear. Further if $\left(\vec{a}+\vec{b}\right)$ is collinear with $\vec{c}$, $\left(\vec{b}+\vec{c}\right)$ is collinear with $\vec{a}$ & $|\vec{a}|=|\vec{b}|=|\vec{c}|=\sqrt{2}$. Then the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$: **VT0033**

(A) is 3 (B) is – 3 (C) is 0 (D) cannot be evaluated

25. The vector equations of two lines L_1 and L_2 are respectively

$\vec{r}=17\hat{i}-9\hat{j}+9\hat{k}+\lambda(3\hat{i}+\hat{j}+5\hat{k})$ and $\vec{r}=15\hat{i}-8\hat{j}-\hat{k}+\mu(4\hat{i}+3\hat{j})$

I L_1 and L_2 are skew lines

II $(11, -11, -1)$ is the point of intersection of L_1 and L_2

III $(-11, 11, 1)$ is the point of intersection of L_1 and L_2

IV $\cos^{-1}\left(3/\sqrt{35}\right)$ is the acute angle between L_1 and L_2

then, which of the following is true?

(A) II and IV (B) I and IV (C) IV only (D) III and IV **VT0034**

26. Let $\vec{u}, \vec{v}, \vec{w}$ be such that $|\vec{u}| = 1,\ |\vec{v}| = 2,\ |\vec{w}| = 3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$, $\vec{w}$ are perpendicular to each other then $|\vec{u} - \vec{v} + \vec{w}|$ equals

(A) 2 (B) $\sqrt{7}$ (C) $\sqrt{14}$ (D) 14 **VT0037**

27. If $\vec{a}$ and $\vec{b}$ are non zero, non collinear, and the linear combination

$(2x - y)\vec{a} + 4\vec{b} = 5\vec{a} + (x - 2y)\vec{b}$ holds for real x and y then x + y has the value equal to

(A) – 3 (B) 1 (C) 17 (D) 3 **VT0038**

28. If $\vec{p}$ & $\vec{s}$ are not perpendicular to each other and $\vec{r} \times \vec{p} = \vec{q} \times \vec{p}$ & $\vec{r} \cdot \vec{s} = 0$, then $\vec{r}$ =

VT0040

(A) $\vec{p} \cdot \vec{s}$ (B) $\vec{q} + \left(\dfrac{\vec{q} \cdot \vec{p}}{\vec{p} \cdot \vec{s}}\right)\vec{p}$ (C) $\vec{q} - \left(\dfrac{\vec{q} \cdot \vec{s}}{\vec{p} \cdot \vec{s}}\right)\vec{p}$ (D) $\vec{q} + \mu\vec{p}$ for all scalars μ

29. If $\vec{u}$ and $\vec{v}$ are two vectors such that $|\vec{u}| = 3$; $|\vec{v}| = 2$ and $|\vec{u} \times \vec{v}| = 6$ then the correct statement is

VT0041

(A) $\vec{u} \wedge \vec{v} \in (0, 90°)$ (B) $\vec{u} \wedge \vec{v} \in (90°, 180°)$ (C) $\vec{u} \wedge \vec{v} = 90°$ (D) $(\vec{u} \times \vec{v}) \times \vec{u} = 6\vec{v}$

30. Given a parallelogram OACB. The lengths of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ & $\overrightarrow{AB}$ are a, b & c respectively. The scalar product of the vectors $\overrightarrow{OC}$ & $\overrightarrow{OB}$ is : **VT0042**

(A) $\dfrac{a^2 - 3b^2 + c^2}{2}$ (B) $\dfrac{3a^2 + b^2 - c^2}{2}$ (C) $\dfrac{3a^2 - b^2 + c^2}{2}$ (D) $\dfrac{a^2 + 3b^2 - c^2}{2}$

31. Vectors $\vec{a}$ & $\vec{b}$ make an angle $\theta = \dfrac{2\pi}{3}$. If $|\vec{a}| = 1$, $|\vec{b}| = 2$ then $\left\{\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)\right\}^2 =$

(A) 225 (B) 250 (C) 275 (D) 300 **VT0043**

32. Given the vectors

$$\vec{u} = 2\hat{i} - \hat{j} - \hat{k}$$
$$\vec{v} = \hat{i} - \hat{j} + 2\hat{k}$$
$$\vec{w} = \hat{i} - \hat{k}$$

If the volume of the parallelopiped having $-c\vec{u}$, $\vec{v}$ and $c\vec{w}$ as concurrent edges, is 8 then 'c' can be equal to **VT0047**

(A) ± 2 (B) 4 (C) 8 (D) can not be determined

33. Given $\bar{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} - \hat{j} + \hat{k}$, $\bar{c} = \hat{i} + 2\hat{j}$; $(\bar{a} \,\hat{}\, \bar{b}) = \pi/2$, $\bar{a} \cdot \bar{c} = 4$ then **VT0048**

(A) $[\bar{a}\ \bar{b}\ \bar{c}]^2 = |\bar{a}|$ (B) $[\bar{a}\ \bar{b}\ \bar{c}] = |\bar{a}|$ (C) $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ (D) $[\bar{a}\ \bar{b}\ \bar{c}] = |\bar{a}|^2$

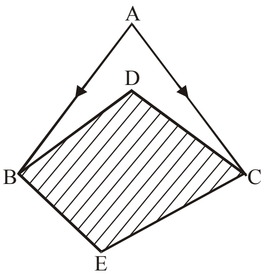
34. For three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ which of the following expressions is not equal to any of the remaining three?

(A) $\vec{u} \cdot (\vec{v} \times \vec{w})$ (B) $(\vec{v} \times \vec{w}) \cdot \vec{u}$ (C) $\vec{v} \cdot (\vec{u} \times \vec{w})$ (D) $(\vec{u} \times \vec{v}) \cdot \vec{w}$ **VT0050**

35. Let $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ & $\vec{c} = \alpha\vec{a} + \beta\vec{b}$. If the vectors, $\hat{i} - 2\hat{j} + \hat{k}$, $3\hat{i} + 2\hat{j} - \hat{k}$ & $\vec{c}$ are coplanar then $\dfrac{\alpha}{\beta}$ is **VT0051**

(A) 1 (B) 2 (C) 3 (D) – 3

36. Let $\overrightarrow{AB} = 3\hat{i} - \hat{j}$, $\overrightarrow{AC} = 2\hat{i} + 3\hat{j}$ and $\overrightarrow{DE} = 4\hat{i} - 2\hat{j}$. The area of the shaded region in the adjacent figure, is-



(A) 5
(B) 6
(C) 7
(D) 8

VT0055

37. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ & $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, then the vector $\vec{c}$ such that $\vec{a}.\vec{c} = 2$ & $\vec{a} \times \vec{c} = \vec{b}$ is -

(A) $\frac{1}{3}\left(3\hat{i} - 2\hat{j} + 5\hat{k}\right)$
(B) $\frac{1}{3}\left(-\hat{i} + 2\hat{j} + 5\hat{k}\right)$
(C) $\frac{1}{3}\left(\hat{i} + 2\hat{j} - 5\hat{k}\right)$
(D) $\frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

VT0061

38. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent vectors & $|\vec{c}| = \sqrt{3}$, then

VT0063

(A) $\alpha = 1, \beta = -1$
(B) $\alpha = 1, \beta = \pm 1$
(C) $\alpha = -1, \beta = \pm 1$
(D) $\alpha = \pm 1, \beta = 1$

[MATRIX MATCH TYPE]

39. If A(0, 1, 0), B(0, 0, 0), C(1, 0, 1) are the vertices of a ΔABC. Match the entries of column-I with column-II.

Column-I	**Column-II**
(A) Orthocentre of ΔABC.	(P) $\frac{\sqrt{2}}{2}$
(B) Circumcentre of ΔABC.	(Q) $\frac{\sqrt{3}}{2}$
(C) Area (ΔABC).	(R) $\frac{\sqrt{3}}{3}$
(D) Distance between orthocentre and centroid.	(S) $\frac{\sqrt{3}}{6}$
(E) Distance between orthocentre and circumcentre.	(T) (0, 0, 0)
(F) Distance between circumcentre and centroid.	(U) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
(G) Incentre of ΔABC.	(V) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
(H) Centroid of ΔABC	(W) $\left(\frac{1}{\sqrt{1}+\sqrt{2}+\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{1}+\sqrt{2}+\sqrt{3}}, \frac{1}{\sqrt{1}+\sqrt{2}+\sqrt{3}}\right)$

VT0066

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Position vectors of the four angular points of a tetrahedron ABCD are A(3, – 2, 1); B(3, 1, 5); C(4, 0, 3) and D(1, 0, 0). Acute angle between the plane faces ADC and ABC is

(A) $\tan^{-1}(5/2)$ (B) $\cos^{-1}(2/5)$ (C) $\operatorname{cosec}^{-1}(5/2)$ (D) $\cot^{-1}(3/2)$ **VT0059**

2. The volume of the tetrahedron formed by the coterminus edges $\vec{a},\ \vec{b},\ \vec{c}$ is 3. Then the volume of the parallelepiped formed by the coterminus edges $\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}$ is

(A) 6 (B) 18 (C) 36 (D) 9 **VT0060**

3. $\vec{a},\vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1,1 and 2 respectively. If $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=0$, then the acute angle between $\vec{a}$ & $\vec{c}$ is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $5\pi/12$ **VT0062**

4. A vector of magnitude $5\sqrt{5}$ coplanar with vectors $\hat{i}+2\hat{j}$ & $\hat{j}+2\hat{k}$ and the perpendicular vector $2\hat{i}+\hat{j}+2\hat{k}$ is

(A) $\pm 5(5\hat{i}+6\hat{j}-8\hat{k})$ (B) $\pm\sqrt{5}(5\hat{i}+6\hat{j}-8\hat{k})$

(C) $\pm 5\sqrt{5}(5\hat{i}+6\hat{j}-8\hat{k})$ (D) $\pm(5\hat{i}+6\hat{j}-8\hat{k})$ **VT0064**

5. Let $\vec{\alpha}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}+\hat{j}$. If $\vec{\gamma}$ is a unit vector, then the maximum value of $\left[\vec{\alpha}\times\vec{\beta}\ \ \vec{\beta}\times\vec{\gamma}\ \ \vec{\gamma}\times\vec{\alpha}\right]$ is equal to

(A) 2 (B) 3 (C) 4 (D) 9 **VT0065**

[MULTIPLE OBJECTIVE TYPE]

6. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non zero vectors satisfying the condition $\vec{a}\times\vec{b}=\vec{c}$ & $\vec{b}\times\vec{c}=\vec{a}$ then which of the following always hold(s) good?

(A) $\vec{a}$, $\vec{b}$, $\vec{c}$ are orthogonal in pairs (B) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left|\vec{b}\right|$

(C) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left|\vec{c}\right|^2$ (D) $\left|\vec{b}\right|=\left|\vec{c}\right|$ **VT0074**

7. Given the following information about the non zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

(i) $(\vec{A}\times\vec{B})\times\vec{A}=\vec{0}$ (ii) $\vec{B}\cdot\vec{B}=4$ (iii) $\vec{A}\cdot\vec{B}=-6$ (iv) $\vec{B}\cdot\vec{C}=6$

Which one of the following holds good?

(A) $\vec{A}\times\vec{B}=\vec{0}$ (B) $\vec{A}\cdot(\vec{B}\times\vec{C})=0$ (C) $\vec{A}\cdot\vec{A}=8$ (D) $\vec{A}\cdot\vec{C}=-9$ **VT0075**

8. If $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ are four non zero vectors in the same plane no two of which are collinear then which of the following hold(s) good?

(A) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=0$ (B) $(\vec{A}\times\vec{C})\cdot(\vec{B}\times\vec{D})\neq 0$

(C) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})=\vec{0}$ (D) $(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{D})\neq\vec{0}$ **VT0076**

9. If $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are the pv's of the points A, B, C & D respectively in three dimensional space & satisfy the relation $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = 0$, then :

(A) A, B, C & D are coplanar

(B) the line joining the points B & D divides the line joining the point A & C in the ratio 2 : 1.

(C) the line joining the points A & C divides the line joining the points B & D in the ratio 1 : 1

(D) the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are linearly dependent. **VT0077**

10. If a, b, c are different real numbers and $a\hat{i} + b\hat{j} + c\hat{k}$; $b\hat{i} + c\hat{j} + a\hat{k}$ & $c\hat{i} + a\hat{j} + b\hat{k}$ are position vectors of three non-collinear points A, B & C then :

(A) centroid of triangle ABC is $\frac{a+b+c}{3}\left(\hat{i} + \hat{j} + \hat{k}\right)$

(B) $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the three vectors

(C) perpendicular from the origin to the plane of triangle ABC meet at centroid

(D) triangle ABC is an equilateral triangle. **VT0082**

11. A vector of magnitude 10 along the normal to the curve $3x^2 + 8xy + 2y^2 - 3 = 0$ at its point P(1, 0) can be

(A) $6\hat{i} + 8\hat{j}$ (B) $-6\hat{i} + 8\hat{j}$ (C) $6\hat{i} - 8\hat{j}$ (D) $-6\hat{i} - 8\hat{j}$ **VT0083**

12. Let OAB be a regular triangle with side length unity (O being the origin). Also M,N are the points of trisection of AB,M being closer to A and N closer to B. Position vectors of A,B,M and N are $\vec{a}, \vec{b}, \vec{m}$ and $\vec{n}$ respectively. Which of the following hold(s) good ?

(A) $\vec{m} = x\vec{a} + y\vec{b} \Rightarrow \frac{2}{3}$ and $y = \frac{1}{3}$ (B) $\vec{m} = x\vec{a} + y\vec{b} \Rightarrow x = \frac{5}{6}$ and $y = \frac{1}{6}$

(C) $\vec{m}.\vec{n}$ equals $\frac{13}{18}$ (D) $\vec{m}.\vec{n}$ equals $\frac{15}{18}$ **VT0084**

13. Given three vectors $\vec{U} = 2\hat{i} + 3\hat{j} - 6\hat{k}$; $\vec{V} = 6\hat{i} + 2\hat{j} + 3\hat{k}$; $\vec{W} = 3\hat{i} - 6\hat{j} - 2\hat{k}$

Which of the following hold good for the vectors $\vec{U}$, $\vec{V}$ and $\vec{W}$? **VT0087**

(A) $\vec{U}$, $\vec{V}$ and $\vec{W}$ are linearly dependent (B) $(\vec{U} \times \vec{V}) \times \vec{W} = \vec{0}$

(C) $\vec{U}$, $\vec{V}$ and $\vec{W}$ form a triplet of mutually perpendicular vectors (D) $\vec{U} \times (\vec{V} \times \vec{W}) = \vec{0}$

14. Let $\vec{a}$ and $\vec{b}$ be two non-zero and non-collinear vectors then which of the following is/are always correct?

(A) $\vec{a} \times \vec{b} = [\vec{a}\ \vec{b}\ \hat{i}]\hat{i} + [\vec{a}\ \vec{b}\ \hat{j}]\hat{j} + [\vec{a}\ \vec{b}\ \hat{k}]\hat{k}$

(B) $\vec{a}.\vec{b} = (\vec{a}.\hat{i})(\vec{b}.\hat{i}) + (\vec{a}.\hat{j}).(\vec{b}.\hat{j}) + (\vec{a}.\hat{k})(\vec{b}.\hat{k})$

(C) if $\vec{u} = \hat{a} - (\hat{a}.\hat{b})\hat{b}$ and $\vec{v} = \hat{a} \times \hat{b}$ then $|\vec{u}| = |\vec{v}|$

(D) if $\vec{c} = \vec{a} \times (\vec{a} \times \vec{b})$ and $\vec{d} = \vec{b} \times (\vec{a} \times \vec{b})$ then $\vec{c} + \vec{d} = \vec{0}$ **VT0089**

[COMPREHENSION TYPE]

Paragraph for questions nos. 15 to 16

Consider three vectors $\vec{p}=\hat{i}+\hat{j}+\hat{k}$, $\vec{q}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{r}=\hat{i}+\hat{j}+3\hat{k}$ and let $\vec{s}$ be a unit vector, then

15. $\vec{p}$, $\vec{q}$ and $\vec{r}$ are

(A) linearly dependent

(B) can form the sides of a possible triangle

(C) such that the vectors $(\vec{q}-\vec{r})$ is orthogonal to $\vec{p}$

(D) such that each one of these can be expressed as a linear combination of the other two **VT0090**

16. If $(\vec{p}\times\vec{q})\times\vec{r}=u\vec{p}+v\vec{q}+w\vec{r}$, then $(u+v+w)$ equals to

(A) 8 (B) 2 (C) – 2 (D) 4 **VT0090**

[MATRIX MATCH TYPE]

17. **Column-I** **Column-II**

(A) P is point in the plane of the triangle ABC. pv's of A,B and C are $\vec{a},\vec{b}$ and $\vec{c}$ respectively with respect to P as the origin.
If $\left(\vec{b}+\vec{c}\right).\left(\vec{b}-\vec{c}\right)=0$ and $\left(\vec{c}+\vec{a}\right).\left(\vec{c}-\vec{a}\right)=0$, then w.r.t. the triangle ABC,P is its **VT0091**

(B) If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the three non collinear points A,B and C respectively such that the vector $\vec{V}=\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is a null vector then w.r.t. the ΔABC, P is its **VT0092**

(C) If P is a point inside the ΔABC such that the vector $\vec{R}=(BC)\overrightarrow{PA}+(CA)\left(\overrightarrow{PB}\right)+(AB)\left(\overrightarrow{PC}\right)$ is a null vector then w.r.t. the ΔABC, P is its **VT0093**

(D) If P is a point in the plane of the triangle ABC such that the scalar product $\overrightarrow{PA}.\overrightarrow{CB}$ and $\overrightarrow{PB}.\overrightarrow{AC}$ vanishes, then w.r.t. the ΔABC, P is its **VT0094**

(P) centroid

(Q) orthocentre

(R) incentre

(S) circumcentre

EXERCISE (S-1)

1. Given the vector $\overrightarrow{PQ}=-6\mathbf{i}-4\mathbf{j}$ and **Q** is the point (3, 3), find the point **P**. **VT0095**

2. In ΔABC, a point P is chosen on side $\overrightarrow{AB}$ so that AP : PB = 1 : 4 and a point Q is chosen on the side $\overrightarrow{BC}$ so that CQ : QB = 1:3. Segment $\overrightarrow{CP}$ and $\overrightarrow{AQ}$ intersect at M. If the ratio $\frac{MC}{PC}$ is expressed as a rational numbers in the lowest term as $\frac{a}{b}$, find (a + b). **VT0098**

3. Let O be an interior point of ΔABC such that $2\overrightarrow{OA}+5\overrightarrow{OB}+10\overrightarrow{OC}=\vec{0}$. If the ratio of the area of ΔABC to the area of ΔAOC is t, where 'O' is the origin. Find [t]. **VT0099**
(where [ ] denotes greatest integer function)

4. If the distance from the point P(1, 1, 1) to the line passing through the points Q(0, 6, 8) and R(–1, 4, 7) is expressed in the form $\sqrt{p/q}$ where p and q are coprime, then the value of $\frac{(p+q)(p+q-1)}{2}$.

VT0100

5. If $\vec{a}$ & $\vec{b}$ are non collinear vectors such that $\vec{p} = (x+4y)\vec{a} + (2x+y+1)\vec{b}$ & $\vec{q} = (y-2x+2)\vec{a} + (2x-3y-1)\vec{b}$, find x & y such that $3\vec{p} = 2\vec{q}$. **VT0103**

6. (a) Show that the points $\vec{a} - 2\vec{b} + 3\vec{c}; 2\vec{a} + 3\vec{b} - 4\vec{c}$ & $-7\vec{b} + 10\vec{c}$ are collinear.

(b) Prove that the points A (1, 2, 3), B(3, 4, 7), C(–3, –2, –5) are collinear & find the ratio in which B divides AC. **VT0104**

7. Find out whether the following pairs of lines are parallel, non-parallel & intersecting, or non-parallel non-intersecting.

(a) $\vec{r}_1 = \hat{i} + \hat{j} + 2\hat{k} + \lambda(3\hat{i} - 2\hat{j} + 4\hat{k})$
$\vec{r}_2 = 2\hat{i} + \hat{j} + 3\hat{k} + \mu(-6\hat{i} + 4\hat{j} - 8\hat{k})$

(b) $\vec{r}_1 = \hat{i} - \hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$
$\vec{r}_2 = 2\hat{i} + 4\hat{j} + 6\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})$

(c) $\vec{r}_1 = \hat{i} + \hat{k} + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$
$\vec{r}_2 = 2\hat{i} + 3\hat{j} + \mu(4\hat{i} - \hat{j} + \hat{k})$

VT0105

8. In a unit cube. Find

(a) The angle between the diagonal of the cube and a diagonal of a face skew to it.

(b) The angle between the diagonals of two faces of the cube through the same vertex.

(c) The angle between a diagonal of a cube and a diagonal of a face intersecting it. **VT0107**

9. **(a)** Find the minimum area of the triangle whose vertices are A(–1,1,2); B(1,2,3) and C(t,1,1) where t is a real number. **VT0111**

(b) Let $\overrightarrow{OA} = \vec{a}; \overrightarrow{OB} = 100\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$ where O,A and C are non collinear points. Let P denotes the area of the parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides and Q denotes the area of the quadrilateral OABC. If Q = λP. Find the value of λ. **VT0112**

10. Given that $\vec{a}$ and $\vec{b}$ are two unit vectors such that angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\vec{c}$ be a vector in the plane of $\vec{a}$ and $\vec{b}$, such that $|\vec{c}| = 4, \vec{c} \times \vec{b} = 2\vec{a} \times \vec{b}$ and $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ then, find **VT0113**

(a) the value of λ, **(b)** the sum of values of μ and **(c)** the product of all possible values of μ.

11. The vector $\overrightarrow{OP} = \hat{i} + 2\hat{j} + 2\hat{k}$ turns through a right angle, passing through the positive x-axis on the way. Find the vector in its new position. **VT0116**

12. If $\vec{x}, \vec{y}$ are two non-zero and non-collinear vectors satisfying

$[(a-2)\alpha^2 + (b-3)\alpha + c]\,\vec{x} + [(a-2)\beta^2 + (b-3)\beta + c]\,\vec{y} + [(a-2)\gamma^2 + (b-3)\gamma + c]\,(\vec{x} \times \vec{y}) = 0$ where α,β,γ are three distinct real numbers, then find the value of $(a^2 + b^2 + c^2)$. **VT0122**

EXERCISE (S-2)

1. A vector $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ satisfies the following conditions :

(i) magnitude of $\vec{V}$ is $7\sqrt{2}$

(ii) $\vec{V}$ is parallel to the plane $x - 2y + z = 6$

(iii) $\vec{V}$ is orthogonal to the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ and (iv) $\vec{V}.\hat{i} > 0$

Find the value of $(v_1 + v_2 + v_3)$. **VT0127**

2. Let $(\vec{p} \times \vec{q}) \times \vec{r} + (\vec{q}.\vec{r})\vec{q} = (x^2 + y^2)\vec{q} + (14 - 4x - 6y)\vec{p}$ and $(\vec{r}.\vec{r})\vec{p} = \vec{r}$ where $\vec{p}$ and $\vec{q}$ are two non-zero non-collinear vectors and x and y are scalars. Find the value of $(x + y)$. **VT0128**

3. Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $|\vec{u} - \hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u} - 2\hat{i}|$ where $\hat{i}$ is the unit vector along x-axis then find the value of $|\vec{u}|$. **VT0130**

4. The position vectors of the points A, B, C are respectively (1, 1, 1) ; (1, –1, 2) ; (0, 2, –1). Find a unit vector parallel to the plane determined by ABC & perpendicular to the vector (1, 0, 1). **VT0133**

5. (a) Find a unit vector $\hat{a}$ which makes an angle $(\pi/4)$ with axis of z & is such that $\hat{a} + \hat{i} + \hat{j}$ is a unit vector. **VT0138**

(c) If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then find the range of $\dfrac{3\left|\vec{a} + \vec{b}\right|}{2} + 2\left|\vec{a} - \vec{b}\right|$. **VT0139**

6. Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $\left(\overrightarrow{PA}.\overrightarrow{PB}\right) + 3\left(\overrightarrow{OA}.\overrightarrow{OB}\right) = 0$. If the maximum and minimum values of $\left|\overrightarrow{PA}\right|\left|\overrightarrow{PB}\right|$ are M and m respectively then find the values of $M^2 + m^2$. **VT0141**

7. Let $\vec{a}, \vec{b}, \vec{c}$ are unit vectors where $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} + \vec{a}|^2 = 3$, then $|\vec{a} + 2\vec{b} + 3\vec{c}|^2$ is equal to **VT0142**

8. The pv's of the four angular points of a tetrahedron are $A\left(\hat{j} + 2\hat{k}\right); B\left(3\hat{i} + \hat{k}\right); C\left(4\hat{i} + 3\hat{j} + 6\hat{k}\right)$ & $D\left(2\hat{i} + 3\hat{j} + 2\hat{k}\right)$. Find :

(i) the perpendicular distance from A to the line BC. **VT0117**

(ii) the volume of the tetrahedron ABCD. **VT0118**

(iii) the perpendicular distance from D to the plane ABC. **VT0119**

(iv) the shortest distance between the lines AB & CD. **VT0120**

9. Let a 3 dimensional vector $\vec{V}$ satisfies the condition $2\vec{V} + \vec{V} \times \left(\hat{i} + 2\hat{j}\right) = 2\hat{i} + \hat{k}$.

If $3\left|\vec{V}\right| = \sqrt{m}$, where m ∈ N, then find m. **VT0121**

10. Vector $\vec{V}$ is perpendicular to the plane of vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ and satisfies the condition $\vec{V}.(\hat{i} + 2\hat{j} - 7\hat{k}) = 10$. Find $\left|\vec{V}\right|^2$. **VT0124**

11. Let two non-collinear vectors $\vec{a}$ and $\vec{b}$ inclined at an angle $\frac{2\pi}{3}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = 4$. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given as $\overrightarrow{OP} = (e^t + e^{-t})\vec{a} + (e^t - e^{-t})\vec{b}$. If the least distance of P from origin is $\sqrt{2}\sqrt{\sqrt{p} - q}$ where p,q ∈ N then find the value of (p + q). **VT0125**

EXERCISE (JM)

1. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality **[AIEEE-2009]**
$[3\vec{u}\ p\vec{v}\ p\vec{w}] - [p\vec{v}\ \vec{w}\ q\vec{u}] - [2\vec{w}\ q\vec{v}\ q\vec{u}] = 0$ holds for :-
(1) More than two but not all values of (p,q) (2) All values of (p, q)
(3) Exactly one value of (p, q) (4) Exactly two values of (p, q) **VT0143**

2. Let $\vec{a} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 3$ is : **[AIEEE-2010]**
(1) $-\hat{i} + \hat{j} - 2\hat{k}$ (2) $2\hat{i} - \hat{j} + 2\hat{k}$ (3) $\hat{i} - \hat{j} - 2\hat{k}$ (4) $\hat{i} + \hat{j} - 2\hat{k}$ **VT0144**

3. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying : $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to :- **[AIEEE-2011]**
(1) $\vec{b} + \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (2) $\vec{c} - \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (3) $\vec{b} - \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (4) $\vec{c} + \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ **VT0145**

4. If $\vec{a} = \frac{1}{\sqrt{10}}\left(3\hat{i} + \hat{k}\right)$ and $\vec{b} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right)$, then the value of $\left(2\vec{a} - \vec{b}\right).\left[\left(\vec{a} \times \vec{b}\right) \times \left(\vec{a} + 2\vec{b}\right)\right]$ is :- **[AIEEE-2011]**
(1) 5 (2) 3 (3) – 5 (4) – 3 **VT0146**

5. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is colliner with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is : **[AIEEE-2011]**
(1) $\vec{a} + \vec{c}$ (2) $\vec{a}$ (3) $\vec{c}$ (4) $\vec{0}$ **VT0147**

6. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is : **[AIEEE-2012]**
(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{3}$ **VT0148**

7. Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{q}$, $\overrightarrow{AD} = \vec{p}$ and ∠BAD be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by : **[AIEEE-2012]**
(1) $\vec{r} = -3\vec{q} + \frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$ (2) $\vec{r} = 3\vec{q} - \frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$
(3) $\vec{r} = -\vec{q} + \left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$ (4) $\vec{r} = \vec{q} - \left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$ **VT0149**

8. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is : **[JEE-MAINS 2013]**

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3) $\sqrt{33}$ (4) $\sqrt{45}$ **VT0150**

9. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vectors of the type $\vec{b} + \lambda\vec{c}$ for some scalar λ, whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$, is : **[JEE-MAINS Online 2013]**

(1) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (2) $2\hat{i} + \hat{j} + 5\hat{k}$ (3) $2\hat{i} - \hat{j} + 5\hat{k}$ (4) $2\hat{i} + 3\hat{j} + 3\hat{k}$ **VT0151**

10. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a} \bullet \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30°, then $\left|\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right|$ equals : **[JEE-MAINS Online 2013]**

(1) $\frac{3}{2}$ (2) 3 (3) $\frac{1}{2}$ (4) $\frac{3\sqrt{3}}{2}$ **VT0152**

11. If $\left[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\right] = \lambda\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ then λ is equal to : **[JEE(Main)-2014]**

(1) 2 (2) 3 (3) 0 (4) 1 **VT0153**

12. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \frac{1}{3}\left|\vec{b}\right|\left|\vec{c}\right|\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of sin θ is :

[JEE(Main)-2015]

(1) $\frac{2}{3}$ (2) $\frac{-2\sqrt{3}}{3}$ (3) $\frac{2\sqrt{2}}{3}$ (4) $\frac{-\sqrt{2}}{3}$ **VT0154**

13. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \frac{\sqrt{3}}{2}\left(\vec{b} + \vec{c}\right)$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is :- **[JEE(Main)-2016]**

(1) $\frac{5\pi}{6}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{2}$ (4) $\frac{2\pi}{3}$ **VT0155**

14. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c} - \vec{a}| = 3$, $\left|(\vec{a} \times \vec{b}) \times \vec{c}\right| = 3$ and the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$ be 30°. Then $\vec{a} \cdot \vec{c}$ is equal to : **[JEE(Main)-2017]**

(1) $\frac{1}{8}$ (2) $\frac{25}{8}$ (3) 2 (4) 5 **VT0156**

15. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b} = 24$, then $|\vec{u}|^2$ is equal to - **[JEE(Main)-2018]**

(1) 315 (2) 256 (3) 84 (4) 336 **VT0157**

16. Let $\vec{a}=\hat{i}-\hat{j}, \vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}$ be a vector such that $\vec{a}\times\vec{c}+\vec{b}=\vec{0}$ and $\vec{a}.\vec{c}=4$, then $|\vec{c}|^2$ is equal to :- **[JEE(Main)-Jan 19]**

(1) $\frac{19}{2}$ (2) 8 (3) $\frac{17}{2}$ (4) 9 **VT0158**

17. Let $\sqrt{3}\hat{i}+\hat{j}, \hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ respectively be the position vectors of the points A, B and C with respect to the origin O. If the distance of C from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$, then the sum of all possible values of β is :- **[JEE(Main)-Jan 19]**

(1) 2 (2) 1 (3) 3 (4) 4 **VT0159**

18. Let $\vec{\alpha}=3\hat{i}+\hat{j}$ and $\vec{\beta}=2\hat{i}-\hat{j}+3\hat{k}$. If $\vec{\beta}=\vec{\beta}_1-\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1\times\vec{\beta}_2$ is equal to **[JEE(Main)-Apr 19]**

VT0160

(1) $-3\hat{i}+9\hat{j}+5\hat{k}$ (2) $3\hat{i}-9\hat{j}-5\hat{k}$ (3) $\frac{1}{2}\left(-3\hat{i}+9\hat{j}+5\hat{k}\right)$ (4) $\frac{1}{2}\left(3\hat{i}-9\hat{j}+5\hat{k}\right)$

19. The distance of the point having position vector $-\hat{i}+2\hat{j}+6\hat{k}$ from the straight line passing through the point (2, 3, –4) and parallel to the vector, $6\hat{i}+3\hat{j}-4\hat{k}$ is : **[JEE(Main)-Apr 19]**

(1) 7 (2) $4\sqrt{3}$ (3) $2\sqrt{13}$ (4) 6 **VT0161**

EXERCISE (JA)

1. **(a)** Two adjacent sides of a parallelogram ABCD are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by -

(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$ **VT0168**

(b) If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $\left(2\vec{a}+\vec{b}\right).\left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-2\vec{b}\right)\right]$ is **[JEE 2010, 5+3]**

VT0169

2. **(a)** Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by **VT0170**

(A) $\hat{i}-3\hat{j}+3\hat{k}$ (B) $-3\hat{i}-3\hat{j}-\hat{k}$ (C) $3\hat{i}-\hat{j}+3\hat{k}$ (D) $\hat{i}+3\hat{j}-3\hat{k}$

(b) The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$ **VT0171**

(c) Let $\vec{a} = -\hat{i} - \hat{k}, \vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$, then the value of $\vec{r}.\vec{b}$ is **[JEE 2011, 3+4+4]**

VT0172

3. **(a)** If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$, then $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ is

VT0173

(b) If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a} + \vec{b}| = \sqrt{29}$ and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a} + \vec{b}).(-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

(A) 0 (B) 3 (C) 4 (D) 8 **VT0174**

[JEE 2012, 4+3]

4. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}, \overrightarrow{PQ}$ and $\overrightarrow{PS}$ is **[JEE-Advanced 2013, 2M]**

(A) 5 (B) 20 (C) 10 (D) 30 **VT0175**

5. Consider the set of eight vectors $V = \left\{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\right\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE-Advanced 2013, 4, (–1)]**

VT0176

6. Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		**List-II**
P.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a} \times \vec{b}), 3(\vec{b} \times \vec{c})$ and $(\vec{c} \times \vec{a})$ is	1.	100
Q.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a} + \vec{b}), (\vec{b} + \vec{c})$ and $2(\vec{c} + \vec{a})$ is	2.	30
R.	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a} + 3\vec{b})$ and $(\vec{a} - \vec{b})$ is	3.	24
S.	Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a} + \vec{b})$ and $\vec{a}$ is	4.	60

Codes :

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

VT0177

[JEE-Advanced 2013, 3, (–1)]

7. Let $\vec{x},\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is nonzero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then **[JEE(Advanced)-2014, 3]**

(A) $\vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$ (B) $\vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$ (C) $\vec{a}.\vec{b}=-(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a}=(\vec{a}.\vec{y})(\vec{z}-\vec{y})$ **VT0178**

8. Let $\vec{a},\vec{b}$, and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is **VT0179**

[JEE(Advanced)-2014, 3]

9. Let ΔPQR be a triangle. Let $\vec{a}=\overrightarrow{QR}, \vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|=12$, $|\vec{b}|=4\sqrt{3}$ and $\vec{b}.\vec{c}=24$, then which of the following is (are) true ? **[JEE 2015, 4M, –2M]**

(A) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ (B) $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$ (C) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ (D) $\vec{a}.\vec{b}=-72$ **VT0180**

10. Suppose that $\vec{p},\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p},\vec{q}$ and $\vec{r}$ be 4,3 and 5, respectively. If the components of this vector $\vec{s}$ along $\left(-\vec{p}+\vec{q}+\vec{r}\right)$, $\left(\vec{p}-\vec{q}+\vec{r}\right)$ and $\left(-\vec{p}-\vec{q}+\vec{r}\right)$ are x,y and z, respectively, then the value of 2x + y + z is **VT0181**

[JEE 2015, 4M, –0M]

11. Let $\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ be a unit vector in $\mathbb{R}^3$ and $\hat{w}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $\left|\hat{u}\times\vec{v}\right|=1$ and $\hat{w}.(\hat{u}\times\vec{v})=1$. Which of the following statement(s) is(are) correct?

(A) There is exactly one choice for such $\vec{v}$

(B) There are infinitely many choice for such $\vec{v}$

(C) If $\hat{u}$ lies in the xy-plane then $|u_1|=|u_2|$ **VT0182**

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1|=|u_3|$ **[JEE(Advanced)-2016, 4(–2)]**

12. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}.\overrightarrow{OQ}+\overrightarrow{OR}.\overrightarrow{OS}=\overrightarrow{OR}.\overrightarrow{OP}+\overrightarrow{OQ}.\overrightarrow{OS}=\overrightarrow{OQ}.\overrightarrow{OR}+\overrightarrow{OP}.\overrightarrow{OS}$. Then the triangle PQR has S as its

[JEE(Advanced)-2017]

(A) incentre (B) orthocenter (C) circumcentre (D) centroid **VT0183**

PARAGRAPH :

Let O be the origin, and $\overrightarrow{OX},\overrightarrow{OY},\overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR},\overrightarrow{RP},\overrightarrow{PQ}$, respectively, of a triangle PQR. **[JEE(Advanced)-2017]**

13. $\left|\overrightarrow{OX}\times\overrightarrow{OY}\right|=$

(A) sin(Q + R) (B) sin(P + R) (C) sin 2R (D) sin(P + Q) **VT0184**

14. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$ **VT0184**

15. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b}=0$. For some x, y ∈ $\mathbb{R}$, let $\vec{c}=x\vec{a}+y\vec{b}+(\vec{a}\times\vec{b})$. If $|\vec{c}|=2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ______ **[JEE(Advanced)-2018, 3(0)]**

VT0185

16. Three lines $L_1 : \vec{r}=\lambda\hat{i},\ \lambda \in \mathbb{R}$,

$L_2 : \vec{r}=\hat{k}+\mu\hat{j},\ \mu \in \mathbb{R}$ and

$L_3 : \vec{r}=\hat{i}+\hat{j}+v\hat{k},\ v \in \mathbb{R}$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $\hat{k}+\hat{j}$ (2) $\hat{k}$ (3) $\hat{k}+\frac{1}{2}\hat{j}$ (4) $\hat{k}-\frac{1}{2}\hat{j}$ **VT0186**

17. Let $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ be two vectors. Consider a vector $\vec{c}=\alpha\vec{a}+\beta\vec{b}$, $\alpha,\beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $\left(\vec{a}+\vec{b}\right)$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c}-\left(\vec{a}\times\vec{b}\right)\right).\vec{c}$ equals

[JEE(Advanced)-2019, 3(0)]

VT0187

ANSWER KEY

EXERCISE (O-1)

1. B **2.** A **3.** D **4.** B **5.** D **6.** D **7.** D **8.** D
9. B **10.** D **11.** C **12.** A **13.** A **14.** (i) D, (ii) B, (iii) B **15.** D
16. A **17.** D **18.** B **19.** C **20.** B **21.** B **22.** D **23.** A
24. B **25.** A **26.** C **27.** B **28.** C **29.** C **30.** D **31.** D
32. A **33.** D **34.** C **35.** D **36.** C **37.** B **38.** D

39. (A) T; (B) U ; (C) P; (D) R ; (E) Q; (F) S; (G) W; (H) V

EXERCISE (O-2)

1. A **2.** C **3.** A **4.** D **5.** B **6.** A,C

7. A,B,D **8.** B, C **9.** A,C,D **10.** A,B,C,D **11.** A,D **12.** A,C
13. B,C,D **14.** A,B,C **15.** C **16.** B **17.** (A) S; (B) P; (C) R; (D) Q

EXERCISE (S-1)

1. (9, 7) **2.** 13 **3.** 3 **4.** 4950

5. x = 2, y = –1 **6.** **(b)** externally in the ratio 1 : 3

7. **(i)** parallel **(ii)** the lines intersect at the point p.v. $-2\hat{i}+2\hat{j}$ **(iii)** lines are skew

8. **(a)** $\cot^{-1}(0)$; **(b)** $\cot^{-1}\frac{1}{\sqrt{3}}$; **(c)** $\cot^{-1}\sqrt{2}$ **9.** **(a)** $\frac{\sqrt{3}}{2}$, **(b)** 51

10. **(a)** 2, **(b)** – 1, **(c)** –12 **11.** $\frac{4}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}-\frac{1}{\sqrt{2}}\hat{k}$ **12.** 13

EXERCISE (S-2)

1. 12 **2.** 5 **3.** $\sqrt{2}-1$ **4.** $\pm\frac{1}{3\sqrt{3}}\left(\hat{i}+5\hat{j}-\hat{k}\right)$

5. **(a)** $\frac{-1}{2}\hat{i}-\frac{1}{2}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$, **(c)** Range : [3,5] **6.** 34 **7.** 19

8. **(i)** $\frac{6}{7}\sqrt{14}$ **(ii)** 6 **(iii)** $\frac{3}{5}\sqrt{10}$ **(iv)** $\sqrt{6}$ **9.** 6 **10.** 75 **11.** 488

EXERCISE (JM)

1. 3 **2.** 1 **3.** 2 **4.** 3 **5.** 4 **6.** 4 **7.** 3 **8.** 3
9. 1 **10.** 1 **11.** 4 **12.** 3 **13.** 1 **14.** 3 **15.** 4 **16.** 1
17. 2 **18.** 3 **19.** 1

EXERCISE (JA)

1. **(a)** B; **(b)** 5 **2.** **(a)** C; **(b)** A,D; **(c)** 9 **3.** **(a)** 3; **(b)** C **4.** C **5.** 5
6. C **7.** A,B,C **8.** 4 **9.** A,C,D **10.** Bonus **11.** B,C **12.** B
13. D **14.** B **15.** 3 **16.** 3,4 **17.** 18.00

3D-COORDINATE GEOMETRY

POINT

1. INTRODUCTION :

In earlier classes we have learnt about points, lines, circles and conic section in two dimensional geometry. In two dimensions a point represented by an ordered pair (x, y) (where x & y are both real numbers)

In space, each body has length, breadth and height i.e. each body exist in three dimensional space. Therefore three independent quantities are essential to represent any point in space. Three axes are required to represent these three quantities.

2. RECTANGULAR CO-ORDINATE SYSTEM :

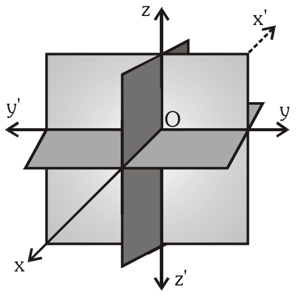
In cartesian system of three lines which are mutually perpendicular, such a system is called rectangular cartesian co-ordinate system.

Co-ordinate axes and co-ordinate planes :

When three mutually perpendicular planes intersect at a point, then mutually perpendicular lines are obtained and these lines also pass through that point. If we assume the point of intersection as origin, then the three planes are known as co-ordinate planes and the three lines are known as co-ordinate axes.

Octants :

Every plane bisects the space. Hence three co-ordinate plane divide the space in eight parts. These parts are known as octants.

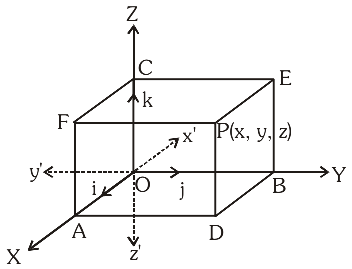


3. COORDINATES OF A POINT IN SPACE :

Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right handed system.

The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy planes be x, y, z respectively (with proper signs) then we say that coordinates of P are (x, y, z). Also OA = |x|, OB = |y|, OC = |z|



4. DISTANCE FORMULA :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

(a) Distance from Origin :

Let O be the origin and P (x, y, z) be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$

(b) Distance of a point from coordinate axes :

Let P(x, y, z) be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively. Then

$$PA = \sqrt{(y^2+z^2)}\,;\ PB = \sqrt{(z^2+x^2)}\,;\ PC = \sqrt{(x^2+y^2)}$$

Illustration 1 : Prove by using distance formula that the points P (1, 2, 3), Q (–1, –1, –1) and R(3, 5, 7) are collinear.

Solution : We have PQ = $\sqrt{(-1-1)^2+(-1-2)^2+(-1-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

$$QR = \sqrt{(3+1)^2+(5+1)^2+(7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

and PR = $\sqrt{(3-1)^2+(5-2)^2+(7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since QR = PQ + PR. Therefore the given points are collinear. **Ans.**

Illustration 2 : Find the locus of a point the sum of whose distances from (1, 0, 0) and (–1, 0, 0) is equal to 10.

Solution : Let the points A(1,0,0), B (–1,0,0) and P(x,y,z)

Given : PA + PB =10

$$\sqrt{(x-1)^2+(y-0)^2+(z-0)^2} + \sqrt{(x+1)^2+(y-0)^2+(z-0)^2} = 10$$

$$\Rightarrow \quad \sqrt{(x-1)^2+y^2+z^2} = 10 - \sqrt{(x+1)^2+y^2+z^2}$$

Squaring both sides, we get ;

$$\Rightarrow \quad (x-1)^2+y^2+z^2 = 100+(x+1)^2+y^2+z^2-20\sqrt{(x+1)^2+y^2+z^2}$$

$$\Rightarrow \quad -4x-100 = -20\sqrt{(x+1)^2+y^2+z^2} \qquad \Rightarrow \quad x+25 = 5\sqrt{(x+1)^2+y^2+z^2}$$

Again squaring both sides we get $x^2+50x+625 = 25\,\{(x^2+2x+1)+y^2+z^2\}$

$\Rightarrow \quad 24x^2+25y^2+25z^2-600 = 0$

i.e. required equation of locus **Ans.**

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5. SECTION FORMULAE :

Let P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$. Then co-ordinates of R(x, y, z) = $\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

Mid-Point : Mid point of PQ is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Illustration 3 : Find the ratio in which the plane x – 2y + 3z = 17 divides the line joining the points (–2, 4, 7) and (3, –5, 8).

Solution : Let the required ratio be k : 1

The co-ordinates of the point which divides the join of (–2, 4, 7) and (3, –5, 8) in the ratio k : 1 are $\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right)$

Since this point lies on the plane x – 2y + 3z – 17 = 0

$$\therefore \quad \left(\frac{3k-2}{k+1}\right)-2\left(\frac{-5k+4}{k+1}\right)+3\left(\frac{8k+7}{k+1}\right)-17=0$$

$$\Rightarrow \quad (3k-2)-2(-5k+4)+3(8k+7)=17k+17$$

$$\Rightarrow \quad 3k+10k+24k-17k=17+2+8-21$$

$$\Rightarrow \quad 37k-17k=6 \quad \Rightarrow \quad 20k=6\ ;\ \ k=\frac{6}{20}=\frac{3}{10}$$

Hence the required ratio = k : 1 $= \frac{3}{10}$: 1 = 3 : 10 **Ans.**

6. CENTROID OF A TRIANGLE :

Let A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) be the vertices of a triangle ABC. Then its centroid G is given by $G=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Illustration 4 : If the centre of a tetrahedron OABC where A, B, C, are given by (a, 2, 3), (1, b, 2) and (2, 1, c) respectively is (1, 2, – 1), then distance of P (a, b, c) from origin is -

(A) $\sqrt{107}$ (B) $\sqrt{14}$ (C) $\sqrt{107}/14$ (D) none of these

Solution : Centre is $\left(\frac{1}{4}\Sigma x, \frac{1}{4}\Sigma y, \frac{1}{4}\Sigma z\right)=(1, 2, -1)$

$$\Rightarrow \quad \frac{a+1+2+0}{4}=1, \frac{2+b+1+0}{4}=2, \frac{3+2+c+0}{4}=-1 \quad \Rightarrow \quad a = 1, b = 5, c = -9$$

$$\therefore \quad OP=\sqrt{a^2+b^2+c^2}=\sqrt{107}$$ **Ans. (A)**

Do yourself 1:

(i) Find the distance between the points P(3, 4, 5) and Q(–1, 2, –3).

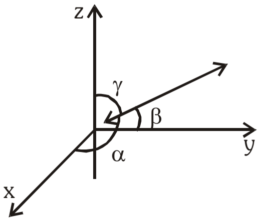
(ii) Show that the points A(0, 7, 10), B(–1, 6, 6) and C(–4, 9, 6) are vertices of an isosceles right angled triangle.

(iii) Find the locus of a point such that the difference of the square of its distance from the points A(3, 4, 5) and B(–1, 3, –7) is equal to $2k^2$.

(iv) Find the co-ordinates of points which trisects the line joining the points A(–3, 2, 4) and B(0, 4, 7)

(v) Find the ratio in which the planes (a) xy (b) yz divide the line joining the points P(–2, 4, 7) and Q(3, –5, 8).

7. DIRECTION COSINES OF LINE :



If α, β, γ be the angles made by a line with x-axis, y-axis & z-axis respectively then cosα, cosβ & cosγ are called direction cosines of a line, denoted by ℓ, m & n respectively.

Note :

(i) If line makes angles α, β, γ with x, y & z axis respectively then π – α, π – β & π – γ is another set of angle that line makes with principle axes. Hence if ℓ, m & n are direction cosines of line then $-\ell$, –m & –n are also direction cosines of the same line.

(ii) Since parallel lines have same direction. So, in case of lines, which do not pass through the origin. We can draw a parallel line passing through the origin and direction cosines of that line can be found.

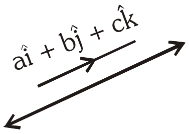
Important points :

(i) Direction cosines of a line :

Take a vector $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ parallel to a line whose D.C's are to be found out.

$\vec{A}.\hat{i} = a$

$|\vec{A}|\cos\alpha = a$



$\cos\alpha = \frac{a}{|\vec{A}|}$ similarly, $\cos\beta = \frac{b}{|\vec{A}|}$; $\cos\gamma = \frac{c}{|\vec{A}|}$

$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \Rightarrow \quad \ell^2 + m^2 + n^2 = 1$

(ii) Direction cosine of axes :

Since the positive x-axes makes angle 0°, 90°, 90° with axes of x, y and z respectively,

∴ D.C.'s of x axes are 1, 0, 0.

D.C.'s of y-axis are 0, 1, 0

D.C.'s of z-axis are 0, 0, 1

8. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line. i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

There can be infinitely many sets of direction ratios for a given line.

Direction ratios and Direction cosines of the line joining two points :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and the d.c.'s of AB are $\frac{1}{r}(x_2 - x_1)$, $\frac{1}{r}(y_2 - y_1)$, $\frac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]}$

9. RELATION BETWEEN D.C'S & D.R'S :

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2}$$

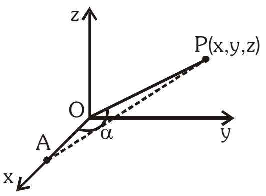
$$\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \; ; m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} \; ; n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

Important point :

Direction cosines of a line have two sets but direction ratios of a line have infinite possible sets.

7. PROJECTIONS :

(a) Projection of line segment OP on co-ordinate axes :



Let line segment make angle α with x-axis

Thus, the projections of line segment OP on axes are the absolute values of the co-ordinates of P. i.e.

Projection of OP on x-axis = $|x|$

Projection of OP on y-axis = $|y|$

Projection of OP on z-axis = $|z|$

Now, in ΔOAP, angle A is a right angle and OA = x

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|OP|}$$

if $|OP| = r$, then $x = |OP|\cos\alpha = \ell r$

Similarly $y = |OP|\cos\beta = mr$, $z = nr$, where ℓ, m, n are DC's of line

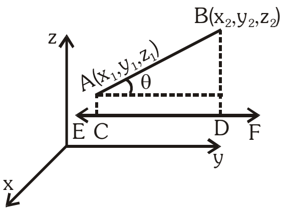
(b) Projection of a line segment AB on coordinate axes :

Projection of the point $A(x_1, y_1, z_1)$ on x-axis is $E(x_1, 0, 0)$. Projection of point $B(x_2, y_2, z_2)$ on x-axis is $F(x_2, 0, 0)$.

Hence projection of AB on x-axis is $EF = |x_2 - x_1|$.

Similarly, projection of AB on y and z-axis are $|y_2 - y_1|$, $|z_2 - z_1|$ respectively.

(c) Projection of line segment AB on a line having direction cosines ℓ, m, n :



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Now projection of AB on EF = CD = AB $\cos\theta$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \times \frac{|(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

$$= |(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|$$

Illustration 5 : A line OP makes with the x-axis an angle of measure 120° and with y-axis an angle of measure 60°. Find the angle made by the line with the z-axis.

Solution : $\alpha = 120°$ and $\beta = 60°$

$\therefore \cos\alpha = \cos 120° = -\frac{1}{2}$ and $\cos\beta = \cos 60° = \frac{1}{2}$ but $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \quad \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}}$$

$\therefore \gamma = 45°$ or $135°$ **Ans.**

Illustration 6 : Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Solution : The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

$$\therefore \ell = \frac{6}{7},\ m = \frac{2}{7},\ n = \frac{3}{7}$$

The required length of projection is given by

$$= |\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)| = \left|\frac{6}{7}[2-(-1)] + \frac{2}{7}(5-0) + \frac{3}{7}(1-3)\right|$$

$$= \left|\frac{6}{7}\times 3 + \frac{2}{7}\times 5 + \frac{3}{7}\times -2\right| = \left|\frac{18}{7} + \frac{10}{7} - \frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}$$ **Ans.**

Do yourself - 2 :

(i) Find the length of projections of the line segment joining the origin O to the point P(3, 2, –5) on the axes.

(ii) Find the length of projections of the line joining the points P(3, 2, 5) and Q(0, –2, 8) on the axes.

(iii) Find the direction ratios & direction cosines of the line joining the points O(0, 0, 0) and P(2, 3, 4).

11. ANGLE BETWEEN TWO LINES :

Let θ be the angle between the lines with d.c.'s ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$.

If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by

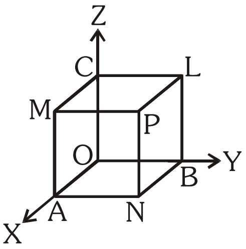
$$\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Illustration 7 : If a line makes angles α, β, γ, δ with four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ equals -

(A) 3 (B) 4 (C) 4/3 (D) 3/4

Solution : Let OA, OB, OC be coterminous edges of a cube and OA = OB = OC = a, then co-ordinates of its vertices are O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), L(0, a, a), M(a, 0, a), N(a, a, 0) and P (a, a, a)

Direction ratio of diagonal AL, BM, CN and OP are



$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

Let ℓ, m, n be the direction cosines of the given line, then

$$\cos\alpha = \ell\left(-\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{-\ell+m+n}{\sqrt{3}}$$

Similarly $\cos\beta = \dfrac{\ell-m+n}{\sqrt{3}}$, $\cos\gamma=\dfrac{\ell+m-n}{\sqrt{3}}$ and $\cos\delta=\dfrac{\ell+m+n}{\sqrt{3}}$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$ **Ans. (C)**

Illustration 8 : (a) Find the acute angle between two lines whose direction ratios are 2, 3, 6 and 1, 2, 2 respectively.

(b) Find the measure of the angle between the lines whose direction ratios are 1, –2, 7 and 3, –2, –1.

Solution : (a) $a_1 = 2, b_1 = 3, c_1 = 6; a_2 = 1, b_2 = 2, c_2 = 2.$

If θ be the angle between two lines whose d.r's are given, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{2\times1+3\times2+6\times2}{\sqrt{2^2+3^2+6^2}\sqrt{1^2+2^2+2^2}}=\frac{2+6+12}{7\times3}=\frac{20}{21}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{20}{21}\right)$$

(b) $\sqrt{1^2+(-2)^2+7^2}=\sqrt{54}$

$\sqrt{3^2+(-2)^2+(-1)^2}=\sqrt{14}$

$\therefore$ The actual direction cosines of the lines are

$$\frac{1}{\sqrt{54}},\frac{-2}{\sqrt{54}},\frac{7}{\sqrt{54}} \quad \text{and} \quad \frac{3}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{-1}{\sqrt{14}}$$

If θ is the angle between the lines, then

$$\cos\theta=\left(\frac{1}{\sqrt{54}}\right)\left(\frac{3}{\sqrt{14}}\right)+\left(\frac{-2}{\sqrt{54}}\right)\left(\frac{-2}{\sqrt{14}}\right)+\left(\frac{7}{\sqrt{54}}\right)\left(\frac{-1}{\sqrt{14}}\right)$$

$$=\frac{3+4-7}{\sqrt{54}.\sqrt{14}}=0 \Rightarrow \theta=90^\circ$$ **Ans.**

12. PERPENDICULAR AND PARALLEL LINES :

Let the two lines have their d.c.'s given by ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note: If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Illustration 9 : If the lines whose direction cosines satisfies the equations $a\ell + bm + cn = 0$ and $fmn + gn\ell + h\ell m = 0$ are perpendicular, then $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}$ equals -

(A) 0 (B) –1 (C) 1 (D) none of these

Solution : Eliminating n between the given relations, we find that $(fm + g\ell)\left(\frac{-a\ell - bm}{c}\right) + h\ell m = 0$

or $$ag\left(\frac{\ell}{m}\right)^2 + (af + bg - ch)\left(\frac{\ell}{m}\right) + bf = 0 \qquad \text{........(i)}$$

Let $\frac{\ell_1}{m_1}$ and $\frac{\ell_2}{m_2}$, are roots of (i), then $\frac{\ell_1}{m_1}\cdot\frac{\ell_2}{m_2} = \frac{bf}{ag}$

$$\Rightarrow \frac{\ell_1\ell_2}{f/a} = \frac{m_1m_2}{g/b} \qquad \text{........(ii)}$$

Similarly $$\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c} \qquad \text{........(iii)}$$

From (ii) and (iii), we get $\frac{\ell_1\ell_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c} = \lambda$

$\Rightarrow \ell_1\ell_2 = \lambda.f/a$; $m_1m_2 = \lambda.g/b$; $n_1n_2 = \lambda.h/c$

$$\Rightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = \lambda\left(\frac{f}{a}+\frac{g}{b}+\frac{h}{c}\right)$$

$\Rightarrow \frac{f}{a}+\frac{g}{b}+\frac{h}{c} = 0$ $\{\because \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0\}$ **Ans. (A)**

Do yourself - 3 :

(i) Find the angle between the lines whose direction ratios are 1, –2, 1 and 4, 3, 2.

(ii) If a line makes α, β and γ angle with axes, then prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to (1, –2, –2) & (0, 2, 1).

PLANE

13. DEFINITION :

A plane is a surface such that a line segment joining any two points on the surface lies wholly on it.

14. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form $ax + by + cz + d = 0$, in which a, b, c are constants, not all zero simultaneously.

(a) Equation of plane passing through a fixed point :

Vector form : If $\vec{a}$ is the position vector of a point on the plane and $\vec{n}$ is a vector normal to the plane then its vectorial equation is given by $(\vec{r}-\vec{a}).\vec{n} = 0 \Rightarrow \mathbf{\vec{r}.\vec{n} = d}$, where $d = \vec{a}.\vec{n} =$ constant.

Cartesian form : If $\vec{a}(x_1, y_1, z_1)$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, then cartesian equation of plane will be

$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$

(b) Plane Parallel to the Coordinate Planes :

(i) Equation of yz plane is x = 0.

(ii) Equation of zx plane is y = 0.

(iii) Equation of xy plane is z = 0.

(iv) Equation of the plane parallel to xy plane at a distance c is z = c or z = –c.

(v) Equation of the plane parallel to yz plane at a distance c is x = c or x = –c

(vi) Equation of the plane parallel to zx plane at a distance c is y = c or y = –c.

(c) Equations of Planes Parallel to the Axes :

If a = 0, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is **by + cz + d = 0**.

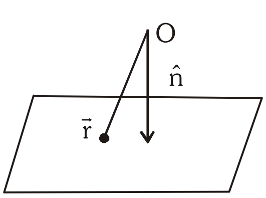
Similarly, equations of planes parallel to y-axis and parallel to z-axis are **ax + cz + d = 0** and **ax + by + d = 0**, respectively.

(d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes x, y, z respectively is

$$\mathbf{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}.$$

(e) Equation of a Plane in Normal Form :



Vector form : If $\hat{n}$ is a unit vector normal to the plane from the origin and d be the perpendicular distance of plane from origin then its vector equation is $\vec{\mathbf{r}}.\hat{\mathbf{n}} = \mathbf{d}$.

Cartesian form : If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (ℓ, m, n), then the equation of the plane is $\boldsymbol{\ell}\mathbf{x} + \mathbf{my} + \mathbf{nz} = \mathbf{p}$.

(f) Equation of a Plane through three points :

Vector form : If A, B, C are three points having P.V.'s $\vec{a}, \vec{b}, \vec{c}$ respectively, then vector equation of the plane is $[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{b}\ \vec{c}]+[\vec{r}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$.

Cartesian form : The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

Illustration 10 : Find the equation of the plane through the points A(2, 2, –1), B(3, 4, 2) and C(7, 0, 6).

Solution : The general equation of a plane passing through (2, 2, –1) is

$$a(x-2) + b(y-2) + c(z+1) = 0 \qquad \text{........(i)}$$

It will pass through B (3, 4, 2) and C (7, 0, 6) if

$$a(3-2) + b(4-2) + c(2+1) = 0 \quad \text{or} \quad a + 2b + 3c = 0 \qquad \text{........(ii)}$$

and $a(7-2) + b(0-2) + c(6+1) = 0$ or $5a - 2b + 7c = 0$ ....(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10} \text{ or } \frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda \quad \text{(say)}$$

$\Rightarrow$ $a = 5\lambda$, $b = 2\lambda$ and $c = -3\lambda$

Substituting the values of a, b and c in (i), we get

$$5\lambda(x-2) + 2\lambda(y-2) - 3\lambda(z+1) = 0$$

or $5(x-2) + 2(y-2) - 3(z+1) = 0$

$\Rightarrow$ $5x + 2y - 3z = 17$, which is the required equation of the plane **Ans.**

Illustration 11 : A plane meets the co-ordinates axes in A,B,C such that the centroid of the Δ ABC is the point (p,q,r) show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

Solution : Let the required equation of plane be :

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ......(i)$$

Then, the co-ordinates of A, B and C are A(a, 0, 0), B(0, b, 0), C(0, 0, c) respectively

So the centroid of the triangle ABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

But the co-ordinate of the centroid are (p,q,r)

$$\frac{a}{3}=p,\ \frac{b}{3}=q,\ \frac{c}{3}=r$$

Putting the values of a, b and c in (i), we get the required plane as $\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1$

$\Rightarrow \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$ **Ans.**

Do yourself - 4 :

(i) Equation of a plane is 3x + 4y + 5z = 7.

- (a) Find the direction cosines of its normal
- (b) Find the points where it intersects the axes.
- (c) Find its intercept form.
- (d) Find its equation in normal form (in cartesian as well as in vector form)

(ii) Find the equation of the plane passing through the points (2, 3, 1), (3, 0, 2) and (–1, 2, 3).

15. ANGLE BETWEEN TWO PLANES :

Vector form : If $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ be two planes, then angle between these planes is the angle between their normals

$$\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$\therefore$ Planes are perpendicular if $\vec{n}_1.\vec{n}_2=0$ and they are parallel if $\vec{n}_1=\lambda\vec{n}_2$.

Cartesian form : Consider two planes ax + by + cz + d = 0 and a' x + b' y + c' z + d' = 0. Angle between these planes is the angle between their normals.

$$\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$$

$\therefore$ Planes are perpendicular if aa' + bb' + cc' = 0 and they are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

Planes parallel to a given Plane :

Equation of a plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + d' = 0. d' is to be found by other given condition.

Illustration 12 : Find the angle between the planes x + y + 2z = 9 and 2x – y + z = 15

Solution : We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Therefore, angle between x + y + 2z = 9 and 2x – y + z = 15 is given by

$$\cos\theta = \frac{(1)(2)+(1)(-1)+(2)(1)}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+(-1)^2+1^2}} = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{3}$$ **Ans.**

Illustration13: Find the equation of the plane through the point (1, 4, –2) and parallel to the plane –2x + y – 3z = 7.

Solution : Let the equation of a plane parallel to the plane –2x + y – 3z = 7 be –2x + y – 3z + k = 0

This passes through (1, 4, –2), therefore (–2) (1) + 4 – 3 (–2) + k = 0

$\Rightarrow \quad -2 + 4 + 6 + k = 0 \quad \Rightarrow \quad k = -8$

Putting k = –8 in (i), we obtain –2x + y – 3z – 8 = 0 or –2x + y – 3z = 8 **Ans.**

This is the equation of the required plane.

Do yourself - 5 :

(i) Prove that the planes 3x – 2y + z + 17 = 0 and 4x + 3y – 6z – 25 = 0 are perpendicular.

(ii) Find the angle between the planes 3x + 4y + z + 7 = 0 and –x + y – 2z = 5

16. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes u ≡ ax + by + cz + d = 0 and v ≡ a' x + b' y + c' z + d' = 0.

The equation u + λv = 0, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

Illustration 14 : Find the equation of plane containing the line of intersection of the plane x + y + z – 6 = 0 and 2x + 3y + 4z + 5 = 0 and passing through (1,1,1).

Solution : The equation of the plane through the line of intersection of the given planes is,

$$(x + y + z - 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \qquad ......(i)$$

If it is passes through (1,1,1)

$$\Rightarrow \quad (1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0 \quad \Rightarrow \quad \lambda = \frac{3}{14}$$

Putting λ =3/14 in (i); we get $(x + y + z - 6) + \dfrac{3}{14}(2x + 3y + 4z + 5) = 0$

$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$ **Ans.**

17. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Vector form : If $\vec{r}\,.\,\vec{n} = d$ be the plane, then perpendicular distance p, of the point $A(\vec{a})$

$$p = \frac{|\vec{a}\,.\,\vec{n} - d|}{|\vec{n}|}$$

Distance between two parallel planes $\vec{r}\,.\,\vec{n} = d_1$ & $\vec{r}\,.\,\vec{n} = d_2$ is $\left|\frac{d_1 - d_2}{|\vec{n}|}\right|$.

Cartesian form : Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by $p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$

Distance between two parallel planes $ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

Illustration 15 : Find the perpendicular distance of the point (2, 1, 0) from the plane $2x + y + 2z + 5 = 0$

Solution : We know that the perpendicular distance of the point (x_1, y_1, z_1) from the plane

$ax + by + cz + d = 0$ is $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

so required distance = $\frac{|2 \times 2 + 1 \times 1 + 2 \times 0 + 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{10}{3}$ **Ans.**

Illustration 16: Find the distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$.

Solution : Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$, then $2x_1 - y_1 + 2z_1 + 3 = 0$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$ is

$$\left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right| = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \text{ [using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$ **Ans.**

Do yourself - 6 :

(i) Find the perpendicular distance of the point P(1, 2, 3) from the plane $2x + y + z + 1 = 0$.

(ii) Find the equation of the plane passing through the line of intersection of the planes $x + y + z = 5$ and $2x + 3y + z + 5 = 0$ and passing through the point (0, 0, 0).

18. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

(a) **Equation of bisector of the angle containing origin :** First make both constant terms positive. Then positive sign give the bisector of the angle which contains the origin.

(b) **Bisector of acute/obtuse angle :** First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \quad \Rightarrow \quad$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \quad \Rightarrow \quad$ origin lies in acute angle

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Illustration 17: Find the equation of the bisector planes of the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.

Solution : The two given planes are $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$

where $d_1, d_2 > 0$

and $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 > 0$

$$\therefore \quad \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{(acute angle bisector)}$$

$$\text{and} \quad \frac{a_1x + b_1y + c_1z + d}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{(obtuse angle bisector)}$$

$$\text{i.e.,} \quad \frac{2x - y + 2z + 3}{\sqrt{4 + 1 + 4}} = \pm\frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}}$$

$\Rightarrow \quad (14x - 7y + 14z + 21) = \pm (9x - 6y + 18z + 24)$

Taking positive sign on the right hand side,

we get $\quad 5x - y - 4z - 3 = 0 \quad$ (obtuse angle bisector)

and taking negative sign on the right hand side,

we get $\quad 23x - 13y + 32z + 45 = 0 \quad$ (acute angle bisector) **Ans.**

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19. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P & Q externally or internally according to P and Q lying on same or opposite sides of the plane.

Do yourself - 7 :

(i) Find the position of the point $P(2, -2, 1)$, $Q(3, 0, 1)$ and $R(-12, 1, 8)$ w.r.t. the plane $2x - 3y + 4z - 7 = 0$.

(ii) Two given planes are $-2x + y - 2z + 5 = 0$ and $6x - 2y + 3z - 7 = 0$. Find

(a) equation of plane bisecting the angle between the planes.

(b) equation of a plane parallel to the plane bisecting the angle between both the two planes and passing through the point (3, 2, 0).

(c) specify which plane is acute angle bisector and which one is obtuse angle bisector.

STRAIGHT LINE

20. DEFINITION :

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes : $a_1x + b_1y + c_1z + d_1 = 0$; $a_2x + b_2y + c_2z + d_2 = 0$

This form is also known as **unsymmetrical form**.

Some particular straight lines :

	Straight lines	Equation
(i)	Through the origin	$y = mx,\ z = nx$
(ii)	x-axis	$y = 0,\ z = 0$ or $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$
(iii)	y-axis	$x = 0,\ z = 0$ or $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$
(iv)	z-axis	$x = 0,\ y = 0$ or $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$
(v)	parallel to x-axis	$y = p,\ z = q$
(vi)	parallel to y-axis	$x = h,\ z = q$
(vii)	parallel to z-axis	$x = h,\ y = p$

21. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) **One point form :** Let $A(x_1, y_1, z_1)$ be a given point on the straight line and ℓ, m, n be the d.c's of the line, then its equation is

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \quad \text{(say)}$$

It should be noted that $P(x_1 + \ell r, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. $AP = r$. One should note that for $AP = r$; ℓ, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r \quad \text{but here } AP \neq r$$

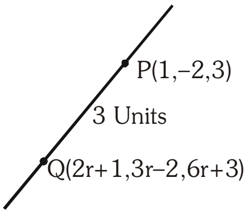
(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Illustration 18 : Find the co-ordinates of those points on the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{6}$ which is at a distance of 3 units from point (1,–2, 3).

Solution : Here, $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{6}$ ....(i)

is the given straight line



Let, P = (1,–2,3) on the straight line

Here direction ratios of line (i) are (2,3,6)

$\therefore$ Direction cosines of line (i) are : $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

$\Rightarrow$ Equations of line (i) can be written as

$$\frac{x-1}{2/7} = \frac{y+2}{3/7} = \frac{z-3}{6/7} \qquad ......(ii)$$

Co-ordinates of any point on the line (ii) can be taken as $\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Let, $Q\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

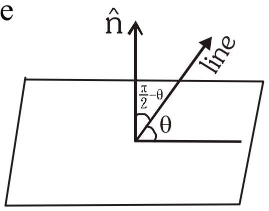
Given $|\vec{r}| = 3, \therefore r = \pm 3$

Putting the value of r, we have $Q\left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ or $Q = \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$ **Ans.**

---

22. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and ax + by + cz + d = 0 respectively and θ be the angle which line makes with the plane. Then (π/2 – θ) is the angle between the line and the normal to the plane.



So, $\sin\theta = \frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(\ell^2+m^2+n^2)}}$

Line is parallel to plane if θ = 0 i.e. if $a\ell + bm + cn = 0$.

Line is perpendicular to the plane if line is parallel to the normal of the plane i.e. if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$.

Illustration 19 : Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane 3x + 4y + z + 5 = 0.

Solution : The given line is $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ .... (i)

and the given plane is 3x + 4y + z + 5 = 0 .... (ii)

If the line (i) makes angle θ with the plane (ii), then the line (i) will make angle (90° – θ) with the normal to the plane (i). Now direction-ratios of line (i) are 3, –1, –2 and direction-ratios of normal to plane (ii) are 3, 4, 1

$$\therefore \quad \cos(90^\circ - \theta) = \frac{(3)(3)+(-1)(4)+(-2)(1)}{\sqrt{9+1+4}\sqrt{9+16+1}} \quad \Rightarrow \quad \sin\theta = \frac{9-4-2}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}\sqrt{26}}$$

Hence $\theta = \sin^{-1}\left(\frac{3}{\sqrt{14}\sqrt{26}}\right)$ **Ans.**

Do yourself - 8 :

(i) Find the equation of the line passing through the point (4, 2, 3) and having direction ratios 1, –1, 2

(ii) Find the symmetrical form of the line $x - y + 2z = 5$, $3x + y + z = 6$.

(iii) Find the angle between the plane $3x + 4y + 5 = 0$ and the line $\frac{x-1}{2} = \frac{y-2}{0} = \frac{z-1}{1}$.

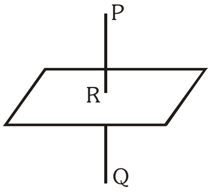
(iv) Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ is parallel to the plane $4x + 4y - 5z + 2 = 0$.

23. CONDITION THAT A LINE LIES ON THE GIVEN PLANE :

The line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ will lie on the plane $Ax + By + Cz + D = 0$ if

(a) $A\ell + Bm + Cn = 0$ and **(b)** $Ax_1 + By_1 + Cz_1 + D = 0$

24. IMAGE OF A POINT IN THE PLANE :



In order to find the image of a point $P(x_1, y_1, z_1)$ in a plane $ax + by + cz + d = 0$, assume it as a mirror. Let $Q(x_2, y_2, z_2)$ be the image of the point $P(x_1, y_1, z_1)$ in the plane, then

(a) Line PQ is perpendicular to the plane. Hence equation of PQ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$

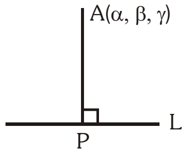
(b) Hence, Q satisfies the equation of line then $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = r$. The plane passes through the middle point of line PQ, therefore the middle point satisfies the equation of the plane i.e. $a\left(\frac{x_2+x_1}{2}\right) + b\left(\frac{y_2+y_1}{2}\right) + c\left(\frac{z_2+z_1}{2}\right) + d = 0$. The co-ordinates of Q can be obtained by solving these equations.

25. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE:

Let equation of the line be $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ ....(i)

and A (α, β, γ) be the point. Any point on the line (i) is $P(\ell r + x_1, mr + y_1, nr + z_1)$ ....(ii)

If it is the foot of the perpendicular, from A on the line, then AP is ⊥ to the line, so



$$\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)m + (\gamma - z_1)n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

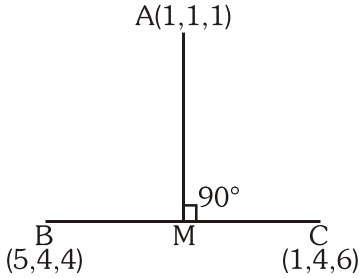
Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r + x_1 - \alpha} = \frac{y-\beta}{mr + y_1 - \beta} = \frac{z-\gamma}{nr + z_1 - \gamma}$$

Illustration 20 : Find the co-ordinates of the foot of the perpendicular from (1, 1, 1) on the line joining (5, 4, 4) and (1, 4, 6).

Solution : Let A (1, 1, 1), B (5, 4, 4) and C (1, 4, 6) be the given points. Let M be the foot of the perpendicular from A on BC.

If M divides BC in the ratio λ : 1, then



co-ordinates of M are $\left(\frac{\lambda+5}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$

Direction ratios of BC are 1 – 5, 4 – 4, 6 – 4

i.e. –4, 0, 2

D.R.'s of AM are $\frac{\lambda+5}{\lambda+1} - 1, \frac{4\lambda+4}{\lambda+1} - 1, \frac{6\lambda+4}{\lambda+1} - 1$

$\Rightarrow \quad \frac{4}{\lambda+1}, \frac{3\lambda+3}{\lambda+1}, \frac{5\lambda+3}{\lambda+1} \quad \Rightarrow \quad 4, 3\lambda + 3, 5\lambda + 3$

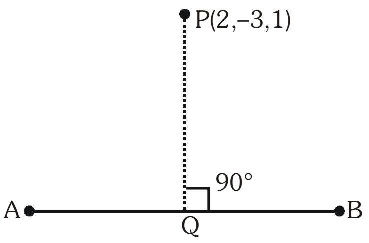
Since AM ⊥ BC

$\therefore \quad 2(4) + 0(3\lambda + 3) - 1(5\lambda + 3) = 0 \quad \Rightarrow 8 - 5\lambda - 3 = 0 \quad \Rightarrow \quad \lambda = 1$

Hence the co-ordinates of M are (3, 4, 5) **Ans.**

Illustration 21 : Find the length of perpendicular from P(2, – 3,1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

Solution : Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ ....(i)



and P(2, – 3,1)

Co-ordinates of any point on (i) may be taken as

(2r–1,3r+3,–r–2)

Let Q = (2r –1, 3r + 3,–r – 2)

Direction ratio's of PQ are : (2r –3, 3r +6, –r –3)

Direction ratio's of AB are : (2, 3, –1)

Since, $PQ \perp AB$

$$2(2r-3)+3(3r+6)-1(-r-3)=0$$

$$\Rightarrow \quad r=-\frac{15}{14}$$

$$\therefore \quad Q=\left(-\frac{22}{7},-\frac{3}{14},-\frac{13}{14}\right)$$

$$PQ^2=\left(2+\frac{22}{7}\right)^2+\left(-3+\frac{3}{14}\right)^2+\left(1+\frac{13}{14}\right)^2=\frac{531}{14}$$

$$PQ=\sqrt{\frac{531}{14}} \text{ units}$$ **Ans.**

Do yourself - 9 :

(i) Find the image of point P(1, 3, 2) in the plane $2x-y+z+3=0$ as well as the foot of the perpendicular drawn from the point (1, 3, 2).

(ii) Find the distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$

(iii) Prove that $\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4}$ lies in the plane $x+2y-z=0$.

26. EQUATION OF PLANE CONTAINING TWO INTERSECTING LINES :

Let the two lines be

$$\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad ....(i)$$

and

$$\frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \quad ....(ii)$$

These lines will coplanar if $\begin{vmatrix}\alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1\\ \ell_1 & m_1 & n_1\\ \ell_2 & m_2 & n_2\end{vmatrix}=0$ (It is condition for intersection of two lines)

the plane containing the two lines is $\begin{vmatrix}x-\alpha_1 & y-\beta_1 & z-\gamma_1\\ \ell_1 & m_1 & n_1\\ \ell_2 & m_2 & n_2\end{vmatrix}=0$

Illustration 22 : Find the equation of the plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ and parallel to the line $\frac{x-4}{2}=\frac{y-1}{-3}=\frac{z+3}{5}$.

Solution : Any plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ is

$$a(x-1)+b(y+6)+c(z+1)=0 \quad \text{....... (i)}$$

where, $3a+4b+2c=0$ .... (ii)

Also, it is parallel to the second line and hence, its normal is perpendicular to this line

$\therefore \quad 2a-3b+5c=0$ .... (iii)

Solving (ii) & (iii) by cross multiplication, we get $\frac{a}{26}=\frac{b}{-11}=\frac{c}{-17}=k$

$\Rightarrow \quad a=26k,\ b=-11k\ \&\ c=-17k$

Putting these values in (i), we get $26k(x-1)-11k(y+6)-17k(z+1)=0$

$\Rightarrow \quad 26x-11y-17z=109$, which is the required equation of the plane.

27. LINE OF GREATEST SLOPE :

Consider two planes G-plane and H-plane. H-plane is treated as a horizontal plane or reference plane. G-plane is a given plane. Let AB be the line of intersection of G-plane & H-plane. Line of greatest slope is a line which is contained by G-plane & perpendicular to line of intersection of G-plane & H-plane. Obviously, infinitely many such lines of greatest slopes are contained by G-plane. Generally an additional information is given in problem so that a unique line of greatest slope can be found out.

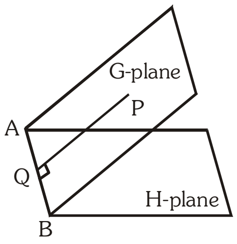


Illustration 23 : Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equation of the line of greatest slope on the plane $2x+y-5z=0$, passing through the point (2,1,1).

Solution : The required line passing through the point P (2,1,1) in the plane $2x+y-5z=0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$2x+y-5z=0$ ......(i)

and $4x-3y+7z=0$ ......(ii)

Let the d.r'.s of the line of intersection of (i) and (ii) be a, b, c

$\Rightarrow \quad 2a+b-5c=0$ and $4a-3b+7c=0$

{as dr'.s of straight line (a, b,c) is perpendicular to d.r'.s of normal to both the planes}

$\Rightarrow \quad \frac{a}{4}=\frac{b}{17}=\frac{c}{5}$

Now let the direction ratio of required line be proportional to ℓ, m, n then its equation be $\frac{x-2}{\ell}=\frac{y-1}{m}=\frac{z-1}{n}$

where $2\ell + m - 5n = 0$ and $4\ell + 17m + 5n = 0$

so, $\frac{\ell}{3} = \frac{m}{-1} = \frac{n}{1}$

Thus the required line is $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$ **Ans.**

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28. AREA OF TRIANGLE :

To find the area of a triangle in terms of its projections on the co-ordinates planes.

Let Δ_x, Δ_y, Δ_z be the projections of the plane area of the triangle on the planes yOz, zOx, xOy respectively.Let ℓ, m, n be the direction cosines of the normal to the plane of the triangle.

Then the angle between the plane of the triangle and yOz plane is the angle between the normal to the plane of the triangle and the x-axis.

$\therefore \quad \Delta_x = \Delta\ell$

Similarly $\Delta_y = \Delta m$; $\Delta_z = \Delta n \Rightarrow \Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the three vertices of the triangle then

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Do yourself - 10 :

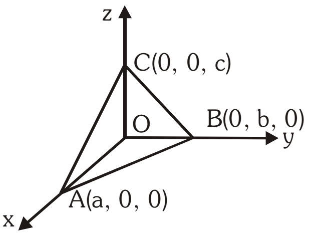
(i) Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ are coplanar. Find their point of intersection.

(ii) Find the area of the triangle whose vertices are the points (1, 2, 3), (–2, 1, –4), (3, 4, –2).

Miscellaneous Illustrations :

Illustration 24: If a variable plane cuts the coordinate axes in A, B and C and is at a constant distance p from the origin, find the locus of the centre of the tetrahedron OABC.

Solution : Let $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$ and $C \equiv (0, 0, c)$



$\therefore$ Equation of plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Now p = length of perpendicular from O to this plane

$$= \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \quad \text{or} \quad p^2 = \frac{1}{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2} \quad ...(i)$$

Let G(α, β, γ) be the centre of the tetrahedron OABC, then

$$\alpha = \frac{a}{4},\ \beta = \frac{b}{4},\ \gamma = \frac{c}{4} \qquad \left[\because\ \alpha = \frac{a+0+0+0}{4} = \frac{a}{4}\right]$$

or, $a = 4\alpha,\ b = 4\beta,\ c = 4\gamma$

Putting these values of a, b, c in equation (i), we get

$$p^2 = \frac{16}{\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)} \quad \text{or} \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2}$$

∴ locus of (α, β, γ) is $x^{-2} + y^{-2} + z^{-2} = 16\,p^{-2}$ **Ans.**

Illustration 25 : Through a point P(h, k, ℓ) a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\frac{p^5}{2|hk\ell|}$.

Solution : $OP = \sqrt{h^2+k^2+\ell^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$

$$\therefore \quad A \equiv \left(\frac{p^2}{h},\ 0,\ 0\right),\ B \equiv \left(0,\ \frac{p^2}{k},\ 0\right),\ C \equiv \left(0,\ 0,\ \frac{p^2}{\ell}\right)$$

Now area of ΔABC, $\Delta^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \frac{1}{2}\frac{p^4}{|k\ell|}$ and $A_{zx} = \frac{1}{2}\frac{p^4}{|\ell h|}$

$$\therefore \quad \Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2\ell^2} + \frac{1}{4}\frac{p^8}{h^2\ell^2} = \frac{p^{10}}{4h^2k^2\ell^2}$$

or $\Delta = \frac{p^5}{2|hk\ell|}$ **Ans.**

Illustration 26: Find the locus of a point, the sum of squares of whose distances from the planes : $x - z = 0$, $x - 2y + z = 0$ and $x + y + z = 0$ is 36

Solution : Given planes are $x - z = 0$, $x - 2y + z = 0$ and, $x + y + z = 0$

Let the point whose locus is required be $P(\alpha,\beta,\gamma)$. According to question

$$\frac{|\alpha-\gamma|^2}{2}+\frac{|\alpha-2\beta+\gamma|^2}{6}+\frac{|\alpha+\beta+\gamma|^2}{3}=36$$

or $3(\alpha^2+\gamma^2-2\alpha\gamma)+\alpha^2+4\beta^2+\gamma^2-4\alpha\beta-4\beta\gamma+2\alpha\gamma+2(\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma)=36\times 6$

or $6\alpha^2+6\beta^2+6\gamma^2=36\times 6$

or $\alpha^2+\beta^2+\gamma^2=36$

Hence, the required equation of locus is $x^2+y^2+z^2=36$ **Ans.**

Illustration 27 : Direction ratios of normal to the plane which passes through the point (1, 0, 0) and (0, 1, 0) which makes angle $\pi/4$ with $x + y = 3$ are -

(A) 1, 1, 2 (B) $\sqrt{2}$, 1, 1 (C) 1, $\sqrt{2}$, 1 (D) 1, 1, $\sqrt{2}$

Solution : The plane by intercept form is $\frac{x}{1}+\frac{y}{1}+\frac{z}{c}=1$

direction ratios of normal of this plane are $1, 1, \frac{1}{c}$ and that of given plane are 1, 1, 0.

$$\therefore \quad \cos\frac{\pi}{4}=\frac{1.1+1.1+0\cdot\frac{1}{c}}{\sqrt{1+1+\frac{1}{c^2}}\sqrt{1+1+0}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2+\frac{1}{c^2}}\sqrt{2}} \Rightarrow 2+\frac{1}{c^2}=4 \Rightarrow \quad c=\pm\frac{1}{\sqrt{2}}$$

$\therefore$ d.r.'s are 1, 1, $\sqrt{2}$ **Ans. (D)**

ANSWERS FOR DO YOURSELF

1: **(i)** $2\sqrt{21}$ **(iii)** $8x + 2y + 24z \pm 2k^2 + 9 = 0$ **(iv)** $\left(-2, \frac{8}{3}, 5\right)$ & $\left(-1, \frac{10}{3}, 6\right)$

(v) **(a)** 7 : 8, externally **(b)** 2 : 3 internally

2 : **(i)** 3, 2, 5 **(ii)** 3, 4, 3 **(iii)** 2, 3, 4 & $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$

3 : **(i)** $\theta = \frac{\pi}{2}$ **(iii)** $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

4 : **(i)** **(a)** $\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$ **(b)** $\left(\frac{7}{3}, 0, 0\right), \left(0, \frac{7}{4}, 0\right)$ & $\left(0, 0, \frac{7}{5}\right)$ **(c)** $\frac{x}{7/3} + \frac{y}{7/4} + \frac{z}{7/5} = 1$

(d) $\frac{3x}{5\sqrt{2}} + \frac{4y}{5\sqrt{2}} + \frac{z}{\sqrt{2}} = \frac{7}{5\sqrt{2}}$ & $\vec{r}.\left(\frac{3}{5\sqrt{2}}\hat{i} + \frac{4}{5\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = \frac{7}{5\sqrt{2}}$

(ii) $x + y + 2z = 7$

5: **(ii)** $\theta = \cos^{-1}\left(\frac{1}{\sqrt{156}}\right)$

6 : **(i)** $\frac{8}{\sqrt{6}}$ **(ii)** $3x + 4y + 2z = 0$

7 : **(i)** P, Q same side & R opposite side

(ii) **(a)** $4x + y - 5z + 14 = 0$ & $32x - 13y + 23z - 56 = 0$

(b) $4x + y - 5z - 14 = 0$ & $32x - 13y + 23z - 70 = 0$

(c) $4x + y - 5z + 14 = 0$ (obtuse angle bisector) & $32x - 13y + 23z - 56 = 0$ (acute angle bisector)

8 : **(i)** $\frac{x-4}{1} = \frac{y-2}{-1} = \frac{z-3}{2}$ **(ii)** $\frac{x-11/4}{-3} = \frac{y+9/4}{5} = \frac{z-0}{4}$ **(iii)** $\theta = \sin^{-1}\left(\frac{6}{5\sqrt{5}}\right)$

9 : **(i)** $\left(\frac{-5}{3}, \frac{13}{3}, \frac{2}{3}\right)$ & $\left(\frac{-1}{3}, \frac{11}{3}, \frac{4}{3}\right)$ **(ii)** 1

10: **(i)** $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ **(ii)** $\frac{\sqrt{1218}}{2}$

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. Consider three vectors $\vec{p}=\hat{i}+\hat{j}+\hat{k}$, $\vec{q}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{r}=\hat{i}+\hat{j}+3\hat{k}$. If $\vec{p}$, $\vec{q}$ and $\vec{r}$ denotes the position vector of three non-collinear points then the equation of the plane containing these points is **TD0001**

(A) $2x-3y+1=0$ (B) $x-3y+2z=0$ (C) $3x-y+z-3=0$ (D) $3x-y-2=0$

2. If the distance between the planes

$$8x+12y-14z=2$$

and

$$4x+6y-7z=2$$

can be expressed in the form $\frac{1}{\sqrt{N}}$ where N is natural then the value of $\frac{N(N+1)}{2}$ is **TD0003**

(A) 4950 (B) 5050 (C) 5150 (D) 5151

3. A plane passes through the point P(4, 0, 0) and Q(0, 0, 4) and is parallel to the y-axis. The distance of the plane from the origin is **TD0004**

(A) 2 (B) 4 (C) $\sqrt{2}$ (D) $2\sqrt{2}$

4. If the plane $2x-3y+6z-11=0$ makes an angle $\sin^{-1}(k)$ with x-axis, then k is equal to **TD0006**

(A) $\sqrt{3}/2$ (B) 2/7 (C) $\sqrt{2}/3$ (D) 1

5. The plane XOZ divides the join of (1, –1, 5) and (2, 3, 4) in the ratio $\lambda : 1$, then λ is **TD0007**

(A) – 3 (B) – 1/3 (C) 3 (D) 1/3

6. Equation of the line which passes through the point with p. v. (2, 1, 0) and perpendicular to the plane containing the vectors $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is **TD0010**

(A) $\vec{r}=(2,1,0)+t(1,-1,1)$ (B) $\vec{r}=(2,1,0)+t(-1,1,1)$

(C) $\vec{r}=(2,1,0)+t(1,1,-1)$ (D) $\vec{r}=(2,1,0)+t(1,1,1)$

where t is a parameter

7. Which of the following planes are parallel but not identical?

$P_1 : 4x-2y+6z=3$
$P_2 : 4x-2y-2z=6$
$P_3 : -6x+3y-9z=5$
$P_4 : 2x-y-z=3$

(A) P_2 & P_3 (B) P_2 & P_4 (C) P_1 & P_3 (D) P_1 & P_4 **TD0011**

8. Vector equation of the plane $\vec{r}=\hat{i}-\hat{j}+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k})$ in the scalar dot product form is **TD0013**

(A) $\vec{r}.(5\hat{i}-2\hat{j}+3\hat{k})=7$ (B) $\vec{r}.(5\hat{i}+2\hat{j}-3\hat{k})=7$

(C) $\vec{r}.(5\hat{i}-2\hat{j}-3\hat{k})=7$ (D) $\vec{r}.(5\hat{i}+2\hat{j}+3\hat{k})=7$

9. The vector equations of the two lines L_1 and L_2 are given by

$L_1 : \vec{r}=2\hat{i}+9\hat{j}+13\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k})$; $L_2 : \vec{r}=-3\hat{i}+7\hat{j}+p\hat{k}+\mu(-\hat{i}+2\hat{j}-3\hat{k})$

then the lines L_1 and L_2 are

(A) skew lines for all $p\in R$

(B) intersecting for all $p\in R$ and the point of intersection is (–1, 3, 4) **TD0014**

(C) intersecting lines for $p=-2$

(D) intersecting for all real $p\in R$

10. Consider the plane $(x, y, z) = (0, 1, 1) + \lambda(1, -1, 1) + \mu(2, -1, 0)$. The distance of this plane from the origin is

(A) 1/3 (B) $\sqrt{3}/2$ (C) $\sqrt{3/2}$ (D) $2/\sqrt{3}$ **TD0015**

11. The value of 'a' for which the lines $\frac{x-2}{1}=\frac{y-9}{2}=\frac{z-13}{3}$ and $\frac{x-a}{-1}=\frac{y-7}{2}=\frac{z+2}{-3}$ intersect, is

(A) – 5 (B) – 2 (C) 5 (D) – 3 **TD0016**

12. Given A(1, –1, 0) ; B(3, 1, 2) ; C(2, –2, 4) and D(–1, 1, –1) which of the following points neither lie on AB nor on CD? **TD0017**

(A) (2, 2, 4) (B) (2, –2, 4) (C) (2, 0,1) (D) (0, –2, –1)

13. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect?

(A) it lies in the plane $x - 2y + z = 0$ (B) it is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ **TD0018**

(C) it passes through (2, 3, 5) (D) it is parallel to the plane $x - 2y + z - 6 = 0$

14. Given planes

$P_1 : cy + bz = x$

$P_2 : az + cx = y$

$P_3 : bx + ay = z$

P_1, P_2 and P_3 pass through one line, if **TD0019**

(A) $a^2 + b^2 + c^2 = ab + bc + ca$ (B) $a^2 + b^2 + c^2 + 2abc = 1$

(C) $a^2 + b^2 + c^2 = 1$ (D) $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca + 2abc = 1$

15. The line $\frac{x-x_1}{0}=\frac{y-y_1}{1}=\frac{z-z_1}{2}$ is **TD0020**

(A) parallel to x-axis (B) perpendicular to x-axis

(C) perpendicular to YOZ plane (D) parallel to y-axis

16. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if **TD0021**

(A) k = 0 or – 1 (B) k = 1 or – 1 (C) k = 0 or – 3 (D) k = 3 or – 3

17. The line which contains all points (x, y, z) which are of the form $(x, y, z) = (2, -2, 5) + \lambda(1, -3, 2)$ intersects the plane $2x - 3y + 4z = 163$ at P and intersects the YZ plane at Q. If the distance PQ is $a\sqrt{b}$ where a, b $\in$ N and a > 3 then (a + b) equals

(A) 23 (B) 95 (C) 27 (D) none **TD0022**

18. Let L_1 be the line $\vec{r}_1 = 2\hat{i}+\hat{j}-\hat{k}+\lambda(\hat{i}+2\hat{k})$ and let L_2 be the line $\vec{r}_2 = 3\hat{i}+\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$.

Let Π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane Π from the origin is

(A) 1/7 (B) $\sqrt{2/7}$ (C) $\sqrt{6}$ (D) none **TD0023**

19. The distance of the point (–1, –5, – 10) from the point of intersection of the line $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 5$ is **TD0025**

(A) $2\sqrt{11}$ (B) $\sqrt{126}$ (C) 13 (D) 14

[MATRIX MATCH TYPE]

20. Consider the following four pairs of lines in **column-I** and match them with one or more entries in **column-II**.

	Column-I		**Column-II**	
(A)	$L_1 : x = 1 + t,\ y = t,\ z = 2 - 5t$ $L_2 : \vec{r} = (2,1,-3) + \lambda(2, 2, -10)$	(P)	non coplanar lines	**TD0027**
(B)	$L_1 : \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}$ $L_2 : \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{3}$	(Q)	lines lie in a unique plane	**TD0028**
(C)	$L_1 : x = -6t,\ y = 1 + 9t,\ z = -3t$ $L_2 : x = 1 + 2s,\ y = 4 - 3s,\ z = s$	(R)	infinite planes containing both the lines	**TD0029**
(D)	$L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ $L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$	(S)	lines are not intersecting at a unique point	**TD0030**

EXERCISE (O-2)

[MULTIPLE OBJECTIVE TYPE]

1. Consider Lines $L_1 : \frac{x-\alpha}{1} = \frac{y}{-2} = \frac{z+\beta}{2}$, $L_2 : x = \alpha, \frac{y}{-\alpha} = \frac{z+\alpha}{2-\beta}$, plane $P : 2x + 2y + z + 7 = 0$. Let line L_2 lies in plane P, then

(A) $\alpha = -7$

(B) $\alpha = 7$

(C) minimum distance between line L_1 and plane P is 11.

(D) minimum distance between line L_1 and plane P is $\frac{23}{3}$ **TD0032**

2. The point $\vec{A}\,(3,4,7)$, $\vec{B}\,(4,5,9)$ and $\vec{C}\,(1,2,-1)$ are three vertices of a parallelogram ABCD, then -

(A) vector equation of line AB is $\vec{r} = 4\hat{i} + 5\hat{j} + 9\hat{k} + \lambda\left(\hat{i} + \hat{j} + 2\hat{k}\right)$

(B) cartesian equation of line BC is $\frac{x-4}{3} = \frac{x-5}{3} = \frac{z-9}{10}$

(C) coordinates of D are (0,1,–3)

(D) ABCD is rectangle **TD0033**

3. Let two planes $P_1 : 2x - y + z - 2 = 0$ and $P_2 : x + 2y - z - 3 = 0$ are given then-

(A) The equation of the plane through line of intersection of $P_1 = 0$ and $P_2 = 0$ and the point (3,2,1) is $x - 3y + 2z + 1 = 0$

(B) The equation of the plane through line of intersection of $P_1 = 0$ and $P_2 = 0$ and the point (3,2,1) is $3x - y + 2z - 9 = 0$

(C) The equation of acute angle bisector plane of $P_1 = 0$ and $P_2 = 0$ is $x - 3y + 2z + 1 = 0$

(D) The equation of acute angle bisector plane of $P_1 = 0$ and $P_2 = 0$ is $x + 3y + 2z + 2 = 0$ **TD0034**

4. A line segment has length 6 and direction ratios are –3,4,6, then the component of the line vector are-

TD0037

(A) $\frac{-18}{\sqrt{61}},\frac{24}{\sqrt{61}},\frac{36}{\sqrt{61}}$ (B) 27,–18,–54 (C) 27,–18,54 (D) $\frac{18}{\sqrt{61}},\frac{-24}{\sqrt{61}},\frac{-36}{\sqrt{61}}$

5. Which of the following is (are) correct -

(A) If two lines in space are not intersecting, then they must be skew lines.

(B) If two lines are parallel to a plane 'P', then their direction ratios will be proportional

(C) If two lines are perpendicular to a plane 'P', then their direction ratios will be proportional

(D) Equation $\frac{x+1}{a}=\frac{y-1}{b}=\frac{z}{c}$, where a, b, c are real parameters, represents a family of concurrent lines in space

TD0038

6. Given the equations of the line $3x-y+z+1=0,\ 5x+y+3z=0$.

Then which of the following is correct ?

(A) Symmetrical form of the equations of line is $\frac{x}{2}=\frac{y-\frac{1}{8}}{-1}=\frac{z+\frac{5}{8}}{1}$

TD0039

(B) symmetrical form of the equations of line is $\frac{x+\frac{1}{8}}{1}=\frac{y-\frac{5}{8}}{1}=\frac{z}{-2}$

(C) equation of the plane through (2, 1, 4) and prependicular to the given lines is $2x-y+z-7=0$

(D) equation of the plane through (2, 1, 4) and prependicular to the given lines is $x+y-2z+5=0$

7. A line L passing through the point P(1,4,3), is perpendicular to both the lines

$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4}$ and $\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$.

TD0042

If the position vector of point Q on L is(a_1,a_2,a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be-

(A) 16 (B) 15 (C) 2 (D) 1

[MATRIX MATCH TYPE]

8. P(0, 3, – 2); Q(3, 7, – 1) and R(1, – 3, – 1) are 3 given points. Let L_1 be the line passing through P and Q and L_2 be the line through R and parallel to the vector $\vec{V}=\hat{i}+\hat{k}$.

	Column-I		**Column-II**
(A)	perpendicular distance of P from L_2	(P)	$7\sqrt{3}$
(B)	shortest distance between L_1 and L_2	(Q)	2
(C)	area of the triangle PQR	(R)	6
(D)	distance from (0, 0, 0) to the plane PQR	(S)	$\frac{19}{\sqrt{147}}$

TD0043

EXERCISE (S-1)

1. Find the angle between the two straight lines whose direction cosines ℓ, m, n are given by $2\ell + 2m - n = 0$ and $mn + n\ell + \ell m = 0$. **TD0044**

2. The plane denoted by $\Pi_1 : 4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $\Pi_2 : 5x + 3y + 10z = 25$. If the plane in its new position be denoted by Π, and the distance of this plane from the origin is $\sqrt{k}$ where $k \in N$, then find k. **TD0045**

3. Find the equations of the straight line passing through the point (1,2,3) to intersect the straight line $x + 1 = 2(y - 2) = z + 4$ and parallel to the plane $x + 5y + 4z = 0$. **TD0046**

4. Find the value of p so that the lines $\frac{x+1}{-3} = \frac{y-p}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are in the same plane. for this value of p, find the coordinates of their point of intersection and the equation of the plane containing them. **TD0048**

5. Let L be the line given by $\vec{r} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and let P be the point (2,–1,1). Also suppose that E be the plane containing three non collinear points A(0,1,1) ; B(1,2,2) and C(1,0,1).

   Find

   (a) Distance between the point P and the line L.

   (b) Equation of the plane E.

   (c) Equation the plane F containing the line L and the point P.

   (d) Acute angle between the plane E and F.

   (e) Volume of the parallelopiped by A,B,C and the point D(–3,0,1). **TD0050**

6. Find the equation of the line which is reflection of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane $3x - 3y + 10z = 26$. **TD0053**

7. Find the equation of the plane containing the line $\frac{x-1}{2} = \frac{y}{3} = \frac{z}{2}$ and parallel to the line $\frac{x-3}{2} = \frac{y}{5} = \frac{z-2}{4}$.

   Find also the S.D. between the two lines. **TD0054**

8. The equation of the plane which has the property that the point Q(5,4,5) is the reflection of point P(1,2,3) through that plane, is $ax + by + cz = d$ where $a,b,c,d \in N$. Find the least value of $(a + b + c + d)$. **TD0056**

9. Find the point where the line of intersection of the planes $x - 2y + z = 1$ and $x + 2y - 2z = 5$, intersects the plane $2x + 2y + z + 6 = 0$. **TD0058**

EXERCISE (S-2)

1. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at an angle of $\frac{\pi}{3}$. **TD0061**

2. The line $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Find the equation of the remaining sides. **TD0064**

3. Planes P_1', P_2',P_3' are drawn parallel to the planes $P_1 : x + y + z = 3$, $P_2 : x - y + z = 1$ & $P_3 : x + y - z = 2$ respectively from the point (2,2,3). If d_1,d_2,d_3 are distances of P_1',P_2',P_3' from (1,1,2) respectively then $\left(d_1^2+\frac{1}{d_2^2}+\frac{1}{d_3^2}\right)$ is equal to **TD0068**

4. Let the equation of the plane containing the line $x-y-z-4=0=x+y+2z-4$ and is parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$ Compute the value of $|A+B+C|$. **TD0128**

5. Consider the plane

$$E : \vec{r}=\begin{bmatrix}-1\\1\\1\end{bmatrix}+\lambda\begin{bmatrix}1\\2\\0\end{bmatrix}+\mu\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Let F be the plane containing the point A(–4,2,2) and parallel to E.

Suppose the point B is on the plane E such that B has a minimum distance from the point A. If C(–3,0,4) lies in the plane F. Find the area of the triangle ABC. **TD0129**

6. Find the equation of the line passing through the point (4,–14,4) and intersecting the line of intersection of the planes : $3x+2y-z=5$ and $x-2y-2z=-1$ at right angles. **TD0130**

7. Feet of the perpendicular drawn from the point P(2,3,–5) on the axes of coordinates are A,B and C. Find the equation of the plane passing through their feet and the area of ΔABC. **TD0131**

8. Find the equation of the plane containing the straight line $\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z}{5}$ and perpendicular to the plane $x-y+z+2=0$. **TD0132**

EXERCISE (JM)

1. Let the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lie in the plane $x+3y-\alpha z+\beta=0$. Then (α, β) equals **[AIEEE-2009]**

(1) $(5, -15)$ (2) $(-5, 5)$ (3) $(6, -17)$ (4) $(-6, 7)$ **TD0133**

2. The projections of a vector on the three coordinate axis are 6, –3, 2 respectively. The direction cosines of the vector are :- **[AIEEE-2009]**

(1) $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$ (2) $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}$ (3) 6, –3, 2 (4) $\frac{6}{5}, \frac{-3}{5}, \frac{2}{5}$ **TD0134**

3. **Statement–1 :** The point A(3, 1, 6) is themirror image of the point B(1, 3, 4) in the plane $x-y+z=5$. **[AIEEE-2010]**

Statement–2 : The plane $x-y+z=5$ bisects the line segment joining A(3, 1, 6) and B(1, 3, 4).

(1) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false. **TD0135**

(4) Statement–1 is false, Statement–2 is true.

4. If the angle between the line $x=\frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x+2y+3z=4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals:- **[AIEEE-2011]**

(1) $\frac{2}{5}$ (2) $\frac{5}{3}$ (3) $\frac{2}{3}$ (4) $\frac{3}{2}$ **TD0136**

5. **Statement-1 :** The point A(1, 0, 7) is the mirror image of the point B(1, 6, 3) in the line : $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Statement-2 : The line : $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line segment joining A (1, 0, 7) and B(1, 6, 3). **[AIEEE-2011]**

(1) Statement-1 is true, Statement-2 is false. **TD0137**

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

6. The distance of the point (1, –5, 9) from the plane $x-y+z=5$ measured along a straight line $x=y=z$ is : **[AIEEE-2011]**

(1) $3\sqrt{5}$ (2) $10\sqrt{3}$ (3) $5\sqrt{3}$ (4) $3\sqrt{10}$ **TD0138**

7. An equation of a plane parallel to the plane $x-2y+2z-5=0$ and at a unit distance from the origin is : **[AIEEE-2012]**

(1) $x-2y+2z+5=0$ (2) $x-2y+2z-3=0$ **TD0073**

(3) $x-2y+2z+1=0$ (4) $x-2y+2z-1=0$

8. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to :

[AIEEE-2012]

(1) 0 (2) – 1 (3) $\frac{2}{9}$ (4) $\frac{9}{2}$ **TD0074**

9. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is :-

[JEE-MAIN 2013]

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) $\frac{9}{2}$ **TD0075**

10. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, then k can have :

[JEE-MAIN 2013]

(1) any value (2) exactly one value **TD0076**

(3) exactly two values (4) exactly three values.

11. A vector $\vec{n}$ is inclined to x-axis at 45°, to y–axis at 60° and at an acute angle to z-axis. If $\vec{n}$ is a normal to a plane passing through the point $(\sqrt{2}, -1, 1)$, then the equation of the plane is :

[JEE-MAIN Online 2013]

(1) $\sqrt{2}x-y-z=2$ (2) $\sqrt{2}x+y+z=2$ **TD0077**

(3) $3\sqrt{2}x-4y-3z=7$ (4) $4\sqrt{2}x+7y+z=2$

12. The acute angle between two lines such that the direction cosines ℓ, m, n of each of them satisfy the equations $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$ is :- **[JEE-MAIN Online 2013]**

(1) 30° (2) 45° (3) 60° (4) 15° **TD0078**

13. Let Q be the foot of perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and R be a point $(-1, 1, -6)$ on the plane. Then length QR is :- **[JEE-MAIN Online 2013]**

(1) $3\sqrt{\frac{7}{2}}$ (2) $\sqrt{14}$ (3) $\sqrt{\frac{19}{2}}$ (4) $\frac{3}{\sqrt{2}}$ **TD0079**

14. If the projections of a line segment on thex, y and z-axes in 3-dimensional space are 2, 3 and 6 respectively, then the length ofthe line segment is : **[JEE-MAIN Online 2013]**

(1) 7 (2) 9 (3) 12 (4) 6 **TD0080**

15. If two lines L_1 and L_2 in space, are definedby **[JEE-MAIN Online 2013]**

$L_1 = \{x = \sqrt{\lambda}\,y + (\sqrt{\lambda} - 1)$

$z = (\sqrt{\lambda} - 1)\,y + \sqrt{\lambda}\}$ and

$L_2 = \{x = \sqrt{\mu}\,y + (1 - \sqrt{\mu})$ **TD0081**

$z = (1 - \sqrt{\mu})\,y + \sqrt{\mu}\}$, then L_1 is perpendicular to L_2, for all non-negative reals λ and μ, such that :

(1) $\lambda = \mu$ (2) $\lambda \neq \mu$ (3) $\sqrt{\lambda} + \sqrt{\mu} = 1$ (4) $\lambda + \mu = 0$

16. The equation of a plane through the line of intersection of the planes $x+2y=3$, $y-2z+1=0$, and perpendicular to the first plane is : **[JEE-MAIN Online 2013]**
TD0082

(1) $2x-y+7z=11$ (2) $2x-y+10z=11$ (3) $2x-y-9z=10$ (4) $2x-y-10z=9$

17. Let ABC be a triangle with vertices at points A (2, 3, 5), B (-1, 3, 2) and C $(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then $(\lambda, \mu.)$ is equal to :
[JEE-MAIN Online 2013]

(1) (10, 7) (2) (7.5) (3) (7, 10) (4) (5,7) **TD0083**

18. The angle between the lines whose direction cosines satisfy the equations $\ell + m + n = 0$ and $\ell^2 = m^2 + n^2$ is : **[JEE-MAIN 2014]**

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$ **TD0084**

19. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line : **[JEE-MAIN 2014]**

(1) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (2) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$ **TD0085**

(3) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (4) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

20. The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$, is : **[JEE(Main)-2015]**

(1) $x + 3y + 6z = 7$ (2) $2x + 6y + 12z = -13$ **TD0086**

(3) $2x + 6y + 12z = 13$ (4) $x + 3y + 6z = -7$

21. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 16$, is : **[JEE(Main)-2015]**

(1) $3\sqrt{21}$ (2) 13 (3) $2\sqrt{14}$ (4) 8 **TD0087**

22. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is : **[JEE(Main)-2016]**

(1) $\frac{20}{3}$ (2) $3\sqrt{10}$ (3) $10\sqrt{3}$ (4) $\frac{10}{\sqrt{3}}$ **TD0088**

23. If the image of the point P(1,–2, 3) in the plane, $2x+3y-4z+22=0$ measured parallel to line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to :- **[JEE(Main)-2017]**

(1) $6\sqrt{5}$ (2) $3\sqrt{5}$ (3) $2\sqrt{42}$ (4) $\sqrt{42}$ **TD0089**

24. The distantce of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is :- **[JEE(Main)-2017]**

(1) $\frac{10}{\sqrt{74}}$ (2) $\frac{20}{\sqrt{74}}$ (3) $\frac{10}{\sqrt{83}}$ (4) $\frac{5}{\sqrt{83}}$ **TD0090**

25. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x + y + z = 7$ is : **[JEE(Main)-2018]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\sqrt{\frac{2}{3}}$ (4) $\frac{2}{\sqrt{3}}$ **TD0091**

26. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is : **[JEE(Main)-2018]**

(1) $\frac{1}{3\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{4\sqrt{2}}$ **TD0092**

27. The equation of the line passing through $(-4, 3, 1)$, parallel to the plane $x + 2y - z - 5 = 0$ and intersecting the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z-2}{-1}$ is : **[JEE(Main)-Jan 19]**

TD0093

(1) $\frac{x+4}{-1} = \frac{y-3}{1} = \frac{z-1}{1}$ (2) $\frac{x+4}{3} = \frac{y-3}{-1} = \frac{z-1}{1}$ (3) $\frac{x+4}{1} = \frac{y-3}{1} = \frac{z-1}{3}$ (4) $\frac{x-4}{2} = \frac{y+3}{1} = \frac{z+1}{4}$

28. If the lines $x = ay + b$, $z = cy + d$ and $x = a'z + b'$, $y = c'z + d'$ are perpendicular, then : **TD0094**

[JEE(Main)-Jan 19]

(1) $cc' + a + a' = 0$ (2) $aa' + c + c' = 0$ (3) $ab' + bc' + 1 = 0$ (4) $bb' + cc' + 1 = 0$

29. A tetrahedron has vertices $P(1, 2, 1)$, $Q(2, 1, 3)$, $R(-1,1,2)$ and $O(0, 0, 0)$. The angle between the faces OPQ and PQR is : **[JEE(Main)-Jan 19]**

(1) $\cos^{-1}\left(\frac{9}{35}\right)$ (2) $\cos^{-1}\left(\frac{19}{35}\right)$ (3) $\cos^{-1}\left(\frac{17}{31}\right)$ (4) $\cos^{-1}\left(\frac{7}{31}\right)$ **TD0095**

30. A plane which bisects the angle between the two given planes $2x - y + 2z - 4 = 0$ and $x + 2y + 2z - 2 = 0$, passes through the point : **[JEE(Main)-Apr 19]**

(1) $(2,4,1)$ (2) $(2, -4, 1)$ (3) $(1, 4, -1)$ (4) $(1, -4, 1)$ **TD0096**

EXERCISE (JA)

1. **(a)** Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is **TD0103**

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$

(C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

(b) If the distance of the point $P(1,-2,1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is- **TD0104**

(A) $\left(\frac{8}{3},\frac{4}{3},-\frac{7}{3}\right)$ (B) $\left(\frac{4}{3},-\frac{4}{3},\frac{1}{3}\right)$ (C) $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ (D) $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

(c) If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is **TD0105**

(d) Match the statements in **Column-I** with the values in **Column-II**.

Column-I	**Column-II**
(A) A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is **TD0106**	(p) –4
(B) The values of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}\left(\frac{3}{5}\right)$ are **TD0107**	(q) 0
(C) Non-zero vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b}=0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c})=0$ and $2\left\|(\vec{b}+\vec{c})\right\|=\left\|(\vec{b}-\vec{a})\right\|$. If $\vec{a}=\mu\vec{b}+4\vec{c}$, then the possible values of μ are **TD0108**	(r) 4
(D) Let f be the function on $[-\pi,\pi]$ given by	(s) 5
$f(0)=9$ and $f(x)=\sin\left(\frac{9x}{2}\right)\Big/\sin\left(\frac{x}{2}\right)$ for $x \neq 0$. **TD0109**	(t) 6
The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	

[JEE 2010, 3+5+3+(2+2+2+2)]

2. **(a)** The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1,–1,4) with the plane $5x-4y-z=1$. If S is the foot of the perpendicular drawn from the point T(2,1,4) to QR, then the length of the line segment PS is - **TD0110**

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

(b) The equation of a plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is **TD0111**

(A) $5x-11y+z=17$ (B) $\sqrt{2}x+y=3\sqrt{2}-1$

(C) $x+y+z=\sqrt{3}$ (D) $x-\sqrt{2}y=1-\sqrt{2}$

(c) If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) **TD0112**

(A) $y+2z=-1$ (B) $y+z=-1$ (C) $y-z=-1$ (D) $y-2z=-1$

[JEE 2012, 3+3+4]

3. Perpendiculars are drawn from points on the line $\frac{x+2}{2}=\frac{y+1}{-1}=\frac{z}{3}$ to the plane $x+y+z=3$. The feet of perpendiculars lie on the line **[JEE-Advanced 2013, 2]**

(A) $\frac{x}{5}=\frac{y-1}{8}=\frac{z-2}{-13}$ (B) $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{-5}$ **TD0113**

(C) $\frac{x}{4}=\frac{y-1}{3}=\frac{z-2}{-7}$ (D) $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

4. A line ℓ passing through the origin is perpendicular to the lines

$$\ell_1:(3+t)\hat{i}+(-1+2t)\hat{j}+(4+2t)\hat{k},-\infty<t<\infty$$

$$\ell_2:(3+2s)\hat{i}+(3+2s)\hat{j}+(2+s)\hat{k},-\infty<s<\infty$$

Then , the coordinate(s) of the point(s) on ℓ_2 at a distance of $\sqrt{17}$ from the point of intersection of ℓ and ℓ_1 is(are) - **[JEE-Advanced 2013, 4, (–1)]**

(A) $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ (B) (–1,–1,0) (C) (1,1,1) (D) $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$ **TD0114**

5. Two lines $L_1:x=5,\frac{y}{3-\alpha}=\frac{z}{-2}$ and $L_2:x=\alpha,\frac{y}{-1}=\frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[JEE-Advanced 2013, 3, (–1)]

(A) 1 (B) 2 (C) 3 (D) 4 **TD0115**

6. Consider the lines $L_1:\frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}, L_2:\frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$ and the planes $P_1:7x+y+2z=3$, $P_2:3x+5y-6z=4$. Let $ax+by+cz=d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2. **TD0116**

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		**List-II**
P.	a =	1.	13
Q.	b =	2.	–3
R.	c =	3.	1
S.	d =	4.	–2

Codes :

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

[JEE-Advanced 2013, 3, (–1)]

7. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y=x, z=1$ and $y=-x, z=-1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

(A) $\sqrt{2}$ (B) 1 (C) –1 (D) $-\sqrt{2}$ **TD0117**

[JEE(Advanced)-2014, 3]

8. In $\mathbb{R}^3$, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0,1,0) from P_3 is 1 and the distance of a point (α,β,γ) from P_3 is 2, then which of the following relations is (are) true ? **[JEE 2015, 4M, –2M]**

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$ **TD0118**

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

9. In $\mathbb{R}^3$, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M ? **[JEE 2015, 4M, –2M]**

(A) $\left(0,-\frac{5}{6},-\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6},-\frac{1}{3},\frac{1}{6}\right)$ (C) $\left(-\frac{5}{6},0,\frac{1}{6}\right)$ (D) $\left(-\frac{1}{3},0,\frac{2}{3}\right)$ **TD0119**

10. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP= 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$. **[JEE(Advanced)-2016, 4(–2)]**

(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$ **TD0120**

11. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is **TD0121**

[JEE(Advanced)-2016, 3(–1)]

(A) $x + y - 3z = 0$ (B) $3x + z = 0$ (C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

12. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is- **[JEE(Advanced)-2017]**

(A) $14x + 2y + 15z = 31$ (B) $14x + 2y - 15z = 1$ **TD0122**

(C) $-14x + 2y + 15z = 3$ (D) $14x - 2y + 15z = 27$

13. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60°

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$ **TD0123**

14. Let P be a point in the first octant, whose image Q in the plane x + y = 3 (that is, the line segment PQ is perpendicular to the plane x + y = 3 and the mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____ . **[JEE(Advanced)-2018, 3(0)]**

TD0124

15. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p}=\overrightarrow{SP}$, $\vec{q}=\overrightarrow{SQ}$, $\vec{r}=\overrightarrow{SR}$ and $\vec{t}=\overrightarrow{ST}$, then the value of $\left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right|$ is _____ . **TD0125**

[JEE(Advanced)-2018, 3(0)]

16. Let L_1 and L_2 denotes the lines

$$\vec{r}=\hat{i}+\lambda(-\hat{i}+2\hat{j}+2\hat{k}),\lambda\in\mathbb{R}$$

$$\text{and}\quad \vec{r}=\mu(2\hat{i}-\hat{j}+2\hat{k}),\mu\in\mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $\vec{r}=\frac{1}{3}(2\hat{i}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(2) $\vec{r}=\frac{2}{9}(2\hat{i}-\hat{j}+2\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(3) $\vec{r}=t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$

(4) $\vec{r}=\frac{2}{9}(4\hat{i}+\hat{j}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),t\in\mathbb{R}$ **TD0126**

17. Three lines are given by

$$\vec{r}=\lambda\hat{i},\ \lambda\in\mathbb{R}$$

$$\vec{r}=\mu(\hat{i}+\hat{j}),\mu\in\mathbb{R}\ \text{ and}$$

$$\vec{r}=\nu(\hat{i}+\hat{j}+\hat{k}),\nu\in\mathbb{R}.$$

TD0127

Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ___ **[JEE(Advanced)-2019, 3(0)]**

ANSWER KEY

EXERCISE (O-1)

1. D **2.** D **3.** D **4.** B **5.** D **6.** A **7.** C **8.** C
9. C **10.** C **11.** D **12.** A **13.** C **14.** B **15.** B **16.** C
17. A **18.** B **19.** C **20.** (A) R,S; (B) Q; (C) Q,S; (D) P,S

EXERCISE (O-2)

1. A,D **2.** A,B,C **3.** A,C **4.** A,D **5.** C,D **6.** B,D **7.** B,D

8. (A) R; (B) Q; (C) P ; (D) S

EXERCISE (S-1)

1. $\theta = 90°$ **2.** 212 **3.** $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-3}$ **4.** p = 3, (2,1,–3); $x + y + z = 0$

5. **(a)** $\sqrt{3}$; **(b)** $x + y - 2z + 1 = 0$; **(c)** $x - 2y + z = 5$; **(d)** $\pi/3$; **(e)** 4

6. $\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$ **7.** $x - 2y + 2z - 1 = 0$; 2 units **8.** 17 **9.** (1, –2, –4)

EXERCISE (S-2)

1. $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ or $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$ **2.** $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2};\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$ **3.** 9

4. 11 **5.** 9/2 **6.** $\frac{x-4}{3}=\frac{y+14}{10}=\frac{z-4}{4}$ **7.** $\frac{x}{2}+\frac{y}{3}+\frac{z}{-5}=1$, Area $=\frac{19}{2}$ sq. units

8. $2x + 3y + z + 4 = 0$

EXERCISE (JM)

1. 4 **2.** 1 **3.** 2 **4.** 3 **5.** 4 **6.** 2 **7.** 2 **8.** 4

9. 3 **10.** 3 **11.** 2 **12.** 3 **13.** 1 **14.** 1 **15.** 1,4 **16.** 2

17. 3 **18.** 1 **19.** 1 **20.** 1 **21.** 2 **22.** 3 **23. 3** **24.** 3

25. 3 **26.** 1 **27.** 2 **28.** 2 **29.** 2 **30.** 2

EXERCISE (JA)

1. **(a)** C; **(b)** A; **(c)** 6; **(d)** (A) t (B) p,r (C) q (D) r **2.** **(a)** A; **(b)** A; **(c)** B,C **3.** D

4. B,D **5.** A,D **6.** A **7.** C **8.** B,D **9.** A,B **10.** B,C,D **11.** C

12. A **13.** C,D **14.** 8 **15.** 0.5 **16.** 1,2,4 **17.** 0.75

Important Notes