INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + C \Leftrightarrow \frac{d}{dx}\{F(x)+C\} = f(x)$, where C is called the **constant of integration.**

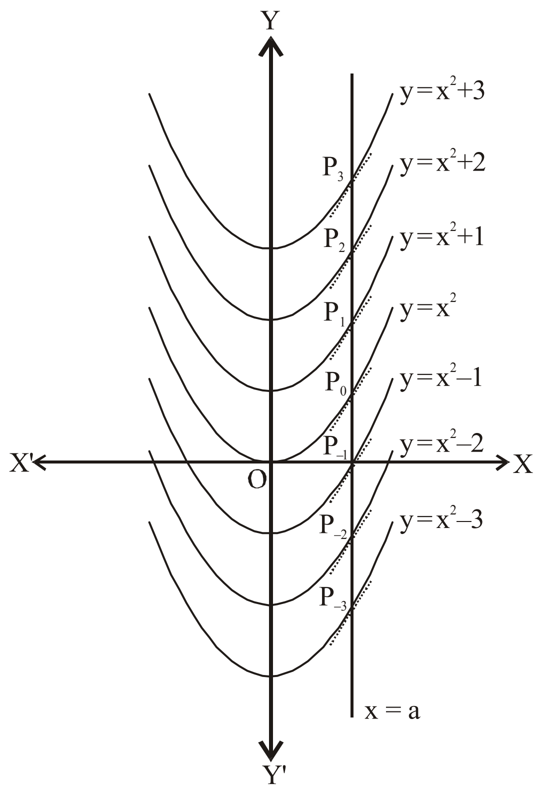
1. GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL :

$\int f(x)dx = F(x) + C = y(say)$, represents a family of curves. The different values of c will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

Let f(x) = 2x. Then $\int f(x)dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

If the line x = a intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals 2a. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\,dx = x^2 + C = f(x) + C$ **(say), implies that the tangents to all the curves f(x) + C, C ∈ R, at the points of intersection of the curves by the line x = a, (a ∈ R) , are parallel.**



2. STANDARD FORMULAE :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n|\,ax+b\,| + C$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a} + C,\ (a>0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n|\sec(ax+b)|+C$ **(viii)** $\int \cot(ax+b)dx = \frac{1}{a}\ell n|\sin(ax+b)|+C$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$

(x) $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$

(xi) $\int \operatorname{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+C$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

(xiii) $\int \sec x\,dx = \ell n|\sec x+\tan x|+C = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$

(xiv) $\int \operatorname{cosec} x\,dx = \ell n|\operatorname{cosec} x-\cot x|+C = \ell n\left|\tan\frac{x}{2}\right|+C = -\ell n|\operatorname{cosec} x+\cot x|+C$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$ **(xvi)** $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$ **(xviii)** $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right]+C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right]+C$ **(xx)** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+C$ **(xxii)** $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right)+C$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right)+C$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

3. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ and $\phi'(x)dx = dt$.

Hence $I = \int f(\phi(x))\phi'(x)dx$ reduces to $\int f(t)dt$.

(i) Fundamental deductions of method of substitution :

$\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put $f(x) = t$ & proceed.

Illustration 1 : Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Solution : $I = \int \frac{(1-\sin^2 x)\cos x}{\sin x(1+\sin x)} dx = \int \frac{1-\sin x}{\sin x} \cos x \, dx$

Put sinx = t $\Rightarrow$ cosx dx = dt

$\Rightarrow \; I = \int \frac{1-t}{t} dt = \ell n\,|t| - t + C = \ell n\,|\sin x| - \sin x + C$ **Ans.**

Illustration 2 : Evaluate $\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}$

Solution : The given integral can be written as

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Let $\left(x+\frac{1}{x}\right) = t$. Differentiating we get $\left(1-\frac{1}{x^2}\right)dx = dt$

Hence $I = \int \frac{dt}{(t^2+1)\tan^{-1} t}$

Now make one more substitution $\tan^{-1}t = u$. Then $\frac{dt}{t^2+1} = du$ and $I = \int \frac{du}{u} = \ell n\,|u| + C$

Returning to t, and then to x, we have

$I = \ell n\,|\tan^{-1} t| + C = \ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + C$ **Ans.**

Do yourself -1 :

(i) Evaluate : $\int \frac{x^2}{9+16x^6} dx$

(ii) Evaluate : $\int \cos^3 x \, dx$

(ii) Standard substitutions :

$\int \frac{dx}{\sqrt{a^2+x^2}}$ or $\int \sqrt{a^2+x^2}dx$; put $x = a\tan\theta$ or $x = a\cot\theta$

$\int \frac{dx}{\sqrt{a^2-x^2}}$ or $\int \sqrt{a^2-x^2}dx$; put $x = a\sin\theta$ or $x = a\cos\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2}dx$; put $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$

$\int \sqrt{\frac{a-x}{a+x}}dx$; put $x = a\cos2\theta$

$\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

Illustration 3 : Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution : Put $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I = \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{\{(a\cos^2\theta + b\sin^2\theta - a)(b - a\cos^2\theta - b\sin^2\theta\}^{\frac{1}{2}}}$$

$$= \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = \left(\frac{b-a}{b-a}\right)\int 2d\theta = 2\theta + C = 2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)} + C$$ **Ans.**

Illustration 4 : Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$

Solution : Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow \quad I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\frac{1}{\cos^2\theta}(-2\sin\theta\cos\theta)d\theta = -\int 2\tan\frac{\theta}{2}\tan\theta d\theta$$

$$= -4\int \frac{\sin^2(\theta/2)}{\cos\theta}d\theta = -2\int \frac{1-\cos\theta}{\cos\theta}d\theta = -2\ell n|\sec\theta + \tan\theta| + 2\theta + C$$

$$= -2\ell n\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| + 2\cos^{-1}\sqrt{x} + C$$

Do yourself -2 :

(i) Evaluate : $\int \sqrt{\frac{x-3}{2-x}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x\sqrt{x^2+4}}$

(b) **Integration by part :** $\int u.v\ dx = u\int v\ dx - \int\left[\frac{du}{dx}.\int v\ dx\right]dx$ where u & v are differentiable functions and are commonly designated as first & second function respectively.

Note : While using integration by parts, choose u & v such that

(i) $\int vdx$ & (ii) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

Illustration 5 : Evaluate : $\int \cos\sqrt{x}\ dx$

Solution : Consider $I = \int \cos\sqrt{x}dx$

Let $\sqrt{x} = t$ then $\frac{1}{2\sqrt{x}}dx = dt$

i.e. $dx = 2\sqrt{x}dt$ or $dx = 2t\ dt$

so $I = \int \cos t.2tdt$

Taking t as first function, integrate it by part

$$\Rightarrow \quad I = 2\left[t\int \cos tdt - \int\left\{\frac{dt}{dt}\int \cos tdt\right\}dt\right]$$

$$I = 2\left[t\sin t - \int 1.\sin tdt\right] = 2[t\sin t + \cos t] + C$$

$$I = 2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$$ **Ans.**

Illustration 6 : Evaluate : $\int \frac{x}{1+\sin x}dx$

Solution : Let $I = \int \frac{x}{1+\sin x}dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$$= \int \frac{x(1-\sin x)}{1-\sin^2 x}dx = \int \frac{x(1-\sin x)}{\cos^2 x}dx = \int x\sec^2 xdx - \int x\sec x\tan xdx$$

$$= \left[x\int \sec^2 xdx - \int\left\{\frac{dx}{dx}\int \sec^2 xdx\right\}dx\right]$$

$$- \left[x\int \sec x\tan xdx - \int\left\{\frac{dx}{dx}\int \sec x\tan xdx\right\}dx\right]$$

$$=\left[x\tan x-\int \tan x dx\right]-\left[x\sec x-\int \sec x dx\right]$$

$$=\left[x\tan x-\ell n|\sec x|\right]-\left[x\sec x-\ell n|\sec x+\tan x|\right]+C$$

$$=x(\tan x-\sec x)+\ell n\left|\frac{(\sec x+\tan x)}{\sec x}\right|+C$$

$$=\frac{-x(1-\sin x)}{\cos x}+\ell n|1+\sin x|+C$$ **Ans.**

Do yourself -3 :

(i) Evaluate : $\int xe^x dx$ **(ii)** Evaluate : $\int x^3\sin(x^2)dx$

Two classic integrands :

(i) $\int e^x[f(x)+f'(x)]dx=e^x.f(x)+C$

Illustration 7 : Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution : $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$

$$=\int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx=\frac{e^x}{1+x^2}+C$$ **Ans.**

Illustration 8 : The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to -

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}+C$ (B) $\frac{e^x(1-x+x^2)}{(1+x^2)^{3/2}}+C$ (C) $\frac{e^x(1-x)}{(1+x^2)^{3/2}}+C$ (D) none of these

Solution : Let $I=\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right).dx$

$$=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}-\frac{x}{(1+x^2)^{3/2}}+\frac{x}{(1+x^2)^{3/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$=\frac{e^x}{(1+x^2)^{1/2}}+\frac{xe^x}{(1+x^2)^{3/2}}+C=\frac{e^x\{1+x^2+x\}}{(1+x^2)^{3/2}}+C$$ **Ans. (D)**

Do yourself - 4 :

(i) Evaluate : $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$ **(ii)** Evaluate : $\int xe^{x^2}\left(\sin x^2+\cos x^2\right)dx$

(ii) $\int[f(x)+xf'(x)]dx=x f(x)+C$

Illustration 9 : Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$

Solution : $I=\int \frac{x+\sin x}{1+\cos x}dx=\int\left(\frac{x+\sin x}{2\cos^2\frac{x}{2}}\right)dx=\int\left(x\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx = x\tan\frac{x}{2}+C$ **Ans.**

Do yourself -5 :

(i) Evaluate : $\int\left(\tan(e^x)+xe^x\sec^2(e^x)\right)dx$ **(ii)** Evaluate : $\int(\ell nx+1)dx$

(c) Integration of trigonometric functions :

(i) $\int \sin^m x\cos^n x dx$

Case-I : When m & n $\in$ natural numbers.

* If one of them is odd, then substitute for the term of even power.
* If both are odd, substitute either of the term.
* If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

Case-II : m + n is a negative even integer.

* In this case the best substitution is tanx = t.

Illustration 10 : Evaluate $\int \sin^3 x\cos^5 x\, dx$

Solution : Put cos x = t; – sin x dx = dt.

so that $I=-\int(1-t^2).t^5dt$

$$=\int(t^7-t^5)dt=\frac{t^8}{8}-\frac{t^6}{6}=\frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+C$$

Alternate :

Put sin x = t; cos x dx = dt

so that $I=\int t^3(1-t^2)^2dt=\int\left(t^3-2t^5+t^7\right)dt$

$$=\frac{\sin^4 x}{4}-\frac{2\sin^6 x}{6}+\frac{\sin^8 x}{8}+C$$

Note : This problem can also be handled by successive reduction or by trigonometric identities.

Illustration 11 : Evaluate $\int \sin^2 x\cos^4 x\, dx$

Solution : $\int \sin^2 x\cos^4 x dx=\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cos 2x+1}{2}\right)^2 dx$

$$=\int\frac{1}{8}\left(1-\cos 2x\right)\left(\cos^2 2x+2\cos 2x+1\right)dx$$

$$=\frac{1}{8}\int\left(\cos^2 2x+2\cos 2x+1-\cos^3 2x-2\cos^2 2x-\cos 2x\right)dx$$

$$=\frac{1}{8}\int\left(-\cos^3 2x-\cos^2 2x+\cos 2x+1\right)dx$$

$$=-\frac{1}{8}\int\left(\frac{\cos 6x+3\cos 2x}{4}+\frac{1+\cos 4x}{2}-\cos 2x-1\right)dx$$

$$=-\frac{1}{32}\left[\frac{\sin 6x}{6}+\frac{3\sin 2x}{2}\right]-\frac{1}{16}x-\frac{\sin 4x}{64}+\frac{\sin 2x}{16}+\frac{x}{8}+C$$

$$=-\frac{\sin 6x}{192}-\frac{\sin 4x}{64}+\frac{1}{64}\sin 2x+\frac{x}{16}+C$$

Illustration 12 : Evaluate $\int\frac{\sqrt{\sin x}}{\cos^{9/2}x}dx$

Solution : Let $I=\int\frac{\sin^{1/2}x}{\cos^{9/2}x}dx=\int\frac{dx}{\sin^{-1/2}x\cos^{9/2}x}$

Here $m+n=\frac{1}{2}-\frac{9}{2}=-4$ (negative even integer).

Divide Numerator & Denominator by $\cos^4 x$.

$$I=\int\sqrt{\tan x}\sec^4 x\,dx=\int\sqrt{\tan x}(1+\tan^2 x)\sec^2 x\,dx$$

$$=\int\sqrt{t}(1+t^2)dt \quad (\text{using } \tan x=t)$$

$$=\frac{2}{3}t^{3/2}+\frac{2}{7}t^{7/2}+C=\frac{2}{3}\tan^{3/2}x+\frac{2}{7}\tan^{7/2}x+C$$

Do yourself -6 :

(i) Evaluate : $\int\frac{\sin^2 x}{\cos^4 x}dx$ **(ii)** Evaluate : $\int\frac{\sqrt{\sin x}\,dx}{\cos^{5/2}x}$ **(iii)** Evaluate : $\int\sin^2 x\cos^5 x\,dx$

(ii) $\int\frac{dx}{a+b\sin^2 x}$ OR $\int\frac{dx}{a+b\cos^2 x}$ OR $\int\frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Divide N^r & D^r by $\cos^2 x$ & put $\tan x=t$.

Illustration 13 : Evaluate : $\int\frac{dx}{2+\sin^2 x}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$I=\int\frac{\sec^2 x\,dx}{2\sec^2 x+\tan^2 x}=\int\frac{\sec^2 x\,dx}{2+3\tan^2 x}$$

Let $\sqrt{3}\tan x=t$ $\therefore$ $\sqrt{3}\sec^2 x\,dx=dt$

So $I=\frac{1}{\sqrt{3}}\int\frac{dt}{2+t^2}=\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C=\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+C$ **Ans.**

Illustration 14 : Evaluate : $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$\therefore \quad I = \int \frac{\sec^2 x dx}{(2\tan x + 3)^2}$$

Let $2\tan x + 3 = t$, $\quad \therefore \; 2\sec^2 x dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + C = -\frac{1}{2(2\tan x + 3)} + C$$ **Ans.**

Do yourself -7 :

(i) Evaluate : $\int \frac{dx}{1 + 4\sin^2 x}$ **(ii)** Evaluate : $\int \frac{dx}{3\sin^2 x + \sin x\cos x + 1}$

(iii) $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles

& put $\tan\frac{x}{2} = t$

In this case $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, $x = 2\tan^{-1} t$; $dx = \frac{2dt}{1+t^2}$

Illustration 15 : Evaluate : $\int \frac{dx}{3\sin x + 4\cos x}$

Solution :

$$I = \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{dx}{3\left\{\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\} + 4\left\{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}} = \int \frac{\sec^2\frac{x}{2}dx}{4 + 6\tan\frac{x}{2} - 4\tan^2\frac{x}{2}}$$

let $\tan\frac{x}{2} = t$, $\quad \therefore \; \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

so $$I = \int \frac{2dt}{4 + 6t - 4t^2} = \frac{1}{2}\int \frac{dt}{1 - \left(t^2 - \frac{3}{2}t\right)} = \frac{1}{2}\int \frac{dt}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{5}{4}\right)}\ell n\left|\frac{\frac{5}{4} + \left(t - \frac{3}{4}\right)}{\frac{5}{4} - \left(t - \frac{3}{4}\right)}\right| + C = \frac{1}{5}\ell n\left|\frac{1 + 2\tan\frac{x}{2}}{4 - 2\tan\frac{x}{2}}\right| + C$$ **Ans.**

Do yourself-8 :

(i) Evaluate : $\int \frac{dx}{3+\sin x}$ **(ii)** Evaluate : $\int \frac{dx}{1+4\sin x+3\cos x}$

(iv) $\int \frac{a\cos x+b\sin x+c}{p\cos x+q\sin x+r}dx$

Express Numerator $(N^r) = \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

Illustration 16 : Evaluate : $\int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}d\theta$

Solution : Write the Numerator = ℓ(denominator) + m(d.c. of denominator) + n

$\Rightarrow 2 + 3\cos\theta = \ell(\sin\theta+2\cos\theta+3) + m(\cos\theta - 2\sin\theta) + n.$

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant terms,

we get $3\ell + n = 2$, $2\ell + m = 3$, $\ell - 2m = 0 \Rightarrow \ell = 6/5$, $m = 3/5$ and $n = -8/5$

Hence $I = \int \frac{6}{5}d\theta + \frac{3}{5}\int \frac{\cos\theta-2\sin\theta}{\sin\theta+2\cos\theta+3}d\theta - \frac{8}{5}\int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

$= \frac{6}{5}\theta + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}I_3$ where $I_3 = \int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

In I_3, put $\tan\frac{\theta}{2} = t \Rightarrow \sec^2\frac{\theta}{2}d\theta = 2dt$

$I_3 = 2\int \frac{dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = 2.\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$

Hence $I = \frac{6\theta}{5} + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right) + C$ **Ans.**

Do yourself -9 :

(i) Evaluate : $\int \frac{\sin x}{\sin x+\cos x}dx$ **(ii)** Evaluate : $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

(d) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

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S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
2.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

Illustration 17 : Evaluate : $\int \frac{x}{(x-2)(x+5)}dx$

Solution :

$$\frac{x}{(x-2)(x+5)}=\frac{A}{x-2}+\frac{B}{x+5}$$

or $x = A(x+5) + B(x-2)$.

by comparing the coefficients, we get

$A = 2/7$ and $B = 5/7$ so that

$$\int \frac{x}{(x-2)(x+5)}dx=\frac{2}{7}\int\frac{dx}{x-2}+\frac{5}{7}\int\frac{dx}{x+5}=\frac{2}{7}\ell n|(x-2)|+\frac{5}{7}\ell n|(x+5)|+C$$ **Ans.**

Illustration 18 : Evaluate $\int \frac{x^4}{(x+2)(x^2+1)}dx$

Solution :

$$\frac{x^4}{(x+2)(x^2+1)}=(x-2)+\frac{3x^2+4}{(x+2)(x^2+1)}$$

Now, $$\frac{3x^2+4}{(x+2)(x^2+1)}=\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$$

So, $$\frac{x^4}{(x+2)(x^2+1)}=x-2+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$$

Now, $$\int\left((x-2)+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}\right)dx$$

$$=\frac{x^2}{2}-2x+\frac{2}{5}\tan^{-1}x+\frac{16}{5}\ell n|x+2|-\frac{1}{10}\ell n(x^2+1)+C$$ **Ans.**

Do yourself - 10 :

(i) Evaluate : $\int \frac{3x+2}{(x+1)(x+3)}dx$ **(ii)** Evaluate : $\int \frac{x^2-1}{(x+1)(x+2)^2}dx$

(ii) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(iii) $\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Express px + q = ℓ (differential coefficient of denominator) + m.

Illustration 19 : Evaluate $\int \frac{dx}{2x^2+x-1}$

Solution :

$$I = \int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2-9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2-(3/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+C \qquad \left\{\text{using}, \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right\}$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+C$$ **Ans.**

Illustration 20 : Evaluate $\int \frac{3x+2}{4x^2+4x+5} dx$

Solution : Express $3x + 2 = \ell(\text{d.c. of } 4x^2 + 4x + 5) + m$

or, $3x + 2 = \ell(8x + 4) + m$

Comparing the coefficients, we get

$$8\ell = 3 \text{ and } 4\ell + m = 2 \Rightarrow \ell = 3/8 \text{ and } m = 2 - 4\ell = 1/2$$

$$\Rightarrow I = \frac{3}{8}\int \frac{8x+4}{4x^2+4x+5} dx + \frac{1}{2}\int \frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\int \frac{dx}{x^2+x+\frac{5}{4}}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right)+C$$ **Ans.**

Do yourself -11 :

(i) Evaluate : $\int \frac{dx}{x^2+x+1}$

(ii) Evaluate : $\int \frac{5x+4}{\sqrt{x^2+4x+1}} dx$

(iv) Integrals of the form $\int \frac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int \frac{x^2-1}{x^4+Kx^2+1}dx$, where K is any constant.

Divide N^r & D^r by x^2 & proceed.

Note : Sometimes it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions e.g.

* $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$

* $\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$

These integrals can be called as **Algebric Twins**.

Illustration 21 : Evaluate : $\int \frac{4}{\sin^4 x + \cos^4 x}dx$

Solution :

$$I = 4\int \frac{1}{\sin^4 x + \cos^4 x}dx = 4\int \frac{\sec^4 x}{1+\tan^4 x}dx$$

$$= 4\int \frac{(\tan^2 x + 1)\sec^2 x}{(\tan^4 x + 1)}dx$$

Now, put tanx = t $\Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = 4\int \frac{1+t^2}{1+t^4}dt = 4\int \frac{1/t^2+1}{t^2+1/t^2}dt$$

Now, put t – 1/t = z $\Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$$\Rightarrow I = 4\int \frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + C = 2\sqrt{2}\tan^{-1}\frac{t-1/t}{\sqrt{2}} + C$$

$$= 2\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1/\tan x}{\sqrt{2}}\right) + C$$ **Ans.**

Illustration 22 : Evaluate : $\int \frac{1}{x^4+5x^2+1}dx$

Solution :

$$I = \frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx + \frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$$

{dividing N^r and D^r by x^2}

$$=\frac{1}{2}\int\frac{(1+1/x^2)}{(x-1/x)^2+7}dx-\frac{1}{2}\int\frac{(1-1/x^2)dx}{(x+1/x)^2+3}$$

$$=\frac{1}{2}\int\frac{dt}{t^2+\left(\sqrt{7}\right)^2}-\frac{1}{2}\int\frac{du}{u^2+\left(\sqrt{3}\right)^2}$$

where $t = x-\frac{1}{x}$ and $u = x+\frac{1}{x}$

$$I=\frac{1}{2}.\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right)-\frac{1}{2}.\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right]+C$$ **Ans.**

Do yourself -12 :

(i) Evaluate : $\int\frac{x^2+1}{x^4-x^2+1}dx$ **(ii)** Evaluate : $\int\frac{1}{1+x^4}dx$

(e) Integration of Irrational functions :

(i) $\int\frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int\frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

(ii) $\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int\frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put $x=\frac{1}{t}$

Illustration 23 : Evaluate $\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$

Solution : Let, $I=\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$ Put $x + 1 = t^2 \Rightarrow dx = 2tdt$

$$\therefore\quad I=\int\frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}.(2t)dt=2\int\frac{t^2+1}{t^4+t^2+1}dt=2\int\frac{1+1/t^2}{t^2+1+1/t^2}dt$$

$$=2\int\frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}.dt=2\int\frac{du}{u^2+(\sqrt{3})^2}\qquad\left\{\text{where } u=t-\frac{1}{t}\right\}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+C$$ **Ans.**

Illustration 24 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Solution : Let, $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$ put $x-1=\frac{1}{t} \Rightarrow dx=-1/t^2\,dt$

$$I=\int \frac{-1/t^2dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}}=-\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+1/12}}=-\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+C$$

$$=-\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+C$$ **Ans.**

Illustration 25 : Evaluate : $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Let, $I=\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ Put $x=\frac{1}{t}$, So that $dx=\frac{-1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2dt}{(1+1/t^2)\sqrt{1-1/t^2}}=-\int \frac{tdt}{(t^2+1)\sqrt{t^2-1}}$$

Again let, $t^2 = u$. So that $2t\,dt = du$.

$=\frac{-1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear

so that we put $u-1=z^2$ so that $du = 2z\,dz$

$$\therefore \quad I=-\frac{1}{2}\int \frac{2zdz}{(z^2+1+1)\sqrt{z^2}}=-\int \frac{dz}{(z^2+2)}$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right)+C=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$$ **Ans.**

Do yourself -13 :

(i) Evaluate : $\int \frac{x}{(x-3)\sqrt{x+1}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x^2\sqrt{1+x^2}}$

(f) Manipulating integrands :

(i) $\int \frac{dx}{x(x^n+1)}$, $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$, $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t^n$.

Illustration 26 : Evaluate : $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

Solution : Let $I = \int \frac{dx}{x^n(1+x^n)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$

Put $1+\frac{1}{x^n} = t^n$, then $\frac{1}{x^{n+1}}dx = -t^{n-1}dt$

$I = -\int \frac{t^{n-1}dt}{t} = -\int t^{n-2}dt = -\frac{t^{n-1}}{n-1} + C = \frac{-1}{n-1}\left(1+\frac{1}{x^n}\right)^{\frac{n-1}{n}} + C$ **Ans.**

Do yourself -14 :

(i) Evaluate : $\int \frac{dx}{x(x^2+1)}$

(ii) Evaluate : $\int \frac{dx}{x^2(x^3+1)^{2/3}}$

(iii) Evaluate : $\int \frac{dx}{x^3(x^3+1)^{1/3}}$

Miscellaneous Illustrations :

Illustration 27 : Evaluate $\int \frac{\cos^4 x dx}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}$

Solution : $I = \int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cos^4 x}{\sin^6 x\{1+\cot^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cot^4 x \operatorname{cosec}^2 x dx}{(1+\cot^5 x)^{3/5}}$

Put $1 + \cot^5 x = t$

$5\cot^4 x \operatorname{cosec}^2 x dx = -dt$

$= -\frac{1}{5}\int \frac{dt}{t^{3/5}} = -\frac{1}{2}t^{2/5} + C = -\frac{1}{2}(1+\cot^5 x)^{2/5} + C$ **Ans.**

Illustration 28 : $\int \frac{dx}{\cos^6 x + \sin^6 x}$ is equal to -

(A) $\ell n|\tan x - \cot x| + C$ (B) $\ell n|\cot x - \tan x| + C$

(C) $\tan^{-1}(\tan x - \cot x) + C$ (D) $\tan^{-1}(-2\cot 2x) + C$

Solution : Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x} dx = \int \frac{(1+\tan^2 x)^2 \sec^2 x dx}{1+\tan^6 x}$

If $\tan x = p$, then $\sec^2 x\, dx = dp$

$$\Rightarrow \quad I = \int \frac{(1+p^2)^2 dp}{1+p^6} = \int \frac{(1+p^2)}{p^4 - p^2 + 1} dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2 + \frac{1}{p^2} - 1\right)} dp$$

$$= \int \frac{dk}{k^2+1} = \tan^{-1}(k) + C \qquad \left(\text{where } p - \frac{1}{p} = k, \left(1+\frac{1}{p^2}\right)dp = dk\right)$$

$$= \tan^{-1}\left(p - \frac{1}{p}\right) + C = \tan^{-1}(\tan x - \cot x) + C = \tan^{-1}(-2\cot 2x) + C$$ **Ans. (C,D)**

Illustration 29 : Evaluate : $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x} dx$

Solution : $I = \int \frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x} dx = \int \frac{(4\sin x - 1)\cos x}{6 - (1-\sin^2 x) - 4\sin x} dx = \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5} dx$

Put sin x = t, so that cos x dx = dt.

$$\therefore \quad I = \int \frac{(4t-1)dt}{(t^2 - 4t + 5)} \qquad \text{...... (i)}$$

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$$2\lambda = 4, -4\lambda + \mu = -1 \qquad \text{...... (ii)}$$

$$\lambda = 2, \mu = 7$$

$$\therefore \quad I = \int \frac{2(2t-4)+7}{t^2 - 4t + 5} dt \qquad \{\text{using (i) and (ii)}\}$$

$$= 2\int \frac{2t-4}{t^2-4t+5} dt + 7\int \frac{dt}{t^2-4t+5} = 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{t^2 - 4t + 4 - 4 + 5}$$

$$= 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{(t-2)^2 + (1)^2} = 2\log|t^2 - 4t + 5| + 7.\tan^{-1}(t-2) + C$$

$$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + C.$$ **Ans.**

Illustration 30 : The value of $\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$, is equal to -

(A) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(B) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(C) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$

(D) none of these

Solution : Here, $I=\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$

Put $x = 3\cos 2\theta \quad \Rightarrow \quad dx = -6\sin 2\theta d\theta$

$$=\int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)d\theta$$

$$=\int \frac{\sin\theta}{\cos\theta}.\sin^{-1}(\sin\theta).(-6\sin 2\theta)d\theta=-6\int \theta.(2\sin^2\theta)d\theta$$

$$=-6\int \theta(1-\cos 2\theta)d\theta=-6\left\{\frac{\theta^2}{2}-\int \theta\cos 2\theta\, d\theta\right\}+C$$

$$=-6\left\{\frac{\theta^2}{2}-\left(\theta\frac{\sin 2\theta}{2}-\int 1.\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\}+C=-3\theta^2+6\left\{\theta\frac{\sin 2\theta}{2}+\frac{\cos 2\theta}{4}\right\}+C$$

$$=\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2+2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right)+2x\right\}+C$$ **Ans. (A)**

Illustration 31 : Evaluate : $\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx$

Solution : $I=\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx=\int \frac{(1-\tan^2 x)dx}{(1+\tan x)^2\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}$

$$I=\int \frac{-\left(1-\frac{1}{\tan^2 x}\right)\sec^2 x dx}{\left(\tan x+2+\frac{1}{\tan x}\right)\sqrt{\tan x+1+\frac{1}{\tan x}}}$$

let, $y=\sqrt{\tan x+1+\frac{1}{\tan x}} \Rightarrow 2y\,dy=\left(\sec^2 x-\frac{1}{\tan^2 x}.\sec^2 x\right)dx$

$\therefore \quad I=\int\frac{-2ydy}{(y^2+1).y}=-2\int\frac{dy}{1+y^2}$

$=-2\tan^{-1}y+c=-2\tan^{-1}\left(\sqrt{\tan x+1+\frac{1}{\tan x}}\right)+C$ **Ans.**

ANSWERS FOR DO YOURSELF

1 : **(i)** $\frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right)+C$ **(ii)** $\sin x-\frac{1}{3}\sin^3 x+C$

2 : **(i)** $\sqrt{(x-2)(3-x)}-\sin^{-1}\sqrt{3-x}+C$ **(ii)** $\frac{1}{2}\ell n\left[\frac{\sqrt{x^2+4}-2}{x}\right]+C$

3 : **(i)** xe^x-e^x+C **(ii)** $\frac{1}{2}\left[-x^2\cos x^2+\sin x^2\right]+C$

4 : **(i)** $e^x\tan^{-1}x+C$ **(ii)** $\frac{1}{2}e^{x^2}\sin(x^2)+C$

5 : **(i)** $x\tan(e^x)+C$ **(ii)** $x\ell nx+C$

6 : **(i)** $\frac{1}{3}\tan^3 x+C$ **(ii)** $\frac{2}{3}\tan^{3/2}x+C$ **(iii)** $\frac{1}{3}\sin^3 x-\frac{2}{5}\sin^5 x+\frac{1}{7}\sin^7 x+C$

7 : **(i)** $\frac{1}{\sqrt5}\tan^{-1}\left(\sqrt5\tan x\right)+C$ **(ii)** $\frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{8\tan x+1}{\sqrt{15}}\right)+C$

8 : **(i)** $\frac{1}{\sqrt2}\tan^{-1}\left(\frac{3\tan x/2+1}{2\sqrt2}\right)+C$ **(ii)** $\frac{1}{2\sqrt6}\ell n\left|\frac{\sqrt6+\tan x/2-2}{\sqrt6-\tan x/2+2}\right|+C$

9 : **(i)** $\frac{1}{2}x-\frac{1}{2}\ell n\left|\sin x+\cos x\right|+C$ **(ii)** $\frac{12}{13}x-\frac{5}{13}\ell n\left|3\cos x+2\sin x\right|+C$

10 : **(i)** $-\frac{1}{2}\ell n|x+1|+\frac{7}{2}\ell n|x+3|+C$ **(ii)** $\ell n|x+2|+\frac{3}{x+2}+C$

11 : **(i)** $\frac{2}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+C$ **(ii)** $5\sqrt{x^2+4x+1}-6\ell n\left[(x+2)+\sqrt{x^2+4x+1}\right]+C$

12 : **(i)** $\tan^{-1}\left(\frac{x^2-1}{x}\right)+C$ **(ii)** $\frac{1}{2\sqrt2}\left[\tan^{-1}\left(\frac{x^2-1}{\sqrt2 x}\right)-\frac{1}{2}\ell n\frac{x^2-\sqrt2x+1}{x^2+\sqrt2x+1}\right]$

13 : **(i)** $2\sqrt{x+1}+\frac{3}{2}\ell n\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C$ **(ii)** $-\frac{1}{x}\sqrt{1+x^2}+C$

14 : **(i)** $-\frac{1}{2}\ell n\left(\frac{x^2+1}{x^2}\right)+C$ **(ii)** $-\left(1+\frac{1}{x^3}\right)^{1/3}+C$ **(iii)** $-\frac{1}{2}\left(1+\frac{1}{x^3}\right)^{2/3}+C$

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. $\int \frac{1-x^7}{x(1+x^7)}dx$ equals -

(A) $\ln x + \frac{2}{7}\ln(1+x^7)+c$

(B) $\ln x - \frac{2}{7}\ln(1-x^7)+c$

(C) $\ln x - \frac{2}{7}\ln(1+x^7)+c$

(D) $\ln x + \frac{2}{7}\ln(1-x^7)+c$ **II0010**

2. Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is -

(A) $\frac{x}{x^4+x+1}+c$ (B) $-\frac{x}{x^4+x+1}+c$ (C) $\frac{x+1}{x^4+x+1}+c$ (D) $-\frac{x+1}{x^4+x+1}+c$

II0007

3. Integral of $\sqrt{1+2\cot x(\cot x+\text{cosec}x)}$ w.r.t. x is

(A) $2\ln\cos\frac{x}{2}+c$

(B) $2\ln\sin\frac{x}{2}+c$

(C) $\frac{1}{2}\ln\cos\frac{x}{2}+c$

(D) $\ln\sin x-\ln(\text{cosec}x-\cot x)+c$ **II0011**

4. $\int x\,.\frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals -

(A) $\sqrt{1+x^2}\,\ln\left(x+\sqrt{1+x^2}\right)-x+c$

(B) $\frac{x}{2}\,.\ln^2\left(x+\sqrt{1+x^2}\right)-\frac{x}{\sqrt{1+x^2}}+c$

(C) $\frac{x}{2}\,.\ln^2\left(x+\sqrt{1+x^2}\right)+\frac{x}{\sqrt{1+x^2}}+c$

(D) $\sqrt{1+x^2}\,\ln\left(x+\sqrt{1+x^2}\right)+x+c$ **II0015**

5. A function y = f(x) satisfies $f''(x)=-\frac{1}{x^2}-\pi^2\sin(\pi x)$; $f'(2)=\pi+\frac{1}{2}$ and $f(1)=0$. The value of $f\left(\frac{1}{2}\right)$ is

(A) $\ln 2$ (B) 1 (C) $\frac{\pi}{2}-\ln 2$ (D) $1-\ln 2$ **II0002**

6. Consider $f(x)=\frac{x^2}{1+x^3}$; $g(t)=\int f(t)dt$. If $g(1)=0$ then g(x) equals-

(A) $\frac{1}{3}\ell n(1+x^3)$ (B) $\frac{1}{3}\ell n\left(\frac{1+x^3}{2}\right)$ (C) $\frac{1}{2}\ell n\left(\frac{1+x^3}{3}\right)$ (D) $\frac{1}{3}\ell n\left(\frac{1+x^3}{3}\right)$

II0003

7. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$

(A) $2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C$

(B) $e^{\sqrt{x}}\left[x-2\sqrt{x}+1\right]+C$

(C) $e^{\sqrt{x}}\left(x+\sqrt{x}\right)+C$

(D) $e^{\sqrt{x}}\left(x+\sqrt{x}+1\right)+C$ **II0012**

8. $\int \frac{dx}{\sqrt[3]{x^{5/2}(x+1)^{7/2}}}$

(A) $-\left(\frac{x+1}{x}\right)^{1/6}+C$ (B) $6\left(\frac{x+1}{x}\right)^{-1/6}+C$ (C) $\left(\frac{x}{x+1}\right)^{5/6}+C$ (D) $-\left(\frac{x}{x+1}\right)^{5/6}+C$

II0009

9. $\int e^x\left(\frac{x^2-3}{(x-1)^2}\right)dx$ is equal to -

(A) $e^x\frac{(x+3)}{(x-1)}+C$ (B) $e^x\left(\frac{x-3}{x-1}\right)+C$ (C) $e^x\left(\frac{x+1}{x-1}\right)+C$ (D) $e^x\left(\frac{1}{x-1}\right)^2+C$

(where C is constant of integration) II0014

10. $\int \frac{x^3}{\left(2x^2+1\right)^3}dx$ is equal to-

(A) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$ (B) $-\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$ (C) $\frac{1}{2}\left(2+\frac{1}{x^2}\right)^{-2}+C$ (D) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{2}+C$

(where 'C' is integration constant) II0006

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. $\int (\sin(101x)\cdot\sin^{99}x)dx$ equals

(A) $\frac{\sin(100x)(\sin x)^{100}}{100}+C$ (B) $\frac{\cos(100x)(\sin x)^{100}}{100}+C$

(C) $\frac{\cos(100x)(\cos x)^{100}}{100}+C$ (D) $\frac{\sin(100x)(\sin x)^{101}}{101}+C$ II0020

2. The integral $\int \sqrt{\cot x}\, e^{\sqrt{\sin x}}\sqrt{\cos x}\, dx$ equals

(A) $\frac{\sqrt{\tan x}\, e^{\sqrt{\sin x}}}{\sqrt{\cos x}}+C$ (B) $2e^{\sqrt{\sin x}}+C$ (C) $-\frac{1}{2}e^{\sqrt{\sin x}}+C$ (D) $\frac{\sqrt{\cot x}\, e^{\sqrt{\sin x}}}{2\sqrt{\cos x}}+C$

II0018

[MULTIPLE OBJECTIVE TYPE]

3. Which one of the following is **FALSE** ?

(A) $x.\int\frac{dx}{x} = x\,ln|x| + C$ (B) $x.\int\frac{dx}{x} = x\,ln|x| + Cx$

(C) $\frac{1}{\cos x}.\int \cos x\, dx = \tan x + C$ (D) $\frac{1}{\cos x}.\int \cos x\, dx = x + C$ II0022

4. Let $f(x)=\sin^3 x+\sin^3\left(x+\frac{2\pi}{3}\right)+\sin^3\left(x+\frac{4\pi}{3}\right)$ then the primitive of $f(x)$ w.r.t. x is

(A) $-\frac{3\sin 3x}{4}+C$ (B) $\frac{1}{2}\cos^2\left(\frac{3x}{2}\right)+C$ (C) $\frac{\sin 3x}{4}+C$ (D) $\frac{\cos 3x}{4}+C$

where C is an arbitrary constant. II0024

5. Suppose $J = \int \frac{\sin^2 x + \sin x}{1 + \sin x + \cos x} dx$ and $K = \int \frac{\cos^2 x + \cos x}{1 + \sin x + \cos x} dx$. If C is an arbitrary constant of integration then which of the following is/are correct?

(A) $J = \frac{1}{2}(x - \sin x + \cos x) + C$

(B) $J = K - (\sin x + \cos x) + C$

(C) $J = x - K + C$

(D) $K = \frac{1}{2}(x - \sin x + \cos x) + C$ **II0025**

6. $\int \frac{\cot^{-1}(e^x)}{e^x} dx$ is equal to -

(A) $\frac{1}{2}\ell n\left(e^{2x}+1\right) - \frac{\cot^{-1}(e^x)}{e^x} + x + c$

(B) $\frac{1}{2}\ell n\left(e^{2x}+1\right) + \frac{\cot^{-1}(e^x)}{e^x} + x + c$

(C) $\frac{1}{2}\ell n\left(e^{2x}+1\right) - \frac{\cot^{-1}(e^x)}{e^x} - x + c$

(D) $\frac{1}{2}\ell n\left(e^{2x}+1\right) + \frac{\cot^{-1}(e^{-x})}{e^x} - \frac{\pi}{2}e^{-x} - x + c$

II0019

7. $\int \sec^2\theta(\sec\theta + \tan\theta)^2 d\theta$

(A) $\frac{(\sec\theta + \tan\theta)}{2}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

(B) $\frac{(\sec\theta + \tan\theta)}{3}[2 + 4\tan\theta\,(\sec\theta + \tan\theta)] + C$

(C) $\frac{(\sec\theta + \tan\theta)}{3}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

(D) $\frac{(\sec\theta + \tan\theta)}{3}[1 + \sec\theta(\sec\theta + \tan\theta)] + C$

II0021

8. If $f'\left(x^2\right) = \frac{\ell nx}{x^2}$ and $f\left(1\right) = -\frac{1}{4}$, then -

(A) $f(e) = 0$

(B) $f'\left(e\right) = \frac{1}{2e}$

(C) $f''(e) = f(e)$

(D) $f''(e) = f'(e)$ **II0089**

9. $I = \int \frac{2x - 1 - x^2}{(1 + x^2)^2} dx$ is equal to

(A) $\alpha - \frac{1}{1+x^2} - \tan^{-1} x$

(B) $\cot^{-1} x - \frac{1}{1+x^2} + \beta$

(C) $\frac{x^2}{1+x^2} - \tan^{-1} x + \gamma$

(D) $\frac{2x^2+1}{1+x^2} - \tan^{-1} x + \delta$ **II0090**

(where $\alpha, \beta, \gamma, \delta$ are arbitrary constants)

10. $\int \frac{x+1}{2x^{3/2}} dx$ is equal to-

(A) $x^{\frac{1}{2}} - x^{-\frac{1}{2}} + C$

(B) $\frac{x^{\frac{3}{2}} - x^{\frac{1}{2}}}{x} + C$

(C) $\frac{x^{\frac{3}{2}} + \sqrt{x}\left(\sqrt{x} - 1\right)}{x} + C$

(D) $\frac{x^2 - 1}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} + C$ **II0091**

(where C is constant of integration)

EXERCISE (S-1)

1. $\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}dx$ II0040

2. A function g defined for all positive real numbers, satisfies $g'(x^2)=x^3$ for all $x>0$ and $g(1)=1$. Compute g(4). II0041

3. $\int\left[\sin\alpha\sin(x-\alpha)+\sin^2\left(\frac{x}{2}-\alpha\right)\right]dx$ II0032

4. $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ell n\,x\,dx$ II0057

5. $\int\frac{(\sqrt{x}+1)dx}{\sqrt{x}\left(\sqrt[3]{x}+1\right)}$ II0043

6. $\int\sin^{-1}\sqrt{\frac{x}{a+x}}dx$ II0045

7. $\int\frac{x\ell nx}{(x^2-1)^{3/2}}.dx$ II0058

8. $\int\left[\frac{\sqrt{x^2+1}\;[\ell n(x^2+1)-2\ell n\,x]}{x^4}\right]dx$ II0039

9. $\int\frac{3x^2+1}{(x^2-1)^3}dx$ II0054

10. $\int\frac{(ax^2-b)\;dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$ II0038

11. $\int\frac{e^{\cos x}(x\sin^3x+\cos x)}{\sin^2x}dx$ II0046

12. $\int\frac{5x^4+4x^5}{(x^5+x+1)^2}dx$ II0047

13. $\int(\sin x)^{-11/3}(\cos x)^{-1/3}dx$ II0048

14. $\int\frac{dx}{\sin x+\sec x}$ II0049

15. $\int\frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$ II0052

16. If the value $\int\frac{1-(\cot x)^{2008}}{\tan x+(\cot x)^{2009}}dx=\frac{1}{k}l n|\sin^k x+\cos^k x|+C$, then find k. II0050

17. $\int\frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$ II0035

EXERCISE (S-2)

1. $\int\frac{\tan(l n\,x)\tan\left(l n\frac{x}{2}\right)\tan(l n\,2)}{x}dx$ II0059

2. $\int\frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}dx$ II0067

3. $\int\frac{dx}{\left(x+\sqrt{x(1+x)}\right)^2}$ II0066

4. $\int\frac{dx}{\sqrt{\sin^3x\sin(x+\alpha)}}$ II0065

5. $\int\frac{(1+x^2)dx}{1-2x^2\cos\alpha+x^4}\;\alpha\in(0,\pi)$ II0064

6. Let f(x) is a quadratic function such that f(0) = 1 and $\int\frac{f(x)dx}{x^2(x+1)^3}$ is a rational function, find the value of $f'(0)$ II0069

7. $\int \frac{\cos x - \sin x}{7 - 9\sin 2x} dx$ **II0061**

8. $\int \cos 2\theta . \ell n \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} d\theta$ **II0062**

EXERCISE (JM)

1. If the integral $\int \frac{5\tan x}{\tan x - 2} dx = x + a \ln|\sin x - 2\cos x| + k$ then a is equal to : **[AIEEE-2012]**

(1) 2 (2) –1 (3) –2 (4) 1 **II0071**

2. If $\int f(x)dx = \Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to : **[JEE-MAIN-2013]**

(1) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx\right] + C$

(2) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\Psi(x^3)dx + C$

(3) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$

(4) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^3\Psi(x^3)dx\right] + C$ **II0072**

3. The integral $\int \left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to : **[JEE-MAIN-2014]**

(1) $(x-1)e^{x+\frac{1}{x}} + c$ (2) $xe^{x+\frac{1}{x}} + c$ (3) $(x+1)e^{x+\frac{1}{x}} + c$ (4) $-xe^{x+\frac{1}{x}} + c$ **II0073**

4. The integral $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals : **[JEE-MAIN-2015]**

(1) $-(x^4+1)^{\frac{1}{4}} + c$ (2) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$ (3) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$ (4) $(x^4+1)^{\frac{1}{4}} + c$ **II0074**

5. The integral $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to :- **[JEE-MAIN 2016]**

(1) $\frac{-x^{10}}{2(x^5+x^3+1)^2} + C$ (2) $\frac{-x^5}{(x^5+x^3+1)^2} + C$ (3) $\frac{x^{10}}{2(x^5+x^3+1)^2} + C$ (4) $\frac{x^5}{2(x^5+x^3+1)^2} + C$

where C is an arbitrary constant. **II0075**

6. Let $I_n = \int \tan^n x\, dx, (n > 1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to :- **[JEE-MAIN 2017]**

(1) $\left(-\frac{1}{5}, 0\right)$ (2) $\left(-\frac{1}{5}, 1\right)$ (3) $\left(\frac{1}{5}, 0\right)$ (4) $\left(\frac{1}{5}, -1\right)$ **II0076**

7. The integral $\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$ is equal to **[JEE-MAIN 2018]**

(1) $\frac{-1}{3(1+\tan^3 x)} + C$ (2) $\frac{1}{1+\cot^3 x} + C$ (3) $\frac{-1}{1+\cot^3 x} + C$ (4) $\frac{1}{3(1+\tan^3 x)} + C$

(where C is a constant of integration) **II0077**

8. For $x^2 \neq n\pi + 1$, $n \in N$ (the set of natural numbers), the integral $\int x\sqrt{\frac{2\sin(x^2-1)-\sin 2(x^2-1)}{2\sin(x^2-1)+\sin 2(x^2-1)}}dx$ is equal to : (where c is a constant of integration) **[JEE (Main) 2019]**

(1) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right|+c$ (2) $\log_e\left|\frac{1}{2}\sec^2(x^2-1)\right|+c$

(3) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right|+c$ (4) $\frac{1}{2}\log_e|\sec(x^2-1)|+c$ **II0078**

9. If $f(x)=\int\frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx, (x \geq 0)$ and $f(0) = 0$, then the value of $f(1)$ is : **[JEE (Main) 2019]**

(1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $-\frac{1}{4}$ (4) $\frac{1}{4}$ **II0092**

10. If $\int\frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$, for a suitable chosen integer m and a function A(x), where C is a constant of integration then $(A(x))^m$ equals : **[JEE (Main) 2019]**

(1) $\frac{-1}{3x^3}$ (2) $\frac{-1}{27x^9}$ (3) $\frac{1}{9x^4}$ (4) $\frac{1}{27x^6}$ **II0079**

11. The integral $\int\frac{3x^{13}+2x^{11}}{(2x^4+3x^2+1)^4}dx$ is equal to : (where C is a constant of integration) **[JEE (Main) 2019]**

(1) $\frac{x^4}{(2x^4+3x^2+1)^3}+C$ (2) $\frac{x^{12}}{6(2x^4+3x^2+1)^3}+C$

(3) $\frac{x^4}{6(2x^4+3x^2+1)^3}+C$ (4) $\frac{x^{12}}{(2x^4+3x^2+1)^3}+C$ **II0093**

12. $\int\frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ is equal to : (where c is a constant of integration) **[JEE (Main) 2019]**

(1) 2x + sinx + 2sin2x + c (2) x + 2sinx + 2sin2x + c

(3) x + 2sinx + sin2x + c (4) 2x + sinx + sin2x + c **II0094**

13. The integral $\int\sec^{2/3}x\,\text{cosec}^{4/3}x\,dx$ is equal to (Hence C is a constant of integration) **[JEE (Main) 2019]**

(1) $3\tan^{-1/3}x + C$ (2) $-\frac{3}{4}\tan^{-4/3}x+C$ (3) $-3\cot^{-1/3}x + C$ (4) $-3\tan^{-1/3}x + C$ **II0095**

14. Let $\alpha \in (0, \pi/2)$ be fixed. If the integral $\int\frac{\tan x+\tan\alpha}{\tan x-\tan\alpha}dx = A(x)\cos 2\alpha + B(x)\sin 2\alpha + C$, where C is a constant of integration, then the functions A(x) and B(x) are respectively : **[JEE (Main) 2019]**

(1) $x - \alpha$ and $\log_e|\cos(x-\alpha)|$ (2) $x + \alpha$ and $\log_e|\sin(x-\alpha)|$ **II0096**

(3) $x - \alpha$ and $\log_e|\sin(x-\alpha)|$ (4) $x + \alpha$ and $\log_e|\sin(x+\alpha)|$

EXERCISE (JA)

1. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$ is equal to - **[JEE 2006, (3M, –1M)]**

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$ (C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

II0084

2. Let $f(x)=\frac{x}{(1+x^n)^{1/n}}$ for $n \ge 2$ and $g(x)=\underbrace{(f\circ f\circ ...\circ f)}_{\text{f occurs n times}}(x)$. Then $\int x^{n-2}g(x)dx$ equals.

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$ (B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+K$ **[JEE 2007, 3]**

(C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$ (D) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+K$ II0085

3. Let F(x) be an indefinite integral of $\sin^2x$. **[JEE 2007, 3]**

Statement-1 : The function F(x) satisfies $F(x+\pi)=F(x)$ for all real x.

because

Statement-2 : $\sin^2(x+\pi)=\sin^2x$ for all real x.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True. II0086

4. Let $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant c, the value of J – I equals **[JEE 2008, 3 (–1)]**

(A) $\frac{1}{2}\ell n\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$ (B) $\frac{1}{2}\ell n\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\frac{1}{2}\ell n\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$ (D) $\frac{1}{2}\ell n\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$ II0087

5. The integral $\int\frac{\sec^2x}{(\sec x+\tan x)^{9/2}}dx$ equals (for some arbitrary constant K) **[JEE 2012, 3M, –1M]**

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$ II0088

ANSWER KEY

INDEFINITE INTEGRATION

EXERCISE (O-1)

1. C **2.** B **3.** B **4.** A **5.** D **6.** B **7.** A **8.** B
9. C **10.** A

EXERCISE (O-2)

1. A **2.** B **3.** A,C,D **4.** B,D **5.** B,C **6.** C,D **7.** C,D
8. A,B,C **9.** A,B,C,D **10.** A,B,C,D

EXERCISE (S-1)

1. $\frac{x^2}{2}-x+C$ **2.** $\frac{67}{5}$ **3.** $\frac{1}{2}(x-\sin x)+C$ **4.** $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$

5. $6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\ln(1+t^2)-\tan^{-1}t\right]+C$ where $t=x^{1/6}$ **6.** $(a+x)\operatorname{arctan}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

7. $\operatorname{arcsec}x-\frac{\ln x}{\sqrt{x^2-1}}+C$ **8.** $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3\ln\left(1+\frac{1}{x^2}\right)\right]$ **9.** $C-\frac{x}{(x^2-1)^2}$

10. $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$ **11.** $C-e^{\cos x}(x+\operatorname{cosec}x)$ **12.** $C-\frac{x+1}{x^5+x+1}$ or $C+\frac{x^5}{x^5+x+1}$

13. $-\frac{3(1+4\tan^2 x)}{8(\tan x)^{8/3}}+C$ **14.** $\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}+\operatorname{arctan}(\sin x+\cos x)+C$

15. $4\ln x+\frac{7}{x}+6\tan^{-1}(x)+\frac{6x}{1+x^2}+C$ **16.** 2010 **17.** $\frac{-2}{\alpha-\beta}\sqrt{\frac{x-\beta}{x-\alpha}}+C$

EXERCISE (S-2)

1. $\ln\left(\frac{\sec(\ln x)}{\sec(\ln x/2)x^{\tan(\ln 2)}}\right)+C$ **2.** $e^x\sqrt{\frac{1+x}{1-x}}+c$ **3.** $2\ln\frac{t}{2t+1}+\frac{1}{2t+1}+C$, when $t=x+\sqrt{x^2+x}$

4. $C-\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}$ **5.** $\frac{1}{2}\left(\operatorname{cosec}\frac{\alpha}{2}\right)\tan^{-1}\left(\left(\frac{x^2-1}{2x}\right)\operatorname{cosec}\frac{\alpha}{2}\right)$ **6.** 3

7. $\frac{1}{24}\ln\left|\frac{(4+3\sin x+3\cos x)}{(4-3\sin x-3\cos x)}\right|+C$ **8.** $\frac{1}{2}(\sin 2\theta)\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln(\sec 2\theta)+C$

EXERCISE (JM)

1. 1 **2.** 3 **3.** 2 **4.** 2 **5.** 3 **6.** 3 **7.** 1 **8.** 1 or 3
9. 4 **10.** 2 **11.** 2 **12.** 3 **13.** 4 **14.** 3

EXERCISE (JA)

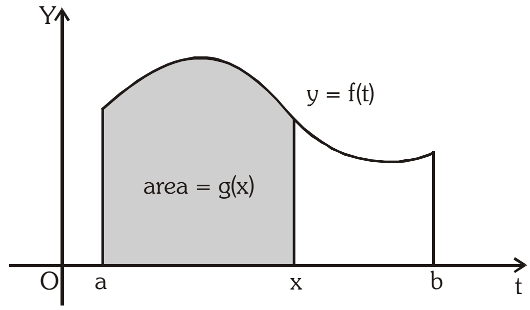
1. D **2.** A **3.** D **4.** C **5.** C

DEFINITE INTEGRATION

A definite integral is denoted by $\int_a^b f(x)dx$ which represent the algebraic area bounded by the curve y = f(x), the ordinates x = a, x = b and the x axis.

1. THE FUNDAMENTAL THEOREM OF CALCULUS :

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.



The Fundamental Theorem of Calculus, Part 1

If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b), and g'(x) = f(x).

The Fundamental Theorem of Calculus, Part 2

If f is continuous on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that F '= f.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation f(x) = 0 has atleast one root lying in (a, b) provided f is a continuous function in (a,b).

2. PROPERTIES OF DEFINITE INTEGRAL :

(a) $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ provided f is same

(b) $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a,b]. This property is to be used when f is piecewise continuous in (a, b).

Illustration 1 : If $f(x) = \begin{cases} x^2, & 0 < x < 2 \\ 3x-4, & 2 \le x < 3 \end{cases}$ then evaluate $\int_0^3 f(x)dx$

Solution : $\int_0^3 f(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^2 x^2dx + \int_2^3 (3x-4)dx$

$$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{3x^2}{2} - 4x\right)_2^3 = \frac{8}{3} + \frac{27}{2} - 12 - 6 + 8 = 37/6$$

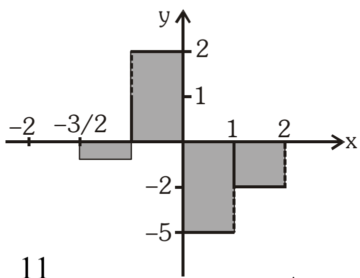
Ans.

Illustration 2 : If $f(x) = \begin{cases} 3[x] - 5\dfrac{|x|}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$ then $\int_{-3/2}^{2} f(x)dx$ is equal to ([.] denotes the greatest integer function)

(A) $-\frac{11}{2}$ (B) $-\frac{7}{2}$ (C) –6 (D) $-\frac{17}{2}$

Solution : $3[x] - 5\frac{|x|}{x} = 3[x] - 5$, if $x > 0$

$= 3[x] + 5$, if $x < 0$



$$\Rightarrow \int_{-3/2}^{2} f(x)dx = \int_{-3/2}^{-1} (-1)dx + \int_{-1}^{0} (2)dx + \int_0^1 (-5)dx + \int_1^2 (-2)dx$$

$$= -1\left(-1 + \frac{3}{2}\right) + 2(1) + 1(-5) + (-2) = -\frac{1}{2} + 2 - 5 - 2 = -\frac{11}{2}$$

Ans. (A)

Illustration 3 : The value of $\int_1^2 (x^{[x^2]} + [x^2]^x)dx$, where [.] denotes the greatest integer function, is equal to -

(A) $\frac{5}{4} + \sqrt{3} + (2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(B) $\frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(C) $\frac{5}{4} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(9 - 3^{\sqrt{3}})$

(D) none of these

Solution : We have, $I=\int_1^2 (x^{[x^2]}+[x^2]^x)dx = \int_1^{\sqrt{2}}(x+1)dx+\int_{\sqrt{2}}^{\sqrt{3}}(x^2+2^x)dx+\int_{\sqrt{3}}^{2}(x^3+3^x)dx$

$$=\left(\frac{x^2}{2}+x\right)_1^{\sqrt{2}}+\left(\frac{x^3}{3}+\frac{2^x}{\log 2}\right)_{\sqrt{2}}^{\sqrt{3}}+\left(\frac{x^4}{4}+\frac{3^x}{\log 3}\right)_{\sqrt{3}}^{2}$$

$$=\frac{5}{4}+\sqrt{3}+\frac{\sqrt{2}}{3}+\frac{1}{\log 2}(2^{\sqrt{3}}-2^{\sqrt{2}})+\frac{1}{\log 3}(3^2-3^{\sqrt{3}})$$ **Ans. (B)**

Illustration 4 : Evaluate : $\int_{-10}^{20}[\cot^{-1}x]dx$. Here [.] is the greatest integer function.

Solution : $I=\int_{-10}^{20}[\cot^{-1}x]dx$, we know $\cot^{-1}x\in(0,\pi)\ \forall\ x\in R$

Thus $[\cot^{-1}x]=\begin{cases}3, & x\in(-\infty,\cot 3)\\ 2, & x\in(\cot 3,\cot 2)\\ 1, & x\in(\cot 2,\cot 1)\\ 0 & x\in(\cot 1,\infty)\end{cases}$

Hence $I=\int_{-10}^{\cot 3}3dx+\int_{\cot 3}^{\cot 2}2dx+\int_{\cot 2}^{\cot 1}1dx+\int_{\cot 1}^{20}0dx = 30+\cot 1+\cot 2+\cot 3$ **Ans.**

Do yourself -1 :
Evaluate :

(i) $\int_0^3|x^2-x-2|dx$

(ii) $\int_0^4\{x\}dx$, where {.} denotes fractional part of x.

(iii) $\int_0^{\pi/2}|\sin x-\cos x|dx$

(iv) If $f(x)=\begin{cases}2 & 0\le x\le 1\\ x+[x] & 1\le x<3\end{cases}$, where [.] denotes the greatest integer function. Evaluate $\int_0^2 f(x)dx$

(d) $\int_{-a}^{a}f(x)\,dx=\int_0^a[f(x)+f(-x)]dx=\begin{bmatrix}0 & ;\text{if } f(x) \text{ is an odd function}\\ 2\int_0^a f(x)dx & ;\text{if } f(x) \text{ is an even function}\end{bmatrix}$

Illustration 5 : Evaluate $\int_{-1/2}^{1/2}\cos x\,\ell n\left(\frac{1+x}{1-x}\right)dx$

Solution : $f(-x)=\cos(-x)\,\ell n\left(\frac{1-x}{1+x}\right)=-\cos\,\ell n\left(\frac{1+x}{1-x}\right)=-f(x)$

$\Rightarrow$ f(x) is odd

Hence, the value of the given integral = 0. **Ans.**

Illustration 6 : If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x)+f''(x))dx$

(A) 1 (B) –1 (C) 2 (D) none of these

Solution : As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$

$\Rightarrow$ $f(-x) = -f(x)$ $\Rightarrow$ f(x) is odd

$\Rightarrow$ f'(x) is even $\Rightarrow$ f''(x) is odd

Thus, f(x) + f''(x) is odd function let,

$\phi(x) = (x^2 + 1).\{f(x) + f''(x)\}$

$\Rightarrow$ $\phi(-x) = -\phi(x)$

i.e. $\phi(x)$ is odd

$\therefore$ $\int_{-\pi/2}^{\pi/2} \phi(x)dx = 0$ **Ans. (D)**

Do yourself -2 :

Evaluate :

(i) $\int_{-\pi/2}^{\pi/2} (x^2\sin^3 x + \cos x)dx$ **(ii)** $\int_{-\pi/2}^{\pi/2} \ell n\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]d\theta$

(e) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Illustration 7 : If f, g, h be continuous functions on [0, a] such that f(a – x) = –f(x), g(a – x) = g(x) and 3h(x) – 4h(a – x) = 5, then prove that $\int_0^a f(x)g(x)h(x)dx = 0$

Solution : $I = \int_0^a f(x)g(x)h(x)dx = \int_0^a f(a-x)g(a-x)h(a-x)dx = -\int_0^a f(x)g(x)h(a-x)dx$

7I = 3I + 4I

$= \int_0^a f(x)g(x)\{3h(x)-4h(a-x)\}\,dx = 5\int_0^a f(x)g(x)dx = 0$

(since f(a – x) g(a – x) = –f(x)g(x))

$\Rightarrow$ I = 0 **Ans.**

Illustration 8 : Evaluate $\int_{-\pi}^{\pi} \frac{x \sin x}{e^x + 1} dx$

Solution : $I = \int_{-\pi}^{0} \frac{x \sin x}{e^x + 1} dx + \int_{0}^{\pi} \frac{x \sin x}{e^x + 1} dx = I_1 + I_2$

where $I_1 = \int_{-\pi}^{0} \frac{x \sin x}{e^x + 1} dx$

Put $x = -t \Rightarrow dx = -\,dt$

$$\Rightarrow \quad I_1 = \int_{\pi}^{0} \frac{(-t)\sin(-t)(-dt)}{e^{-t} + 1} = \int_{0}^{\pi} \frac{t \sin t\, dt}{e^{-t} + 1} = \int_{0}^{\pi} \frac{e^t t \sin t\, dt}{e^t + 1} = \int_{0}^{\pi} \frac{e^x x \sin x\, dx}{e^x + 1}$$

Hence $I = I_1 + I_2 = \int_{0}^{\pi} \frac{e^x x \sin x}{e^x + 1} dx + \int_{0}^{\pi} \frac{x \sin x}{e^x + 1} dx$

$$I = \int_{0}^{\pi} x \sin x dx = \int_{0}^{\pi} (\pi - x) \sin(\pi - x) dx = \pi \int_{0}^{\pi} \sin x dx - I$$

$$\Rightarrow \quad 2I = \pi \int_{0}^{\pi} \sin x dx = \pi |-\cos x|_0^{\pi} = 2\pi \Rightarrow I = \pi$$ **Ans.**

Illustration 9 : Evaluate $\int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{6(1-x)} + 1]}$

Solution : Let $I = \int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{6(1-x)} + 1]}$

Also $I = \int_{0}^{2} \frac{dx}{(17 + 8x - 4x^2)[e^{-6(1-x)} + 1]}$ $\left[\because \int_{0}^{a} f(x)dx = \int_{0}^{a} f(a - x)dx\right]$

Adding, we get

$$2I = \int_{0}^{2} \frac{1}{17 + 8x - 4x^2} \left(\frac{1}{e^{6(1-x)} + 1} + \frac{1}{e^{-6(1-x)} + 1} \right) dx$$

$$= \int_{0}^{2} \frac{1}{17 + 8x - 4x^2} dx = -\frac{1}{4} \int_{0}^{2} \frac{dx}{x^2 - 2x - 17/4}$$

$$= -\frac{1}{4} \int_{0}^{2} \frac{dx}{(x-1)^2 - 21/4} = -\frac{1}{4} \times \frac{1}{2 \times \frac{\sqrt{21}}{2}} \left[\log \left| \frac{x - 1 - \frac{\sqrt{21}}{2}}{x - 1 + \frac{\sqrt{21}}{2}} \right| \right]_0^2$$

$$= -\frac{1}{4\sqrt{21}} \left[\log \left| \frac{2x - 2 - \sqrt{21}}{2x - 2 + \sqrt{21}} \right| \right]_0^2 \Rightarrow I = -\frac{1}{8\sqrt{21}} \left[\log \left| \frac{2 - \sqrt{21}}{2 + \sqrt{21}} \right| - \log \left| \frac{2 + \sqrt{21}}{\sqrt{21} - 2} \right| \right]$$

$$= -\frac{1}{4\sqrt{21}} \left[\log \left| \frac{\sqrt{21} - 2}{2 + \sqrt{21}} \right| \right]$$ **Ans.**

Illustration 10 : $\int_0^1 \cot^{-1}(1-x+x^2)dx$ equals -

(A) $\frac{\pi}{2}+\log 2$ (B) $\frac{\pi}{2}-\log 2$ (C) $\pi-\log 2$ (D) none of these

Solution : $I=\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=\int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$

$=\int_0^1 [\tan^{-1}x+\tan^{-1}(1-x)]\ dx=\int_0^1 \tan^{-1}x\ dx+\int_0^1 \tan^{-1}(1-x)\ dx$

$=2\int_0^1 \tan^{-1}x\ dx=2\left[x\ \tan^{-1}x-\frac{1}{2}\log(1+x^2)\right]_0^1=2\frac{\pi}{4}-\log 2=\frac{\pi}{2}-\log 2$ **Ans. (B)**

Illustration 11 : $\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$

Solution : $I=\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$ .....(i)

$I=\int_0^{\pi/2}\frac{a\sin(\pi/2-x)+b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx=\int_0^{\pi/2}\frac{a\cos x+b\sin x}{\sin x+\cos x}dx$ .....(ii)

$\therefore\ \ 2I=\int_0^{\pi/2}\frac{(a+b)(\sin x+\cos x)}{\sin x+\cos x}dx=\int_0^{\pi/2}(a+b)dx=(a+b)\pi/2 \Rightarrow I=(a+b)\pi/4$ **Ans.**

Illustration 12 : $\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx$ equals -

(A) 2 (B) π (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Solution : $I=\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx=\int_0^{\pi/2}\frac{2^{\sin(\pi/2-x)}}{2^{\sin x(\pi/2-x)}+2^{\cos(\pi/2-x)}}dx=\int_0^{\pi/2}\frac{2^{\cos x}}{2^{\cos x}+2^{\sin x}}dx$

$2I=\int_0^{\pi/2}dx=\frac{\pi}{2}\quad\Rightarrow\quad I=\frac{\pi}{4}$ **Ans. (C)**

Do yourself - 3 :

Evaluate :

(i) $\int_1^5 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}dx$ **(ii)** $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\tan^5 x}$

(f) $\int_0^{2a} f(x)dx=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx=\begin{cases}2\int_0^a f(x)\,dx\ ; & \text{if } f(2a-x)=f(x)\\ 0\ ; & \text{if } f(2a-x)=-f(x)\end{cases}$

Illustration 13 : Evaluate $\int_0^{\pi}\frac{xdx}{1+\cos^2 x}$

Solution : Let I = $\int_0^{\pi}\frac{xdx}{1+\cos^2 x}=\int_0^{\pi}\frac{(\pi-x)dx}{1+\cos^2(\pi-x)}=\int_0^{\pi}\frac{\pi dx}{1+\cos^2 x}-I$

$\Rightarrow\ 2I=\int_0^{\pi}\frac{\pi dx}{1+\cos^2 x}=2\pi\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}=2\pi\int_0^{\pi/2}\frac{\sec^2 xdx}{2+\tan^2 x}$

Let tan x = t so that for $x\to 0,\ t\to 0$ and for $x\to\pi/2,\ t\to\infty$. Hence we can write,

$I=\pi\int_0^{\infty}\frac{dt}{2+t^2}=\pi\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^{\infty}=\frac{\pi^2}{2\sqrt{2}}$ **Ans.**

Illustration 14 : Prove that $\int_0^{\pi/2}\log(\sin x)dx=\int_0^{\pi/2}\log(\cos x)dx=-\frac{\pi}{2}\log 2$

Solution : Let $I=\int_0^{\pi/2}\log(\sin x)dx$ .... (i)

then $I=\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx=\int_0^{\pi/2}\log(\cos x)dx$ .... (ii)

adding (i) and (ii), we get

$2I=\int_0^{\pi/2}\log\sin x\,dx+\int_0^{\pi/2}\log\cos x\,dx=\int_0^{\pi/2}(\log\sin x+\log\cos x)dx$

$\Rightarrow\quad 2I=\int_0^{\pi/2}\log(\sin x\cos x)dx=\int_0^{\pi/2}\log\left(\frac{2\sin x\cos x}{2}\right)dx$

$=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx=\int_0^{\pi/2}\log(\sin 2x)dx-\int_0^{\pi/2}(\log 2)dx=\int_0^{\pi/2}\log\sin 2x.dx-(\log 2)(x)_0^{\pi/2}$

$\Rightarrow\quad 2I=\int_0^{\pi/2}\log(\sin 2x)dx-\frac{\pi}{2}\log 2$ .... (iii)

Let $I_1=\int_0^{\pi/2}\log(\sin 2x)dx$, putting 2x = t, we get

$I_1=\int_0^{\pi}\log(\sin t)\frac{dt}{2}=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt=\frac{1}{2}.2\int_0^{\pi/2}\log(\sin t)dt$

$I_1=\int_0^{\pi/2}\log(\sin x)dx$

$\therefore$ (iii) becomes ; $2I=I-\frac{\pi}{2}\log 2$

Hence $\int_0^{\pi/2}\log\sin x\,dx=-\frac{\pi}{2}\log 2$ **Ans.**

Illustration 15 : $\int_0^{\pi/2}(2\log\sin x-\log\sin 2x)dx$ equals -

(A) $\pi \log 2$ (B) $-\pi \log 2$ (C) $(\pi/2)\log 2$ (D) $-(\pi/2)\log 2$

Solution : $I=\int_0^{\pi/2}(2\log\sin x-\log 2\sin x\cos x)dx=\int_0^{\pi/2}(2\log\sin x-\log 2-\log\sin x-\log\cos x)dx$

$$=\int_0^{\pi/2}\log\sin x dx-\int_0^{\pi/2}\log 2dx-\int_0^{\pi/2}\log\cos x dx=-(\pi/2)\log 2$$ **Ans. (D)**

Do yourself -4 :

Evaluate :

(i) $\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{(1+e^x)(1+x^2)}$ **(ii)** $\int_0^{\pi/2}\ell n\left(\sin^2 x\cos x\right)dx$

(iii) $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ **(iv)** $\int_{-\pi/2}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx$

(g) $\int_0^{nT}f(x)dx=n\int_0^T f(x)dx$, $(n\in I)$; where 'T' is the period of the function i.e. $f(T+x)=f(x)$

Note that : $\int_x^{T+x}f(t)dt$ will be independent of x and equal to $\int_0^T f(t)dt$

(h) $\int_{a+nT}^{b+nT}f(x)dx=\int_a^b f(x)dx$ where f(x) is periodic with period T & $n\in I$.

(i) $\int_{mT}^{nT}f(x)dx=(n-m)\int_0^T f(x)dx$, $(n, m\in I)$ if f(x) is periodic with period 'T'.

Illustration 16 : Evaluate $\int_0^{4\pi}|\cos x|dx$

Solution : Note that $|\cos x|$ is a periodic function with period π. Hence the given integral.

$$I=4\int_0^{\pi}|\cos x|dx=4\left[\int_0^{\frac{\pi}{2}}\cos x dx-\int_{\frac{\pi}{2}}^{\pi}\cos x dx\right]=4\left[[\sin x]_0^{\pi/2}-[\sin x]_{\pi/2}^{\pi}\right]=4[1+1]=8$$ **Ans.**

Illustration 17 : Evaluate $\int_{0}^{16\pi/3} |\sin x|\,dx$

Solution : $\int_{0}^{16\pi/3} |\sin x|\,dx = \int_{0}^{5\pi} |\sin x|\,dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x|\,dx = 5\int_{0}^{\pi} |\sin x|\,dx + \int_{0}^{\pi/3} |\sin x|\,dx$

$= 5[-\cos x]_0^{\pi} + [-\cos x]_0^{\pi/3} = 10 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$ **Ans.**

Illustration 18 : Evaluate : $\int_{0}^{2n\pi} [\sin x + \cos x]dx$. Here [.] is the greatest integer function.

Solution : Let $I = \int_{0}^{2n\pi} [\sin x + \cos x]dx = n\int_{0}^{2\pi} [\sin x + \cos x]dx$

($\because$ [sinx + cosx] is periodic function with period 2π]

$$[\sin x + \cos x] = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \frac{3\pi}{4} \\ -1, & \frac{3\pi}{4} < x \le \pi \\ -2, & \pi < x \le \frac{3\pi}{2} \\ -1, & \frac{3\pi}{2} < x \le \frac{7\pi}{4} \\ 0, & \frac{7\pi}{4} < x \le 2\pi \end{cases}$$

Hence $I = n\left[\int_{0}^{\pi/2} 1dx + \int_{\pi/2}^{3\pi/4} 0dx + \int_{3\pi/4}^{\pi} -1dx + \int_{\pi}^{3\pi/2} -2dx + \int_{3\pi/2}^{7\pi/4} -1dx + \int_{7\pi/4}^{2\pi} 0dx\right]$

$I = n\left[\frac{\pi}{2} + 0 - \pi + \frac{3\pi}{4} - 3\pi + 2\pi - \frac{7\pi}{4} + \frac{3\pi}{2} + 0\right] = -n\pi$ **Ans.**

Do yourself -5 :

Evaluate :

(i) $\int_{-1.5}^{10} \{2x\}dx$, where {.} denotes fractional part of x.

(ii) $\int_{20\pi+\frac{\pi}{6}}^{20\pi+\frac{\pi}{3}} (\sin x + \cos x)dx$

3. WALLI'S FORMULA :

If m, n $\in$ N & m, n $\geq$ 2, then

(a) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \dfrac{(n-1)(n-3).....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)}K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) $\sin^n x.\cos^m x\,dx = \dfrac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2}K$

Where $K = \begin{cases} \dfrac{\pi}{2} & \text{if both m and n are even} \\ 1 & \text{otherwise} \end{cases}$

Illustration 19 : $\int_{-\pi/2}^{\pi/2} \sin^4 x \;\cos^6 x \;\, dx =$

(A) $\dfrac{3\pi}{64}$ (B) $\dfrac{3\pi}{572}$ (C) $\dfrac{3\pi}{256}$ (D) $\dfrac{3\pi}{128}$

Solution : $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \;\cos^6 x \;\, dx = 2\int_0^{\pi/2} \sin^4 x \;\cos^6 x.dx = 2\dfrac{(3.1)(5.3.1)}{10.8.6.4.2}.\dfrac{\pi}{2} = \dfrac{3\pi}{256}$ **Ans. (C)**

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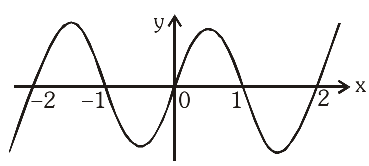
4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x then, $\dfrac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$

Illustration 20 : Find the points of maxima/minima of $\int_0^{x^2} \dfrac{t^2-5t+4}{2+e^t}dt$

Solution : Let $f(x) = \int_0^{x^2} \dfrac{t^2-5t+4}{2+e^t}dt$

$f'(x) = \dfrac{x^4-5x^2+4}{2+e^{x^2}}2x - 0 = \dfrac{(x-1)(x+1)(x-2)(x+2)2x}{2+e^{x^2}}$



Graph of f'(x)

From the wavy curve, it is clear that f'(x) changes its sign at x = ± 2, ±1, 0 and hence the points of maxima are –1, 1 and of the minima are –2, 0, 2.

Illustration 21 : Evaluate $\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log x}dx$

Solution : $\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log x}dx = \frac{1}{\log t^3}.\frac{d}{dt}(t^3) - \frac{1}{\log t^2}.\frac{d}{dt}(t^2) = \frac{3t^2}{3\log t} - \frac{2t}{2\log t} = \frac{t(t-1)}{\log t}$ **Ans.**

Do yourself - 6 :

(i) If $f(x) = \int_{1/x}^{\sqrt{x}} \sin t\, dt$, then find $f\,'(1)$.

(ii) $\int_{\pi/3}^{x}\sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t\,dt = 0$, then evaluate $\frac{dy}{dx}$.

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

An alternative way of describing $\int_a^b f(x)dx$ is that the definite integral $\int_a^b f(x)dx$ is a limiting case of the summation of an infinite series, provided $f(x)$ is continuous on [a,b]

i.e. $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$ where $h = \frac{b-a}{n}$. The converse is also true i.e., if we have an infinite series of the above form, it can be expressed as a definite integral.

Step I : Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

Step II : Then the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty}\frac{1}{n} f\left(\frac{r}{n}\right)$

Step III : Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\sum$ by the sign of $\int$

Step IV : The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Illustration 22 : Evaluate $\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+.........+\frac{1}{6n}\right)$

Solution : Let $S_n = \frac{1}{2n+1}+\frac{1}{2n+2}+.........+\frac{1}{6n} = \sum_{r=1}^{4n}\frac{1}{2n+r} = \sum_{r=1}^{4n}\frac{1}{n}.\frac{1}{2+\left(\frac{r}{n}\right)}$

$\Rightarrow\quad S = \lim_{n\to\infty} S_n = \int_0^4 \frac{dx}{2+x} = [\ell n\,|2+x|]_0^4 = \ell n6 - \ell n2 = \ell n3$ **Ans.**

Illustration 23 : Evaluate $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2}+.......+\frac{1}{49n}\right]$

Solution : Let $p=\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+........+\frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}\right]$

Analyzing the expression with the view of increasing integral value we get the expression in terms of r as

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}}+4\right)^2}=\int_0^1\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4=t$, $\therefore\ \frac{3}{2\sqrt{x}}dx=dt$

Hence $p=\frac{2}{3}\int_4^7\frac{dt}{t^2}=\frac{2}{3}\left[-\frac{1}{t}\right]_4^7=\frac{2}{3}\left(-\frac{1}{7}+\frac{1}{4}\right)=\frac{1}{14}$ **Ans.**

Do yourself - 7 :

Evaluate :

(i) $\lim_{n\to\infty}\left[\frac{1}{n+2.1}+\frac{1}{n+2.2}+\frac{1}{n+2.3}.......\frac{1}{3n}\right]$ **(ii)** $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

6. ESTIMATION OF DEFINITE INTEGRAL :

(a) If f(x) is continuous in [a, b] and it's range in this interval is [m, M], then

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$

Illustration 24 : Prove that $4\le\int_1^3\sqrt{3+x^3}\,dx\le2\sqrt{30}$

Solution : Since the function f(x) = $\sqrt{3+x^3}$ increases monotonically on the interval [1, 3], m = 2, M = $\sqrt{30}$, b – a =2.

Hence, $2.2\le\int_1^3\sqrt{3+x^3}\,dx\le2\sqrt{30}\Rightarrow 4\le\int_1^3\sqrt{3+x^3}\,dx\le2\sqrt{30}$ **Ans.**

(b) If f(x) $\le\phi(x)$ for a $\le x\le$ b then $\int_a^b f(x)dx\le\int_a^b\phi(x)\,dx$

Illustration 25 : Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

Solution : Since $4 - x^2 \ge 4 - x^2 - x^3 \ge 4 - 2x^2 > 0 \ \forall \ x \in [0, 1]$

$\sqrt{4-x^2} \ge \sqrt{4-x^2-x^3} \ge \sqrt{4-2x^2} > 0 \ \forall \ x \in [0, 1]$

$\Rightarrow \quad 0 < \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}} \ \forall \ x \in [0 \ 1]$

$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}} \ \forall x \in [0, 1]$

$\Rightarrow \quad \left[\sin^{-1}\frac{x}{2}\right]_0^1 \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 \Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$ **Ans.**

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

Illustration 26 : Prove that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < 10^{-7}$

Solution : To find $\quad I = \left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ .... (i)

Since $\quad |\sin x| \le 1$ for $x \ge 10$

The inequality $\quad \left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|}$ .... (ii)

also, $\quad 10 \le x \le 19$

$\Rightarrow \quad 1 + x^8 > 10^8$

$\Rightarrow \quad \frac{1}{1+x^8} < \frac{1}{10^8}$ or $\frac{1}{|1+x^8|} < 10^{-8}$ .... (iii)

from (ii) and (iii) ;

$$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8}$$

$$\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < \int_{10}^{19} 10^{-8}dx$$

$\therefore \quad \left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < (19-10).10^{-8} < 10^{-7}$ **Ans.**

Illustration 27 : If $f(x)$ is integrable function such that $|f(x) - f(y)| \le |x^2 - y^2|, \forall x, y \in [a,b]$ then prove that $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \frac{(a-b)^2}{2}$.

Solution : Given, $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \int_a^b \left|\frac{f(x)-f(a)}{x+a}\right|dx$

$$\le \int_a^b \left|\frac{x^2-a^2}{x+a}\right|dx = \int_a^b |x-a|\,dx = \int_a^b (x-a)dx = \frac{(a-b)^2}{2}$$

(d) If $f(x) \geq 0$ on the interval $[a,b]$, then $\int_a^b f(x)dx \ge 0$.

Illustration 28: If $f(x)$ is a continuous function such that $f(x) \ge 0 \ \forall x \in [2,10]$ and $\int_4^8 f(x)\,dx = 0$, then find $f(6)$.

Solution : $f(x)$ is above the x-axis or on the x-axis for all $x \in [2,10]$**.** If $f(x)$ is greater than zero for any sub interval of $[4,8]$, then $\int_4^8 f(x)dx$ must be greater than zero.

But $\int_4^8 f(x)dx = 0 \Rightarrow f(x) = 0 \ \forall x \in [4,8]$

$\Rightarrow f(6) = 0.$

Do yourself - 8 :

(i) Prove that $4 \le \int_1^3 \sqrt{3+x^2}\,dx \le 4\sqrt{3}$

(ii) Prove that $\frac{\pi}{4} \le \int_0^{2\pi} \frac{dx}{5+3\sin x} \le \pi$.

(iii) Show that $\frac{3}{5}\left(2^{1/3}-1\right) \le \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}dx \le 1$

Miscellaneous Illustrations **:**

Illustration 29 : Evaluate : $\int_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)}dx$

Solution : Let $I = \int_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)}dx$ .... (i)

$$= \int_0^{\pi} \frac{(\pi-x)^3 \cos^4(\pi-x)\sin^2(\pi-x)dx}{\pi^2 - 3\pi(\pi-x) + 3(\pi-x)^2} \quad \text{(By. Prop.)}$$

$$= \int_0^{\pi} \frac{(\pi^3 - x^3 - 3\pi^2 x + 3\pi x^2)\cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx \qquad \text{......... (ii)}$$

Adding (i) and (ii) we have

$$2I = \int_0^{\pi} \frac{(\pi^3 - 3\pi^2 x + 3\pi x^2)\cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \cos^4 x \sin^2 x dx \quad \Rightarrow \quad 2I = 2\pi\int_0^{\pi/2} \cos^4 x \sin^2 x dx$$

$$\therefore \quad I = \pi\int_0^{\pi/2} \cos^4 x \sin^2 x \, dx$$

Using walli's formula, we get $I = \pi \frac{(3.1)(1)}{6.4.2}\frac{\pi}{2} = \frac{\pi^2}{32}$ **Ans.**

Illustration 30 : Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x and y with f(0) = 1 and f'(1) = 2 find f(x) and show that $3\int f(x)dx - x(f(x)+2)$ is a constant.

Solution : We have f(x)f(y) + 2 = f(x) + f(y) + f(xy)

Putting x = 1 & y = 1

then f(1)f(1) + 2 = 3f(1)

we get f(1) = 1,2

$f(1) \neq 1$ $\quad (\because f(0) = 1$ & function is injective)

then f(1) = 2

Replacing y by $\frac{1}{x}$ in (1) then

$$f(x)f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad \Rightarrow \quad f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

Hence f(x) is of the type

$f(x) = 1 \pm x^n$

$\because \quad f(1) = 2$

$\therefore \quad f(x) = 1 + x^n$

and $\quad f'(x) = nx^{n-1} \Rightarrow f'(1) = n = 2$

$\quad f(x) = 1 + x^2$

$$\therefore \quad 3\int f(x)dx - x(f(x)+2) = 3\int (1+x^2)dx - x(1+x^2+2)$$

$$= 3\left(x + \frac{x^3}{3}\right) - x(3+x^2) + c = c = \text{constant}$$

Illustration 31 : Evaluate : $\int_{-1}^{1}[x[1+\sin \pi x]+1]dx$, [.] is the greatest integer function.

Solution : Let $I = \int_{-1}^{1}[x[1+\sin \pi x]+1]dx = \int_{-1}^{0}[x[1+\sin \pi x]+1]dx + \int_{0}^{1}[x[1+\sin \pi x]+1]dx$

Now $[1 + \sin\pi x] = 0$ if $-1 < x < 0$

$[1 + \sin\pi x] = 1$ if $0 < x < 1$

$\therefore \quad I = \int_{-1}^{0} 1.dx + \int_{0}^{1}[x+1]dx = 1 + 1\int_{0}^{1} dx = 1 + 1 = 2.$ **Ans.**

Illustration 32 : Find the limit, when $n \to \infty$ of

$$\frac{1}{\sqrt{(2n-1^2)}}+\frac{1}{\sqrt{(4n-2^2)}}+\frac{1}{\sqrt{(6n-3^2)}}+.....+\frac{1}{n}$$

Solution : Let $P = \lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+....+\frac{1}{n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{1(2n)-1^2}}+\frac{1}{\sqrt{2(2n)-2^2}}+\frac{1}{\sqrt{3(2n)-3^2}}+....+\frac{1}{\sqrt{n(2n)-n^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{r(2n)-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n.\sqrt{2\frac{r}{n}-\left(\frac{r}{n}\right)^2}} = \int_{0}^{1}\frac{dx}{\sqrt{(2x-x^2)}}$$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad P = \int_{0}^{1}\frac{2tdt}{t\sqrt{2-t^2}} = \left[2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{\pi}{4}\right)$$

Hence $P = \pi/2$. **Ans.**

Illustration 33 : If $f(x) = \begin{cases} 1-|x|, & |x|\le 1 \\ |x|-1, & |x|>1 \end{cases}$, and $g(x) = f(x-1) + f(x+1)$. Find the value of $\int_{-3}^{5} g(x)\,dx$.

Solution : Given,

$$f(x)=\begin{cases}-x-1, & x<-1\\ 1+x, & -1\le x<0\\ 1-x, & 0\le x\le 1\\ x-1, & x>1\end{cases}\quad;\quad f(x-1)=\begin{cases}-x, & x-1<-1 & \Rightarrow x<0\\ x, & -1\le x-1<0 & \Rightarrow 0\le x<1\\ 2-x, & 0\le x-1\le 1 & \Rightarrow 1\le x\le 2\\ x-2, & x-1>1 & \Rightarrow x>2\end{cases}$$

Similarly

$$f(x+1)=\begin{cases}-x-2, & x+1<-1 & \Rightarrow x<-2\\ x+2, & -1\le x+1<0 & \Rightarrow -2\le x<-1\\ -x, & 0\le x+1\le 1 & \Rightarrow -1\le x\le 0\\ x, & x+1>1 & \Rightarrow x>0\end{cases}$$

$$\Rightarrow \quad g(x)=f(x-1)+f(x+1)=\begin{cases}-2x-2 & x<-2\\ 2, & -2\le x<-1\\ -2x, & -1\le x\le 0\\ 2x, & 0<x<1\\ 2, & 1<x\le 2\\ 2x-2, & 2<x\end{cases}$$

Clearly g(x) is even,

Now $\int_{-3}^{5} g(x)dx = 2\int_{0}^{3} g(x)\,dx + \int_{3}^{5} g(x)\,dx$

$$= 2\left(\int_0^1 2x\ dx+\int_1^2 2dx+\int_2^3 (2x-2)\,dx\right)+\int_3^5 (2x-2)\,dx = 24$$

ANSWERS FOR DO YOURSELF

1 :	(i)	$\frac{31}{6}$	(ii)	2	(iii)	$2(\sqrt{2}-1)$	(iv)	$\frac{9}{2}$
2 :	(i)	2	(ii)	$\pi\ell n2$				
3 :	(i)	2	(ii)	$\pi/12$				
4 :	(i)	$\frac{\pi}{3}$	(ii)	$-\left(\frac{3\pi}{2}\right)\ell n2$	(iii)	0	(iv)	$\frac{4}{3}$
5 :	(i)	$\frac{23}{4}$	(ii)	$\left(\sqrt{3}-1\right)$				
6 :	(i)	$\frac{3}{2}\sin 1$	(ii)	$\frac{dy}{dx}=\frac{-\sqrt{3-\sin^2 x}}{\cos y}$				
7 :	(i)	$\frac{1}{2}\ell n3$	(ii)	$\frac{\pi}{2}$				

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. If $g(x)=\int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals

(A) $g(x)+g(\pi)$ (B) $g(x)-g(\pi)$ (C) $g(x)\,g(\pi)$ (D) $[g(x)/g(\pi)]$ **DI0009**

2. Variable x and y are related by equation $x=\int_0^y \frac{dt}{\sqrt{1+t^2}}$. The value of $\frac{d^2y}{dx^2}$ is equal to

(A) $\frac{y}{\sqrt{1+y^2}}$ (B) y (C) $\frac{2y}{\sqrt{1+y^2}}$ (D) 4y **DI0007**

3. If $\int_0^x f(t)\,dt = x+\int_x^1 t^2\cdot f(t)dt+\frac{\pi}{4}-1$, then the value of the integral $\int_{-1}^1 f(x)\,dx$ is equal to

(A) 0 (B) $\pi/4$ (C) $\pi/2$ (D) π **DI0022**

4. If $I=\int_0^{\pi/2} \ell n(\sin x)dx$ then $\int_{-\pi/4}^{\pi/4} \ell n(\sin x+\cos x)dx$

(A) $\frac{I}{2}$ (B) $\frac{I}{4}$ (C) $\frac{I}{\sqrt{2}}$ (D) I **DI0011**

5. The value of the definite integral $\int_0^{\pi/2} \sin x \sin 2x \sin 3x\, dx$ is equal to :

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{6}$ **DI0006**

6. Value of the definite integral $\int_{-1/2}^{1/2} (\sin^{-1}(3x-4x^3) - \cos^{-1}(4x^3-3x))\,dx$

(A) 0 (B) $-\frac{\pi}{2}$ (C) $\frac{7\pi}{2}$ (D) $\frac{\pi}{2}$ **DI0005**

7. The value of the definite integral $\int_1^e \left((x+1)e^x.\ell nx\right)dx$ is -

(A) e (B) e^{e+1} (C) $e^e(e-1)$ (D) $e^e(e-1)+e$ **DI0012**

8. $\lim_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+.....+\frac{1}{\sqrt{n}\sqrt{4n}}$ is equal to -

(A) 2 (B) 4 (C) $2\left(\sqrt{2}-1\right)$ (D) $2\sqrt{2}-1$ **DI0029**

9. The value of the definite integral $\int_{19}^{37} \left(\{x\}^2+3(\sin 2\pi x)\right)dx$ where $\{x\}$ denotes the fractional part function.

(A) 0 (B) 6 (C) 9 (D) can't be determined
DI0013

10. $\lim\limits_{x\to 0}\dfrac{\int_0^x \sin t^2 dt}{x(1-\cos x)}$ equals -

(A) $\frac{1}{3}$ (B) 2 (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ **DI0024**

11. If g (x) is the inverse of f(x) and f(x) has domain $x \in [1, 5]$, where $f(1)=2$ and $f(5)=10$ then the values of $\int_1^5 f(x)\,dx + \int_2^{10} g(y)\,dy$ equals -

(A) 48 (B) 64 (C) 71 (D) 52 **DI0014**

12. The value of the definite integral $\int_2^4 \left(x(3-x)(4+x)(6-x)(10-x)+\sin x\right)dx$ equals-

(A) cos 2 + cos 4 (B) cos 2 – cos 4 (C) sin 2 + sin 4 (D) sin 2 – sin 4 **DI0003**

13. The true solution set of the inequality, $\sqrt{5x-6-x^2}+\left(\frac{\pi}{2}\int_0^x dz\right) > x\int_0^\pi \sin^2 x\,dx$ is :

(A) R (B) (1,6) (C) (–6,1) (D) (2,3) **DI0025**

14. $\int_{\frac{1}{2}}^{3\frac{1}{2}} \left\{\frac{1}{2}(|x-3|+|1-x|-4)\right\}dx$ equals-

(A) $-\frac{3}{2}$ (B) $\frac{9}{8}$ (C) $\frac{1}{4}$ (D) $\frac{3}{2}$ **DI0001**

Where {.} denotes the fraction part function.

15. Suppose that F(x) is an antiderivative of $f(x)=\frac{\sin x}{x}, x>0$ then $\int_1^3 \frac{\sin 2x}{x}$ can be expressed as -

DI0016

(A) F(6) – F(2) (B) $\frac{1}{2}$(F(6) – F(2)) (C) $\frac{1}{2}$(F(3) – F(1)) (D) 2(F(6) – F(2))

16. $\int_0^{\pi/2} \frac{\cos\theta-\sin\theta}{(1+\cos\theta)(1+\sin\theta)}d\theta$ equals -

(A) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (B) $\cos^{-1}(0)$ (C) $\cos^{-1}(1)$ (D) $\cos^{1}(-1)$ **DI0019**

17. Let $f(x)$ be a continuous function on [0,4] satisfying $f(x)\,f(4-x)=1$.

The value of the definite integral $\int_0^4 \frac{1}{1+f(x)}dx$ equals-

(A) 0 (B) 1 (C) 2 (D) 4 **DI0020**

18. If $g(x)=\int_1^x e^{t^2}dt$ then the value of $\int_3^{x^3} e^{t^2}dt$ equals

(A) $g(x^3)-g(3)$ (B) $g(x^3)+g(3)$ (C) $g(x^3)-3$ (D) $g(x^3)-3g(x)$ **DI0021**

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. The absolute value of $\dfrac{\int_0^{\pi/2}(x\cos x+1)e^{\sin x}dx}{\int_0^{\pi/2}(x\sin x-1)e^{\cos x}dx}$ is equal to -

(A) e (B) πe (C) e/2 (D) π/e **DI0043**

2. Suppose f is continuous and satisfies $f(x)+f(-x)=x^2$ then the integral $\int_{-1}^{1}f(x)dx$ has the value equal to

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{4}{3}$ (D) zero **DI0037**

3. If the value of the integral $\int_1^2 e^{x^2}dx$ is α, then the value of $\int_e^{e^4}\sqrt{\ell nx}\,dx$ is -

(A) $e^4-e-\alpha$ (B) $2e^4-e-\alpha$ (C) $2(e^4-e)-\alpha$ (D) $2e^4-1-\alpha$ **DI0038**

4. The value of $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{r=4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to

(A) $\frac{1}{35}$ (B) $\frac{1}{14}$ (C) $\frac{1}{10}$ (D) $\frac{1}{5}$ **DI0046**

5. The value of $\int_{\pi}^{2\pi}[2\cos x]dx$ where [.] represents the greatest integer function, is -

(A) $-\frac{5\pi}{6}$ (B) $-\frac{\pi}{2}$ (C) $-\pi$ (D) none **DI0031**

6. $\displaystyle\lim_{k\to 0}\frac{1}{k}\int_0^k(1+\sin 2x)^{\frac{1}{x}}dx$ -

(A) 2 (B) 1 (C) e^2 (D) non existent **DI0192**

Paragraph for Question Nos. 7 to 9

Let the function f satisfies

$$f(x)\cdot f'(-x)=f(-x)\cdot f'(x) \text{ for all x and } f(0)=3.$$

7. The value of $f(x)\cdot f(-x)$ for all x, is

(A) 4 (B) 9 (C) 12 (D) 16 **DI0041**

8. $\displaystyle\int_{-51}^{51}\frac{dx}{3+f(x)}$ has the value equal to

(A) 17 (B) 34 (C) 102 (D) 0 **DI0041**

9. Number of roots of $f(x)=0$ in [–2, 2] is

(A) 0 (B) 1 (C) 2 (D) 4 **DI0041**

[MULTIPLE OBJECTIVE TYPE]

10. $\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) is same as $\int_0^{\infty}\frac{dx}{(1+x)(1+x^2)}$ (D) cannot be evaluated **DI0047**

11. Let $u=\int_0^{\infty}\frac{dx}{x^4+7x^2+1}$ & $v=\int_0^{\infty}\frac{x^2dx}{x^4+7x^2+1}$ then -

(A) $v>u$ (B) $6v=\pi$ (C) $3u+2v=5\pi/6$ (D) $u+v=\pi/3$

DI0048

12. Which of the following statement(s) is/are **TRUE ?**

(A) $\int_0^1 \ell nx\,dx=-1$

(B) $\lim_{n\to\infty}\frac{1}{n}ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right)=1+2ln2.$

(C) Let f be a continuous and non-negative function defined on [a.b].

If $\int_a^b f(x)dx=0$ then $f(x)=0\ \forall\ x\in[a,b]$

(D) Let f be a continuous function defined on [a ,b] such that $\int_a^b f(x)dx=0$, then there exists atleast one $c\in(a,b)$ for which $f(c)=0$. **DI0051**

13. Let $f(x)=\begin{cases}x+1, & 0\le x\le 1\\ 2x^2-6x+6, & 1<x\le 2\end{cases}$ and $g(t)=\int_{t-1}^{t}f(x)dx$ for $t\in[1,2]$

Which of the following hold(s) good ?

(A) $f(x)$ is continuous and differentiable in [0,2] (B) g'(t) vanishes for t = 3/2 and 2

(C) g(t) is maximum at t = 3/2 (D) g(t) is minimum at t = 1 **DI0052**

14. Which of the following is/are true ?

(A) $\int_a^{\pi-a} xf(\sin x)dx=\frac{\pi}{2}\int_a^{\pi-a}f(\sin x)dx$ (B) $\int_0^{n\pi}f(\cos^2 x)dx=n\int_0^{\pi}f(\cos^2 x)dx$

(C) $\int_{-a}^{a}f(x^2)dx=2\int_0^a f(x^2)dx$ (D) $\int_0^{b-c}f(x+c)dx=\int_c^b f(x)dx$ **DI0206**

15. Which of the following definite integral reduces to $\frac{\pi}{2}$?

(A) $\int_0^{\pi}\frac{dx}{1+(\sin x)^{\cos x}}$ (B) $\int_0^{\pi/2}\frac{dx}{1+(\tan x)^5}$

(C) $\int_0^{\infty}\frac{x^2+1}{x^4-x^2+1}dx$ (D) $\int_0^{\pi/2}\left(\ell n(\sec x)\right)\left(e^{\ell n(\ell n2)}\right)^{-1}dx$ **DI0207**

16. Let $A=\int_1^e \log^2(x)\,dx$, then -

(A) $A > e-1$ (B) $A < e-1$ (C) $A > \frac{e-1}{2}$ (D) $A < \log^2 2+(e-2)$

DI0193

17. The value of $\int_0^{\pi}\left(\sqrt[2015]{\cos x}+\sqrt[2015]{\sin x}+\sqrt[2015]{\tan x}\right)dx$ is equal to-

(A) 0 (B) $\int_{1/2}^{2}\frac{\ell nx}{1+x^2}dx$ (C) $2\int_0^{\pi/2}(\sin x)^{\frac{1}{2015}}dx$ (D) $2\int_0^{\pi/2}(\cos x)^{\frac{1}{2015}}dx$

DI0194

18. $650\int_0^2 x(2-x)^{24}\,dx$ is divisible by -

(A) 2^{25} (B) 2^{26} (C) 2^{27} (D) 2^{28} **DI0195**

19. Let $f(x) = 2014\tan^{2015}x + 2014\tan^{2013}x - 2010\tan^{2011}x - 2010\tan^{2009}x$ for all $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} x f(x)dx=\frac{1}{2011.2013}$ (B) $\int_0^{\pi/4} f(x)dx=0$

(C) $\int_0^{\pi/4} x f(x)dx=\frac{2}{2011.2013}$ (D) $\int_0^{\pi/4} f(x)dx=1$ **DI0196**

EXERCISE (S-1)

1. Evaluate : **(i)** $\int_0^1 e^{\ell n\tan^{-1}x}.\sin^{-1}(\cos x).dx$ **DI0058** **(ii)** $\int_{1/3}^{3}\frac{\sin^{-1}\frac{x}{\sqrt{1+x^2}}}{x}dx$ **DI0059**

2. Evaluate : $\int_0^1\frac{\sin^{-1}\sqrt{x}}{\sqrt{x\,(1-x)}}dx$ **DI0055**

3. Evaluate : $\int_0^{\pi/2}\frac{1+2\cos x}{(2+\cos x)^2}dx$ **DI0057**

4. Evaluate : $\int_0^{\pi/2} e^x\left\{\cos(\sin x)\cos^2\frac{x}{2}+\sin(\sin x)\sin^2\frac{x}{2}\right\}dx$ **DI0069**

5. Evaluate : $\int_1^e\left\{(1+x)e^x+(1-x)e^{-x}\right\}\ell nx\,dx$ **DI0070**

6. If $P=\int_0^{\infty}\frac{x^2}{1+x^4}dx; Q=\int_0^{\infty}\frac{x\,dx}{1+x^4}$ and $R=\int_0^{\infty}\frac{dx}{1+x^4}$, then prove that :

(a) $Q=\frac{\pi}{4}$, **(b)** $P=R$, **(c)** $P-\sqrt{2}Q+R=\frac{\pi}{2\sqrt{2}}$ **DI0056**

7. $\int_1^2\frac{(x^2-1)dx}{x^3.\sqrt{2x^4-2x^2+1}}=\frac{u}{v}$ where u and v are in their lowest form. Find the value of $\frac{(1000)u}{v}$. **DI0071**

8. Evaluate : $\int_0^{\pi/2}\sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}dx$ **DI0060**

9. If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$\int_0^{\pi/2}(\sin x + a\cos x)^3 dx - \dfrac{4a}{\pi-2}\int_0^{\pi/2} x\cos x dx = 2$ then find the value of $1000\left(a_1^2+a_2^2+a_3^2\right)$. **DI0074**

10. Evaluate : $\int_{-\sqrt{2}}^{\sqrt{2}}\dfrac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}dx$ **DI0073**

11. Evaluate : $\int_{-2}^{2}\dfrac{x^2-x}{\sqrt{x^2+4}}dx$ **DI0072**

12. Evaluate : $\int_0^{\pi/4}\dfrac{x\,dx}{\cos x(\cos x+\sin x)}$ **DI0076**

13. Evaluate : $\int_1^{\frac{1+\sqrt{5}}{2}}\dfrac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\dfrac{1}{x}\right)dx$ **DI0063**

14. Evaluate : $\int_0^{\pi/2}\dfrac{a\sin x+b\cos x}{\sin\left(\dfrac{\pi}{4}+x\right)}dx$ **DI0077**

15. Evaluate : $\int_0^{\pi}\dfrac{(2x+3)\sin x}{(1+\cos^2 x)}dx$ **DI0080**

16. Evaluate : $\int_0^{3}\sqrt{\dfrac{x}{3-x}}\,dx$ **DI0079**

17. Evaluate : $\int_0^{\frac{\ln 3}{2}}\dfrac{e^x+1}{e^{2x}+1}dx$ **DI0066**

18. Let $I=\int_0^1\dfrac{2+3x+4x^2}{2\sqrt{1+x+x^2}}dx$. Find the value of I^2. **DI0067**

19. Evaluate : $\int_{-1}^{1}\dfrac{(2x^{332}+x^{998}+4x^{1668}.\sin x^{691})}{1+x^{666}}dx$ **DI0085**

20. **(a)** Let $g(x) = x^c.\ e^{2x}$ & let $f(x)=\int_0^x e^{2t}.(3t^2+1)^{1/2}dt$. For a certain value of 'c', the limit of $\dfrac{f'(x)}{g'(x)}$ as $x\to\infty$ is finite and non-zero. Determine the value of 'c' and the limit. **DI0092**

(b) Find the constants 'a' $(a>0)$ and 'b' such that $\displaystyle\lim_{x\to 0}\dfrac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx-\sin x}=1$. **DI0093**

21. Evaluate : $\displaystyle\lim_{x\to+\infty}\dfrac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}}\dfrac{3t^4+1}{(t-3)(t^2+3)}dt$ **DI0090**

22. Evaluate **(a)** $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ **DI0094**

(b) $\lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{n+1}+\frac{2}{n+2}+......+\frac{3n}{4n}\right]$ **DI0095**

23. Find a positive real valued continuously differentiable functions f on the real line such that for all x

$f^2(x)=\int_0^x\left((f(t))^2+(f'(t))^2\right)dt+e^2$ **DI0091**

24. Let f(x) be a function defined on R such that $f'(x) = f'(3 - x)\ \forall\ x \in [0,3]$ with $f(0) = -32$ and $f(3) = 46$. Then find the value of $\int_0^3 f(x)dx$. **DI0086**

25. Prove the inequalities :

(a) $\frac{\pi}{6}<\int_0^1\frac{dx}{\sqrt{4-x^2-x^3}}<\frac{\pi\sqrt{2}}{8}$ **DI0097** **(b)** $2e^{-1/4}<\int_0^2 e^{x^2-x}dx<2e^2$. **DI0098**

EXERCISE (S-2)

1. Evaluate : $\int_0^1 x(\tan^{-1}x)^2\,dx$ **DI0115**

2. Evaluate : $\int_0^1 x^5\sqrt{\frac{1+x^2}{1-x^2}}\,dx$ **DI0114**

3. Let $A=\int_{3/4}^{4/3}\frac{2x^2+x+1}{x^3+x^2+x+1}dx$, then find the value of e^A. **DI0113**

4. Evaluate : $\int_0^1\frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$ **DI0112**

5. Evaluate : $\int_0^1\frac{1-x}{1+x}\frac{dx}{\sqrt{x+x^2+x^3}}$ **DI0104**

6. Evaluate : $\int_0^{\sqrt{3}}\sin^{-1}\frac{2x}{1+x^2}dx$ **DI0099**

7. Evaluate : $\int_1^{16}\tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ **DI0100**

8. A curve C_1 is defined by : $\frac{dy}{dx}=e^x\cos x$ for $x\in[0,2\pi]$ and passes through the origin. Prove that the roots of the function y = 0 (other than zero) occurs in the ranges $\frac{\pi}{2}<x<\pi$ and $\frac{3\pi}{2}<x<2\pi$. **DI0129**

9. Let $F(x)=\int_{-1}^{x}\sqrt{4+t^2}\,dt$ and $G(x)=\int_x^1\sqrt{4+t^2}\,dt$ then compute the value of (FG)' (0) where dash denotes the derivative. **DI0130**

10. (a) $\lim\limits_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$ **DI0133**

(b) Let $P_n=\sqrt[n]{\frac{(3n)!}{(2n)!}}$ (n= 1,2,3.......), then find $\lim\limits_{n\to\infty}\frac{P_n}{n}$. **DI0134**

11. If $f(x) = x + \sin x$ and I denotes the value of integral $\int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right) dx$ then the value of $\left[\frac{2I}{3}\right]$

(where [.] denotes greatest integer function) **DI0126**

12. Prove the inequalities :

(a) $\frac{1}{3} < \int_{0}^{1} x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$ **DI0135**

(b) $\frac{1}{2} \le \int_{0}^{2} \frac{dx}{2 + x^2} \le \frac{5}{6}$ **DI0136**

13. Evaluate : $\int_{0}^{2\pi} \frac{dx}{2 + \sin 2x}$ **DI0122**

14. Let $I = \int_{0}^{\pi/2} \frac{\cos x + 4}{3\sin x + 4\cos x + 25} dx$ and $J = \int_{0}^{\pi/2} \frac{\sin x + 3}{3\sin x + 4\cos x + 25} dx$.

If $25I = a\pi + b\ln\frac{c}{d}$ where a,b,c and d $\in$N and $\frac{c}{d}$ is not a perfect square of a rational then find the value of (a + b + c + d). **DI0120**

EXERCISE (JM)

1. Let p(x) be a function defined on R such that p'(x) = p'(1 – x), for all x $\in$ [0, 1], p(0) = 1 and $p(1) = 41$. Then $\int_{0}^{1} p(x)\, dx$ equals :- **[AIEEE-2010]**

(1) $\sqrt{41}$ (2) 21 (3) 41 (4) 42 **DI0197**

2. The value of $\int_{0}^{1} \frac{8\log(1+x)}{1+x^2} dx$ is :- **[AIEEE-2011]**

(1) $\frac{\pi}{2}\log 2$ (2) log 2 (3) π log 2 (4) $\frac{\pi}{8}\log 2$ **DI0138**

3. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2]dx$ is :- **[AIEEE-2011]**

(1) $\frac{5}{4}$ (2) 0 (3) $\frac{3}{2}$ (4) $\frac{3}{4}$ **DI0139**

4. If $g(x) = \int_{0}^{x} \cos 4t\, dt$, then $g(x + \pi)$ equals : **[AIEEE-2012]**

(1) $g(x) \cdot g(\pi)$ (2) $\frac{g(x)}{g(\pi)}$ (3) $g(x) + g(\pi)$ (4) $g(x) - g(\pi)$ **DI0141**

5. **Statement-I :** The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$. **[JEE-MAIN-2013]**

Statement-II : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

(1) Statement-I is true, Statement-II is true; Statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true, Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true, Statement-II is false.
(4) Statement-I is false, Statement-II is true. **DI0142**

6. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals : **[JEE-MAIN-2014]**

(1) $\pi - 4$ (2) $\frac{2\pi}{3}-4-4\sqrt{3}$ (3) $4\sqrt{3}-4$ (4) $4\sqrt{3}-4-\frac{\pi}{3}$ **DI0144**

7. The integral $\int_2^4 \frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to : **[JEE-MAIN-2015]**

(1) 1 (2) 6 (3) 2 (4) 4 **DI0145**

8. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to :- **[JEE-MAIN-2016]**

(1) 3 log3 – 2 (2) $\frac{18}{e^4}$ (3) $\frac{27}{e^2}$ (4) $\frac{9}{e^2}$ **DI0198**

9. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{dx}{1+\cos x}$ is equal to :- **[JEE-MAIN-2017]**

(1) –1 (2) –2 (3) 2 (4) 4 **DI0146**

10. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 x}{1+2^x}dx$ is : **[JEE-MAIN-2018]**

(1) $\frac{\pi}{2}$ (2) 4π (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$ **DI0147**

11. If $\int_0^{\frac{\pi}{3}}\frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta = 1-\frac{1}{\sqrt{2}}, (k>0)$, then the value of k is : **[JEE (Main)-2019]**

(1) 2 (2) $\frac{1}{2}$ (3) 4 (4) 1 **DI0199**

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12. Let $I=\int_a^b (x^4-2x^2)dx$. If I is minimum then the ordered pair (a, b) is : **[JEE (Main)-2019]**

(1) $\left(-\sqrt{2},0\right)$ (2) $\left(-\sqrt{2},\sqrt{2}\right)$ (3) $\left(0,\sqrt{2}\right)$ (4) $\left(\sqrt{2},-\sqrt{2}\right)$ **DI0150**

13. The value of $\int_{-\pi/2}^{\pi/2}\frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest integer less than or equal to t, is :

[JEE (Main)-2019]

(1) $\frac{1}{12}(7\pi+5)$ (2) $\frac{3}{10}(4\pi-3)$ (3) $\frac{1}{12}(7\pi-5)$ (4) $\frac{3}{20}(4\pi-3)$ **DI0148**

14. If $\int_0^x f(t)dt = x^2+\int_x^1 t^2 f(t)dt$, then f'(1/2) is : **[JEE (Main)-2019]**

(1) $\frac{6}{25}$ (2) $\frac{24}{25}$ (3) $\frac{18}{25}$ (4) $\frac{4}{5}$ **DI0149**

15. The value of the integral $\int_{-2}^{2}\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$ (where [x] denotes the greatest integer less than or equal to x) is : **[JEE (Main)-2019]**

(1) 4 (2) 4 – sin4 (3) sin 4 (4) 0 **DI0200**

16. $\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+......+\frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to : **[JEE (Main)-2019]**

(1) $\frac{4}{3}(2)^{4/3}$ (2) $\frac{3}{4}(2)^{4/3}-\frac{4}{3}$ (3) $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$ (4) $\frac{4}{3}(2)^{3/4}$ **DI0201**

17. The integral $\int_{\pi/6}^{\pi/3}\sec^{2/3}x\,\text{cosec}^{4/3}x\,dx$ equal to: **[JEE (Main)-2019]**

(1) $3^{7/6}-3^{5/6}$ (2) $3^{5/3}-3^{1/3}$ (3) $3^{4/3}-3^{1/3}$ (4) $3^{5/6}-3^{2/3}$ **DI0202**

EXERCISE (JA)

1. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\ell n(1+t)}{t^4+4}dt$ is **[JEE 2010, 3 (–1)]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$ **DI0203**

2. The value(s) of $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ is (are) **[JEE 2010, 3]**

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$ **DI0204**

3. Let f be a real-valued function defined on the interval (–1,1) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all x∈(–1,1), and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to- **[JEE2010, 5 (–2)]**

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$ **DI0205**

4. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}dx$ is **[JEE 2011, 3 (–1)]**

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$ **DI0156**

5. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **[JEE 2012, 3, (–1)]**

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$ **DI0158**

6. For a ∈ R (the set of all real numbers), a ≠ –1.

$$\lim_{n\to\infty}\frac{(1^a + 2^a + ...+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+......+(na+n)]} = \frac{1}{60}$$

Then a = **[JEE(Advanced) 2013, 3, (–1)]**

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$ **DI0161**

7. Let f : [a, b] → [1, ∞) be a continuous function and let g : ℝ → ℝ be defined as

$$g(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \int_a^x f(t)\,dt & \text{if} \quad a \le x \le b \\ \int_a^b f(t)\,dt & \text{if} \quad x > b \end{cases}.$$

Then

(A) g(x) is continuous but not differentiable at a

(B) g(x) is differentiable on ℝ

(C) g(x) is continuous but not differentiable at b **DI0162**

(D) g(x) is continuous and differentiable at either a or b but not both. **[JEE(Advanced)-2014, 3]**

8. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **[JEE(Advanced)-2014, 3]**

DI0163

9. Let f : [0, 2] → ℝ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for x ∈ [0, 2]. If $F'(x) = f'(x)$ for all x ∈ (0, 2), then F(2) equals -

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4 **DI0165**

[JEE(Advanced)-2014, 3(–1)]

Paragraph For Questions 10 and 11 :

Given that for each $a \in (0,1)$, $\lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$ exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0,1).

10. The value of $g\left(\frac{1}{2}\right)$ is - **[JEE(Advanced)-2014, 3(–1)]**

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$ **DI0166**

11. The value of $g'\left(\frac{1}{2}\right)$ is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0 **DI0166**

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x] & , \quad x \le 2 \\ 0 & , \quad x > 2 \end{cases}$, where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)}dx$, then the value of (4I – 1) is **[JEE 2015, 4M, –0M]**

DI0172

13. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE 2015, 4M, –0M]**

DI0173

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)\,dt$ for all $x \in [-1,2]$ and $G(x) = \int_{-1}^{x} t\left|f(f(t))\right|dt$ for all $x \in [-1,2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE 2015, 4M, –0M]**

DI0174

15. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)dx = 0$ **[JEE 2015, 4M, –0M]**

(C) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)dx = 1$ **DI0176**

16. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}dx$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$ **DI0179**

17. Let f : R → R be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right)$ = 3 and f'(0) = 1.

If $g(x) = \int_{x}^{\frac{\pi}{2}} \left[f'(t)\text{ cosec } t - \cot t \text{ cosec } t\, f(t)\right] dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$ **DI0182**

[JEE(Advanced)-2017, 3]

18. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$, then **[JEE(Advanced)-2017, 4]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$ **DI0184**

19. For each positive integer n, let $y_n = \frac{1}{n}(n+1)(n+2)...(n+n)^{1/n}$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim\limits_{n\to\infty} y_n = L$, then the value of [L] is ______ **[JEE(Advanced)-2018, 3(0)]**

DI0186

20. The value of the integral $\int_{0}^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}} dx$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

DI0187

21. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals _____ **[JEE(Advanced)-2019, 3(0)]**

DI0189

22. For $a \in \mathbb{R}$, $|a| > 1$, let $\lim\limits_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.......+\frac{1}{(an+n)^2}\right)}\right) = 54$. Then the possible value(s) of a is/are : **[JEE(Advanced)-2019, 4(–1)]**

(1) 8 (2) –9 (3) –6 (4) 7 **DI0190**

23. The value of the integral $\int_{0}^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals **[JEE(Advanced)-2019, 3(0)]**

DI0191

ANSWER KEY

DEFINITE INTEGRATION

EXERCISE (O-1)

1. A **2.** B **3.** C **4.** A **5.** D **6.** B **7.** D **8.** A
9. B **10.** D **11.** A **12.** B **13.** D **14.** C **15.** A **16.** C
17. C **18.** A

EXERCISE (O-2)

1. A **2.** B **3.** B **4.** C **5.** B **6.** C **7.** B **8.** A
9. A **10.** A,C **11.** B,C,D **12.** A,C,D **13.** B,C,D **14.** A,C,D **15.** A,D **16.** B,D
17. C,D **18.** A,B **19.** B,C

EXERCISE (S-1)

1. (i) $\frac{\pi^2}{8}-\frac{\pi}{4}(1+\ln 2)+\frac{1}{2}$; (ii) $\frac{\pi \ln 3}{2}$ **2.** $\frac{\pi^2}{4}$ **3.** $\frac{1}{2}$ **4.** $\frac{1}{2}\left[e^{\pi/2}(\cos 1+\sin 1)-1\right]$

5. $e^{1+e}+e^{1-e}+e^{-e}-e^{e}+e-e^{-1}$ **7.** 125 **8.** $\ln 2$ **9.** 5250 **10.** $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$

11. $4\sqrt{2}-4\ln(\sqrt{2}+1)$ **12.** $\frac{\pi}{8}\ln 2$ **13.** $\frac{\pi}{8}\ln 2$ **14.** $\frac{\pi(a+b)}{2\sqrt{2}}$ **15.** $\frac{\pi(\pi+3)}{2}$

16. $\frac{3\pi}{2}$ **17.** $\frac{1}{2}\left[\frac{\pi}{6}+\ln 3-\ln 2\right]$ **18.** 3 **19.** $\frac{\pi+4}{666}$

20. **(a)** c = 1 and $\lim\limits_{x\to\infty}$ will be $\frac{\sqrt{3}}{2}$; **(b)** a = 4 and b = 1 **21.** 13.5

22. **(a)** $2e^{(1/2)(\pi-4)}$; **(b)** $3-\ln 4$ **23.** $f(x)=e^{x+1}$ **24.** 21

EXERCISE (S-2)

1. $\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\ell n 2$ **2.** $\frac{3\pi+8}{24}$ **3.** $\frac{16}{9}$ **4.** $\frac{\pi}{2}$ **5.** $\frac{\pi}{3}$ **6.** $\frac{\pi\sqrt{3}}{3}$

7. $\frac{16\pi}{3}-2\sqrt{3}$ **9.** 0 **10.** **(a)** $\frac{1}{e}$; **(b)** $\frac{27}{4e}$ **11.** 9 **13.** $\frac{2\pi}{\sqrt{3}}$ **14.** 62

EXERCISE (JM)

1. 2 **2.** 3 **3.** 4 **4.** 3,4 **5.** **4** **6.** 4 **7.** **1** **8.** 3
9. 3 **10.** 3 **11.** 1 **12.** 2 **13.** 4 **14.** 2 **15.** 4 **16.** 3

EXERCISE (JA)

1. B **2.** A **3.** B **4.** A **5.** B **6.** B **7.** A,C **8.** 2
9. B **10.** A **11.** D **12.** 0 **13.** 9 **14.** 7 **15.** A,B **16.** A
17. 2 **18.** BONUS **19.** 1 **20.** 2 **21.** 4.00 **22.** 1,2 **23.** 0.50