FUNCTION

1. CARTESIAN PRODUCT OF TWO SETS :

Given two non-empty sets A and B. The cartesian product A × B is the set of all ordered pairs of the form (a, b) where the first entry comes from set A & second comes from set B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

e.g. $A = \{1, 2, 3\}$ $B = \{p, q\}$

$A \times B = \{(1, p), (1, q), (2, p), (2, q), (3, p), (3, q)\}$

Note :

(i) If either A or B is the null set, then A × B will also be empty set, i.e. $A \times B = \phi$

(ii) If $n(A) = p, n(B) = q$, then $n(A \times B) = pq$, where n(X) denotes the number of elements in set X.

(iii) A **Relation** R from set A to B is any subset of A × B. If A R B & $(a, b) \in R$ then b is image of a under R and a is preimage of b under R.

Note : If $n(A) = m$, $n(B) = n$, then number of relations defined from set A to B are $2^{mn} - 1$.

2. FUNCTION :

A relation R from set A to set B is called a function if each element of A is uniquely associated with some element of B. It is denoted by the symbol :

$f : A \to B$ or $A \xrightarrow{f} B$

which reads 'f' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the 'pre-image of b' or 'argument of b under the function f'. We write it as

$b = f(a)$ or $f : a \to b$ or $f : (a, b)$

Thus a function 'f' from set A to set B is subset of A × B in which each a belonging to A appears in one and only one ordered pair belonging to f.

Representation of Function :

(a) Ordered pair : Every function from $A \to B$ satisfies the following conditions :

(i) $f \subset A \times B$ (ii) $\forall\ a \in A$ there exist $b \in B$ and (iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(b) Formula based (uniformly/nonuniformly) :

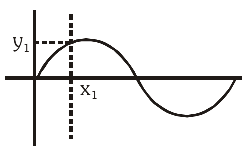
(i) $f : R \to R$, $y = f(x) = 4x$, $f(x) = x^2$ (uniformly defined)

(ii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

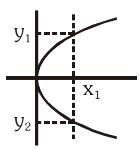
(iii) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation :

If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.



Graph (1)



Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

Every function is a relation but every relation is not necessarily a function.

3. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of f images of all the elements of A is known as the range of f .

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

(a) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined.

(b) For a continuous function, the interval from minimum to maximum value of a function gives the range

(c) It should be noted that range is a subset of co-domain.

Note :

(i) The complete set of all positive real numbers is denoted by R^+.

(ii) The complete set of all negative real numbers is denoted by R^-.

(iii) The complete set of all real numbers other then zero is denoted by R_0.

(iv) The complete set of all integers is denoted by Z.

Illustration 1 : Find the domain of following functions :

(i) $y = \sqrt{5-2x}$ (ii) $y = \dfrac{1}{\sqrt{x-|x|}}$

Solution : (i) $5 - 2x \geq 0 \Rightarrow x \leq \dfrac{5}{2}$ $\therefore$ Domain is $(-\infty, 5/2]$

(ii) $x - |x| > 0 \Rightarrow |x| < x \Rightarrow$ x cannot take any real values $\therefore$ Domain is ϕ

Illustration 2 : Find the range of following functions :

(i) $f(x) = \log_{\sqrt{2}}\left((x-1)^2 + 4\right)$ (ii) $f(x) = 3 - \cos x$

Solution : (i) $f(x) = \log_{\sqrt{2}}\left((x-1)^2 + 4\right)$

$4 \leq (x-1)^2 + 4 < \infty$

$\Rightarrow \log_{\sqrt{2}} 4 \leq \log_{\sqrt{2}} (x-1)^2 + 4 < \infty$

$\Rightarrow 4 \leq \log_{\sqrt{2}} (x-1)^2 + 4 < \infty$

$\therefore$ Range of $f(x) = [4, \infty)$

(ii) $f(x) = 3 - \cos x$

$-1 \leq \cos x \leq 1$

$2 \leq 3 - \cos x \leq 4$

$\therefore$ Range of $f(x) = [2, 4]$

Do yourself - 1 :

(i) Find the domain of following functions :

(a) $y = 1 - \log_{10} x$ (b) $y = \dfrac{1}{\sqrt{x^2 - 4x}}$

(ii) Find the range of the following function :

(a) $\log_4 \left| x + \dfrac{1}{x} \right|$ (b) $f(x) = \sin(3x^2 + 1)$

(c) $f(x) = 2\sin\left(2x + \dfrac{\pi}{4}\right)$ (d) $f(x) = \cos\left(2x + \dfrac{\pi}{4}\right)$

4. IMPORTANT TYPES OF FUNCTIONS :

(a) Polynomial Function :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n. If n is odd, then polynomial is of odd degree, if n is even, then polynomial is of even degree.

Note :

(i) Range of odd degree polynomial is always R.

(ii) Range of even degree polynomial is never R.

(iii) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax,\ a \neq 0$

(iv) $f(x) = ax + b,\ a \neq 0$ is a linear polynomial

(v) $f(x) = c$ is non linear polynomial (its degree is zero)

(vi) $f(x) = 0$ is a polynomial but its degree is **not** defined

(vii) There are two polynomial functions, satisfying the relation ; $f(x) . f(1/x) = f(x) + f(1/x)$. They are : (a) $f(x) = x^n + 1$ & (b) $f(x) = 1 - x^n$, where n is a positive integer.

(b) Algebraic Function :

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) within polynomials.

Example : $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\, y^n + P_1(x)\, y^{n-1} + ... + P_{n-1}(x)\, y + P_n(x) = 0$. Where n is a positive integer and $P_0(x)$, $P_1(x)$....... are Polynomials in x. e.g. $x^3 + y^3 - 3xy = 0$.

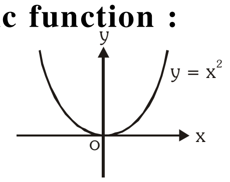
Note :

(i) All polynomial functions are Algebraic but not the converse.

(ii) A function that is not algebraic is called **Transcendental Function.**

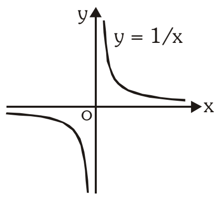
Basic algebraic function :

(i) $y = x^2$



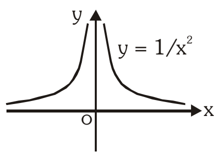
Domain : R
Range : $R^+ \cup \{0\}$ or $[0, \infty)$

(ii) $y = \dfrac{1}{x}$



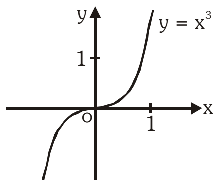
Domain : $R - \{0\}$ or R_0
Range : $R - \{0\}$

(iii) $y = \dfrac{1}{x^2}$



Domain : R_0
Range : R^+ or $(0, \infty)$

(iv) $y = x^3$



Domain : R
Range : R

(c) Rational function :

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials &
$h(x) \neq 0$, **Domain :** $R - \{x \mid h(x) = 0\}$

Any rational function is automatically an algebraic function.

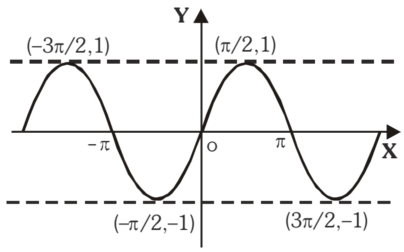
(d) Trigonometric functions :

(i) Sine function

$f(x) = \sin x$

Domain : R

Range : $[-1, 1]$, period 2π

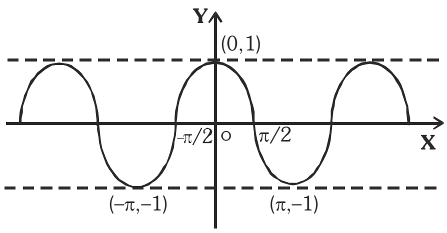


(ii) Cosine function

$f(x) = \cos x$

Domain : R

Range : $[-1, 1]$, period 2π

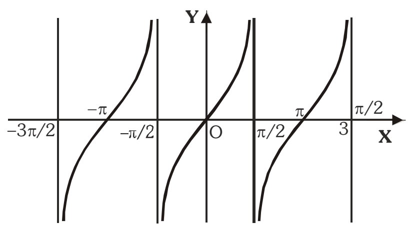


(iii) Tangent function

$f(x) = \tan x$

Domain : $R - \left\{x \mid x = \dfrac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R , period π

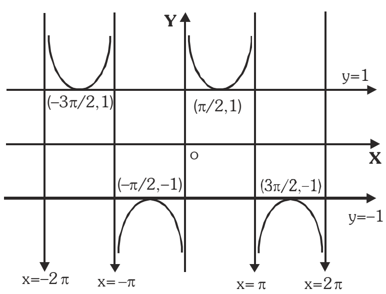


(iv) Cosecant function

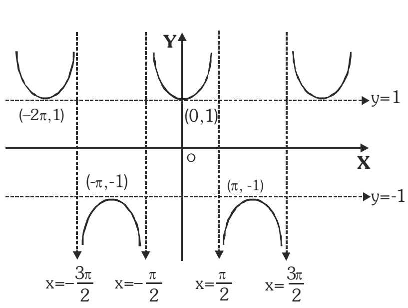
$f(x) = \operatorname{cosec} x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : $R - (-1, 1)$, period 2π



(v) Secant function

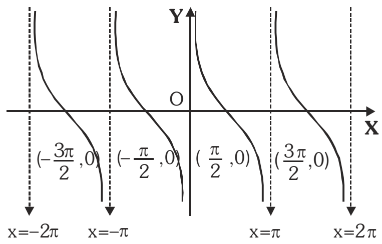


$f(x) = \sec x$

Domain : $R - \{x|x = (2n + 1)\ \pi/2 : n \in I\}$

Range : $R - (-1, 1)$, period 2π

(vi) Cotangent function



$f(x) = \cot x$

Domain : $R - \{x|x = n\pi,\ n \in I\}$

Range : R, period π

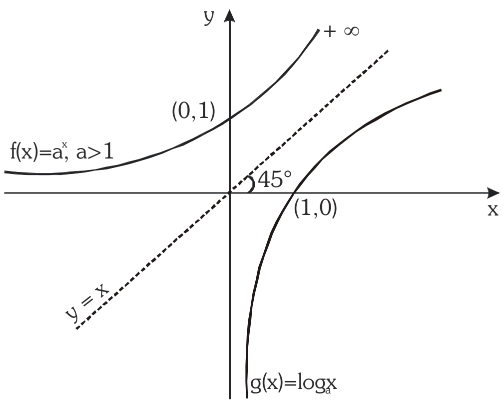
(e) Exponential and Logarithmic Function :

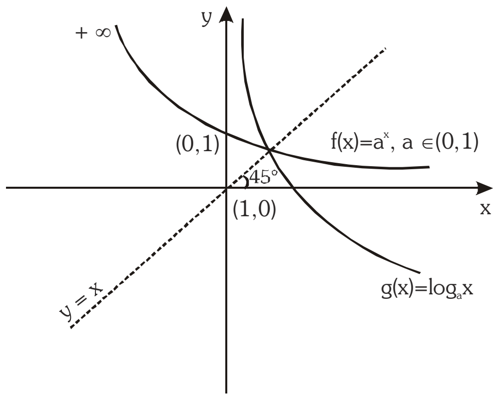
A function $f(x) = a^x (a > 0)$, $a \neq 1$, $x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

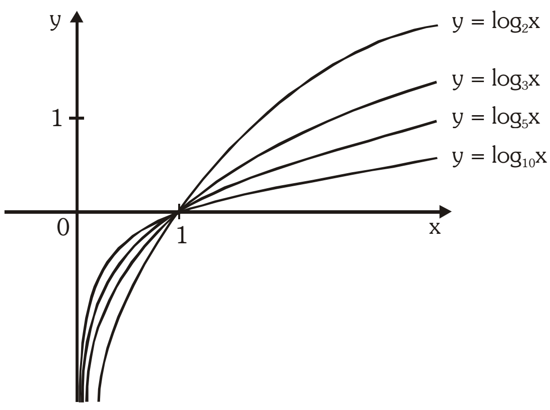
Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line y = x)

Domain of a^x is R **Range** R^+

Domain of $\log_a x$ is R^+ **Range** R







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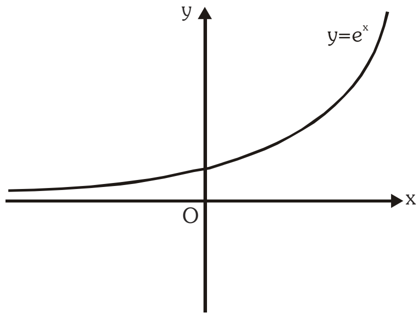
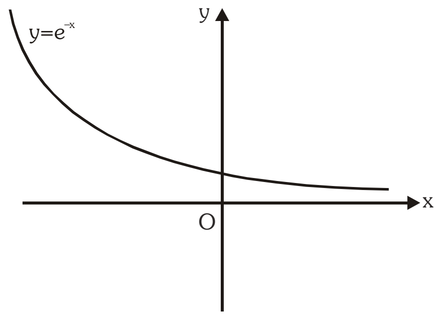
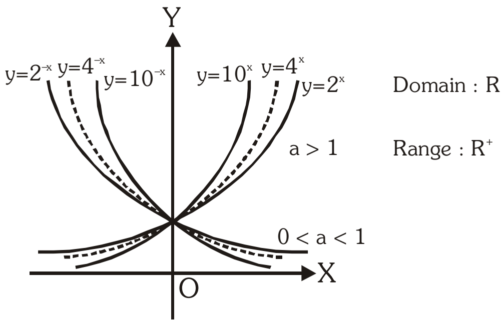






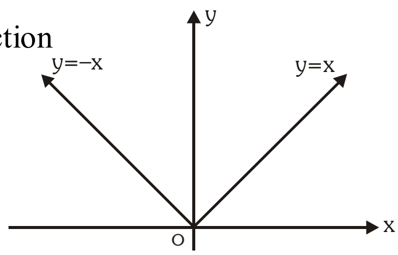
Note-1 : $f(x) = a^{1/x},\ a > 0$ **Domain :** $R - \{0\}$ **Range :** $R^+ - \{1\}$

Note-2 : $f(x) = \log_x a = \dfrac{1}{\log_a x}$ **Domain :** $R^+ - \{1\}$ **Range :** $R - \{0\}$

$(a > 0)\ (a \neq 1)$

(f) Absolute Value Function :

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :



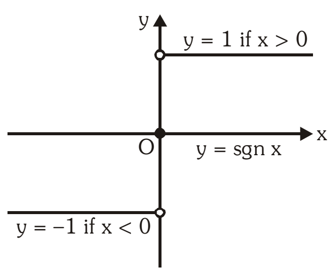
$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For $f(x) = |x|$, domain is R and range is $[0,\infty)$

For $f(x) = \dfrac{1}{|x|}$, domain is $R - \{0\}$ and range is R^+.

(g) Signum Function :

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :



$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{Sgn } x = |x|/x\ ;\ x \neq 0\ ;$

$= 0;\ x = 0$

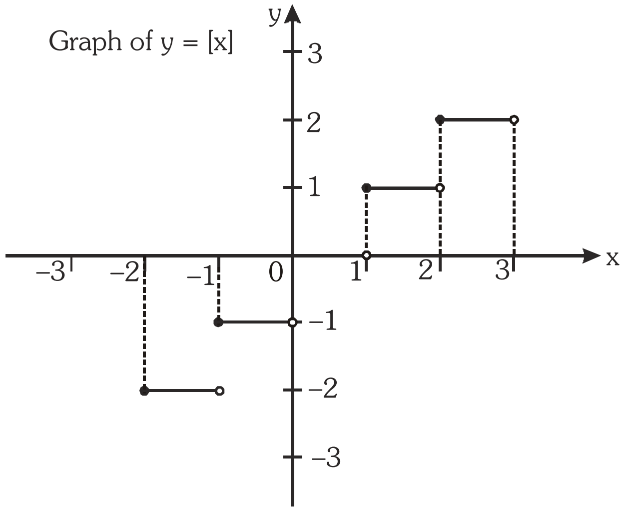
Note : $f(x) = (\text{sgn}(x))x \Rightarrow f(x) = |x|$

Domain : R

Range : $\{-1, 0, 1\}$

(h) Greatest integer or step up function :

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :



x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1

Domain : R

Range : I

Properties of greatest integer function :

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(ii) $[x + m] = [x] + m$, if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Illustration 3 : If y = 2[x] + 3 & y = 3[x – 2] + 5, then find [x + y] where [.] denotes greatest integer function.

Solution : $y = 3[x - 2] + 5 = 3[x] - 1$

so $3[x] - 1 = 2[x] + 3$

$[x] = 4 \Rightarrow 4 \le x < 5$

then y = 11

so x + y will lie in the interval [15, 16)

so [x + y] = 15

Illustration 4 : Find the value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution :

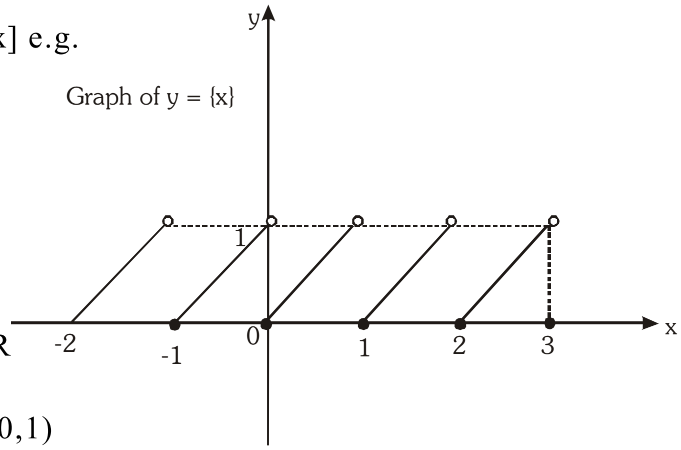
$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+\ldots\ldots$$

$$+\left[\frac{1}{2}+\frac{2499}{1000}\right]+\left[\frac{1}{2}+\frac{2500}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

$= 0 + 1 \times 1000 + 2 \times 1000 + 3 \times 447 = 3000 + 1341 = 4341$ **Ans.**

(i) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$ e.g. the fractional part of the number 2.1 is $2.1-2 = 0.1$ and the fractional part of -3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.



x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

Domain : R

Range : [0,1)

Properties of fractional part function :

(i) $0 \le \{x\} < 1$ (ii) $\{[x]\} = [\{x\}] = 0$ (iii) $\{\{x\}\} = \{x\}$

(iv) $\{x+m\} = \{x\}$, $m \in I$ (v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$

Illustration 5 : Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and {.} denotes fractional part function.

Solution : We are given that, $|2x - 1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$ i.e. $x \le \frac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow \quad 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \Rightarrow \quad 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \dfrac{1-5[x]}{4}$

$\Rightarrow \quad 0 \le \dfrac{1-5[x]}{4} < 1 \Rightarrow 0 \le 1 - 5[x] < 4 \Rightarrow -\dfrac{3}{5} < [x] \le \dfrac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\dfrac{3}{5}$ and $\dfrac{1}{5}$

$\Rightarrow \quad \{x\} = \dfrac{1}{4} \Rightarrow x = \dfrac{1}{4}$ which is less than $\dfrac{1}{2}$, Hence $\dfrac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \dfrac{1}{2}$

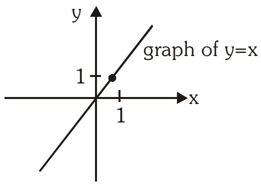
$\Rightarrow \quad 2x - 1 = 3[x] + 2\{x\} \quad \Rightarrow \quad 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad [x] = -1 \Rightarrow -1 \le x < 0$ which is not a solution as $x > \dfrac{1}{2}$

$\Rightarrow \quad x = \dfrac{1}{4}$ is the only solution.

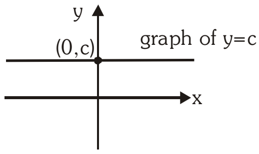
(j) Identity function :

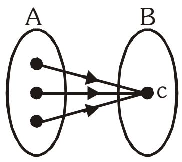
The function $f: A \rightarrow A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.



(k) Constant function :

A function $f: A \rightarrow B$ is said to be a constant function if every element of A has the same f image in B. Thus $f: A \rightarrow B$; $f(x) = c, \ \forall \ x \in A, \ c \in B$ is a constant function. Note that the range of a constant function is a singleton.





Do yourself - 2 :

(i) Let {x} & [x] denotes the fraction and integral part of a real number x respectively, then match the column.

	Column-I		**Column-II**
(A)	$[x^2] \geq 4$	(p)	$x \in [2, 4)$
(B)	$[x]^2 - 5[x] + 6 = 0$	(q)	$x \in (-\infty, -2] \cup [2, \infty)$
(C)	$x = \{x\}$	(r)	$x \in (-\infty, -5)$
(D)	$[x] < -5$	(s)	$x \in \{-2\}$
		(t)	$x \in [0, 1)$

5. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, f + g, f – g, (f. g) & (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x|g(x) = 0\}$

Illustration 6 : Find the domain of the following function :

(i) $y = \log_{(x-4)}(x^2 - 11x + 24)$ (ii) $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

Solution : (i) $y = \log_{(x-4)}(x^2 - 11x + 24)$

Here 'y' would assume real value if,

$x - 4 > 0$ and $\neq 1$, $x^2 - 11x + 24 > 0 \Rightarrow x > 4$ and $\neq 5$, $(x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5$, $x < 3$ or $x > 8 \Rightarrow x > 8 \Rightarrow$ Domain (y) = $(8, \infty)$

(ii) We have $f(x) = \log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

f(x) is defined if $-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1>0$

or if $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)<-1$ or if $\left(1+\frac{1}{\sqrt[4]{x}}\right)>(1/2)^{-1}$

or if $1+\frac{1}{\sqrt[4]{x}}>2$ or if $\frac{1}{\sqrt[4]{x}}>1$ or if $x^{1/4}<1$ or if $0<x<1$

$\therefore$ D(f) = (0, 1)

Illustration 7 : Find the domain $f(x)=\frac{1}{\sqrt{\left|[|x|-5]\right|-11}}$ where [.] denotes greatest integer function.

Solution : $\left|[|x|-5]\right|>11$

so $[|x|-5]>11$ or $[|x|-5]<-11$

$[|x|]>16$ $\quad$ $[|x|]<-6$

$|x|\geq 17$ or $|x|<-6$ (Not Possible)

$\Rightarrow$ $x \leq -17$ or $x \geq 17$

so $x \in (-\infty, -17] \cup [17, \infty)$

Illustration 8 : Find the range of following functions :

(i) $f(x)=\frac{1}{8-3\sin x}$ (ii) $f(x)=\log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

Solution : (i) $f(x)=\frac{1}{8-3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of $f=\left[\frac{1}{11},\frac{1}{5}\right]$

(ii) $f(x)=\log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

$1 \leq 16\sin^2 x + 1 \leq 17$

$\therefore$ $0 \leq \log_2(16\sin^2 x+1) \leq \log_2 17$

$\therefore$ $2-\log_2 17 \leq 2-\log_2(16\sin^2 x+1) \leq 2$

Now consider $0 < 2-\log_2(16\sin^2 x+1) \leq 2$

$\therefore$ $-\infty < \log_{\sqrt{2}}[2-\log_2(16\sin^2 x+1)] \leq \log_{\sqrt{2}} 2 = 2$

$\therefore$ the range is $(-\infty, 2]$

Illustration 9 : Find the range of $f(x)=\dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution : $y=\dfrac{x-[x]}{1+x-[x]}=\dfrac{\{x\}}{1+\{x\}}$

$\therefore\ \dfrac{1}{y}=\dfrac{1}{\{x\}}+1\Rightarrow\dfrac{1}{\{x\}}=\dfrac{1-y}{y}\Rightarrow\{x\}=\dfrac{y}{1-y}$

$0\le\{x\}<1\ \Rightarrow\ 0\le\dfrac{y}{1-y}<1$

Range = [0, 1/2)

Do yourself - 3 :

(i) Find domain of following functions :

(a) $f(x)=\sin\left(\sqrt{1-x^2}\right)+\sqrt{x+2}+\dfrac{1}{\log_{10}^{(x+1)}}$

(b) $f(x)=\sqrt{\dfrac{(2x+1)}{x^3-3x^2+2x}}$

(ii) Find range of following functions :

(a) $f(x)=\log_2\left(\log_{1/2}(x^2+4x+4)\right)$

(b) $f(x)=\dfrac{1}{2-\cos 3x}$

6. EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(a) The domain of f = the domain of g

(b) The co-domain of f = co-domain of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

Illustration 10 : The functions f(x) = log (x – 1) – log (x – 2) and $g(x)=\log\left(\dfrac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) [1, 2] (B) [2, ∞) (C) (2, ∞) (D) (–∞, ∞)

Solution : Since f(x) = log (x – 1) –log (x – 2).

Domain of f(x) is x > 2 or $x\in(2,\infty)$ .....(i)

$g(x)=\log\left(\dfrac{x-1}{x-2}\right)$ is defined if $\dfrac{x-1}{x-2}>0\ \Rightarrow\ x\in(-\infty,1)\cup(2,\infty)$ .....(ii)

From (i) and (ii), $x\in(2,\infty)$. **Ans. (C)**

Do yourself - 4 :

(i) Are the following functions identical ?

(a) $f(x) = \frac{x}{x^2}$ & $\phi(x) = \frac{x^2}{x}$ (b) $f(x) = x$ & $\phi(x) = \sqrt{x^2}$ (c) $f(x) = \log_{10}x^2$ & $\phi(x) = 2\log_{10}|x|$

7. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then f(x, y) is homogeneous function of degree n.

Illustration 11 : Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$ (C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Solution : It is clear that (D) does not have the same degree in each term. **Ans. (D)**

8. BOUNDED FUNCTION :

A function is said to be bounded if there exists a finite M such that $|f(x)| \le M, \forall x \in D_f$.

9. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

Illustration 12 : Which of the following function is implicit function ?

(A) $y = \frac{x^2+e^x+5}{\sqrt{1-\cos^{-1}x}}$ (B) $y = x^2$ (C) $xy - \sin(x+y) = 0$ (D) $y = \frac{x^2\log x}{\sin x}$

Solution : It is clear that in (C) y is not clearly expressed in x. **Ans. (C)**

Do yourself - 5 :

(i) Find the boundness of the function $f(x) = \frac{x^2}{x^4+1}$

(ii) Which of the following function is implicit function ?

(A) $xy - \cos(x+y) = 0$ (B) $y = x^3$

(C) $y = \log(x^2 + x + 1)$ (D) $y = |x|$

(iii) Convert the implicit form into the explicit function :

(a) $xy = 1$ (b) $x^2y = 1$.

10. APPLICATIONS OF FUNCTIONAL RULE :

Illustration 13 : Determine all functions f satisfying the functional relation.

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)} \text{ where } x\in R-\{0,1\}$$

Solution : Given $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=\frac{2}{x}-\frac{2}{1-x}$ ...(i)

Replacing x by $\frac{1}{1-x}$ we obtain

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=2(1-x)-\frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=-2x+\frac{2}{x} \qquad \text{...(ii)}$$

Again replacing x by $\left(1-\frac{1}{x}\right)$ in (i) we obtain

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right)=\frac{2}{1-\frac{1}{x}}-\frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \qquad \text{....(iii)}$$

subtracting (ii) from (i) then

$$f(x)-f\left(1-\frac{1}{x}\right)=2x-\frac{2}{1-x}$$

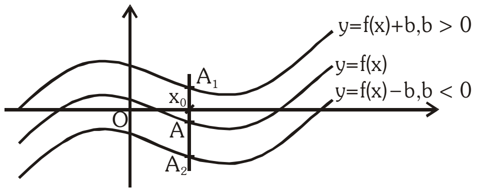
Now adding (iii) and (iv) we get

$$2f(x)=\frac{2x}{x-1}-\frac{2}{1-x}$$

$$\Rightarrow \quad f(x)=\frac{x+1}{x-1}$$

11. BASIC TRANSFORMATIONS ON GRAPHS :

(i) Drawing the graph of y = f(x) + b, b ∈ R, from the known graph of y = f(x)



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y|_{x=x_0} = f(x_0)$.

The corresponding point on f(x) + b would be $f(x_0) + b$.

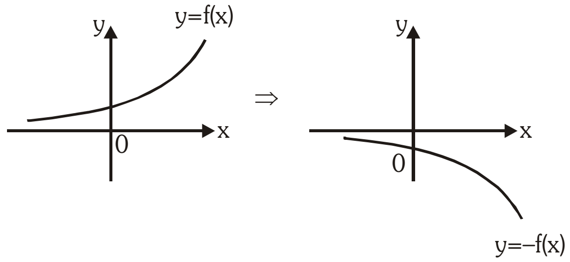
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

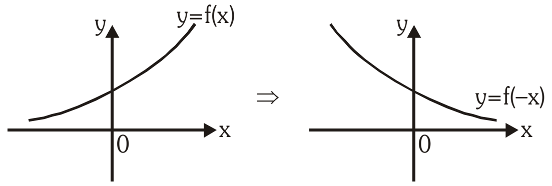
(ii) Drawing the graph of y = –f(x) from the known graph of y = f(x)

To draw y = –f(x), take the image of the curve y = f(x) in the x-axis as plane mirror.



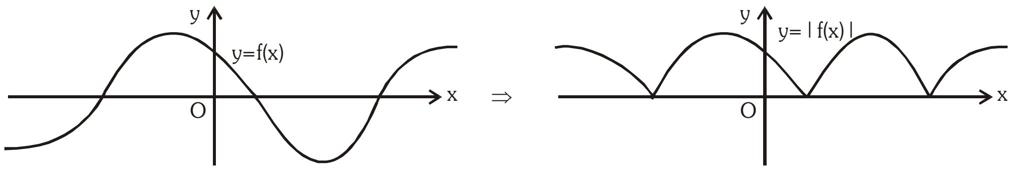
(iii) Drawing the graph of y = f(–x) from the known graph of y = f(x)

To draw y = f(–x), take the image of the curve y = f(x) in the y-axis as plane mirror.



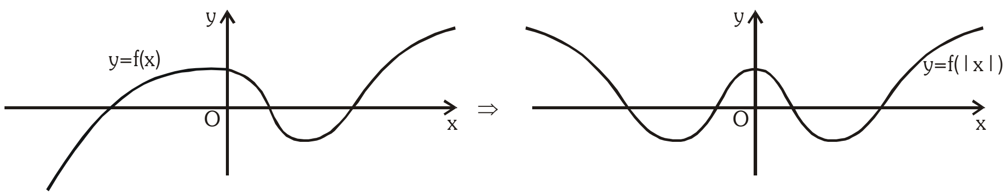
(iv) Drawing the graph of y = |f(x)| from the known graph of y = f(x)

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of f(x) and |f(x)| would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of |f(x)| would be image of y = f(x) in x-axis.



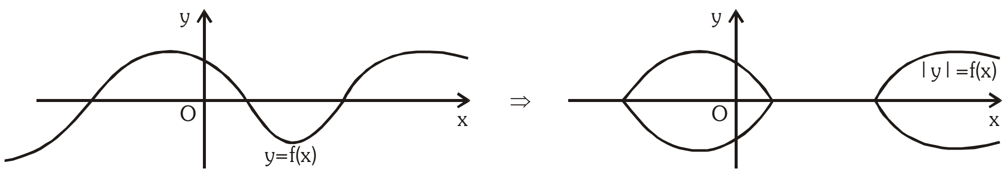
(v) Drawing the graph of y = f(|x|) from the known graph of y = f(x)

It is clear that, $f(|x|)=\begin{cases} f(x), & x \ge 0 \\ f(-x), & x<0 \end{cases}$. Thus f(|x|) would be a even function, graph of f(|x|) and f(x) would be identical in the first and the fourth quadrants (as $x \ge 0$) and as such the graph of f(|x|) would be symmetric about the y-axis (as (|x|) is even).

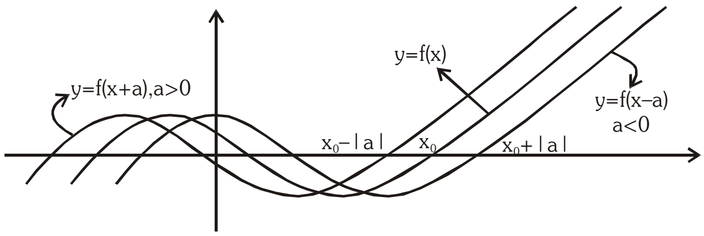


(vi) Drawing the graph of |y| = f(x) from the known graph of y = f(x)

Clearly $|y| \ge 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \ge 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where f(x) is non-negative and will be reflected about the x-axis only in those regions.



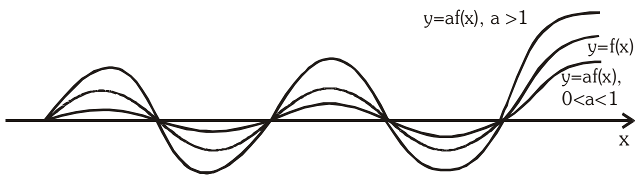
(vii) Drawing the graph of y = f(x + a), a ∈ R from the known graph of y = f(x)



(i) If a > 0, shift the graph of f(x) through 'a' units towards left of f(x).

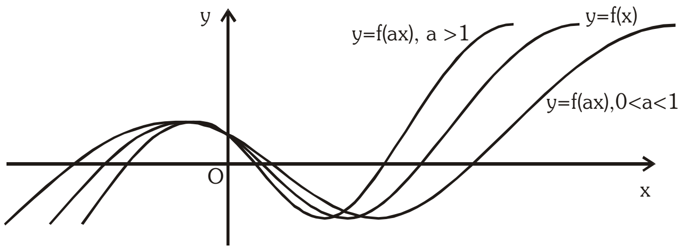
(ii) If a < 0, shift the graph of f(x) through 'a' units towards right of f(x).

(viii) Drawing the graph of y = af(x) from the known graph of y = f(x)



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

(ix) Drawing the graph of y = f(ax) from the known graph of y = f(x).



Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and f (x) will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units along the y-axis.

Note :

(i) A function h(x) is defined as

h(x) = max. {f(x), g(x)} then

$$h(x) = \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & g(x) > f(x) \end{cases}$$

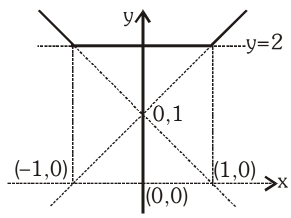
(ii) A function h(x) is defined as

h(x) = min. {f(x), g(x)} then

$$h(x) = \begin{cases} f(x) & f(x) \leq g(x) \\ g(x) & g(x) < f(x) \end{cases}$$

Illustration 14 : Find f(x) = max {1 + x, 1 – x, 2}.

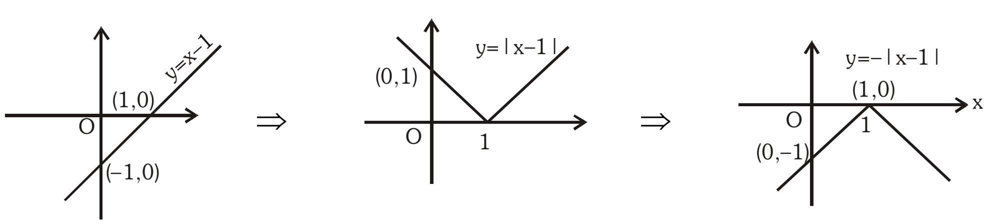
Solution : From the graph it is clear that



$$f(x) = \begin{cases} 1-x\,; & x < -1 \\ 2\;\;\;\;; & -1 \leq x \leq 1 \\ 1+x\,; & x > 1 \end{cases}$$

Illustration 15 : Draw the graph of y = |2 –|x –1||.

Solution :



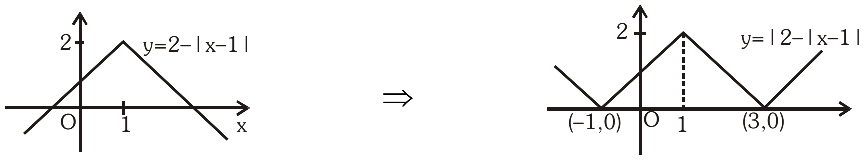
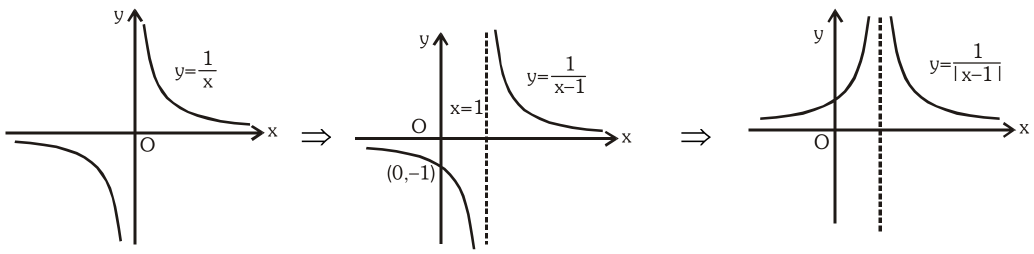


Illustration 16 : Draw the graph of $y = 2-\dfrac{4}{|x-1|}$

Solution :



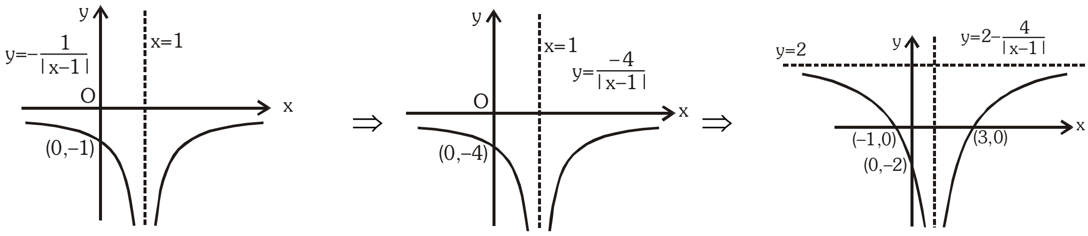
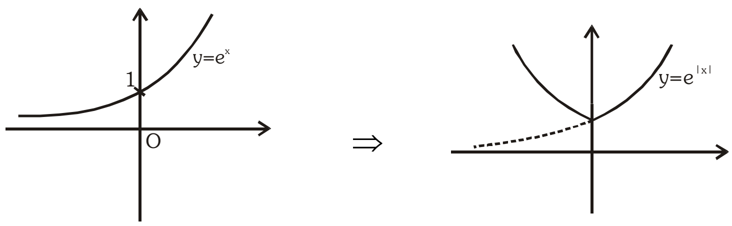


Illustration 17 : Draw the graph of $y = |e^{|x|} - 2|$

Solution :



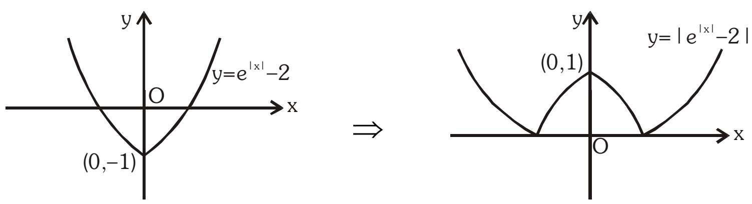
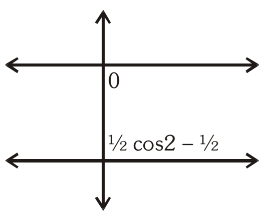


Illustration 18 : Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution : $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$= \frac{1}{2}[\cos(2x+2)+\cos 2]-\frac{1}{2}[\cos(2x+2)+1]$$

$$= \frac{1}{2}\cos 2-\frac{1}{2}<0.$$



Do yourself - 6 :

(i) Draw graph of following functions :

(a) $y = \left|\ell n|x|\right| + 1$ (b) $y = \min.\{x^2 + 1, 3-x\}$

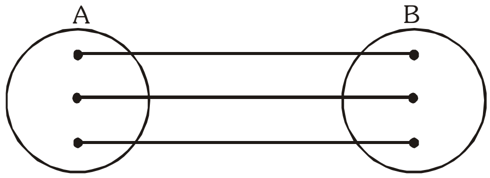
12. CLASSIFICATION OF FUNCTIONS :

One-One Function (Injective mapping) :

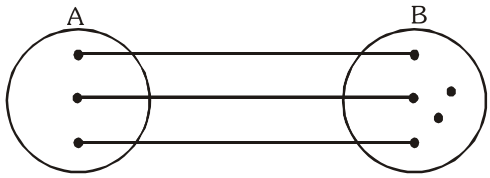
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus there exist $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $\mathbf{f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2}$ **or** $\mathbf{x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)}$.

Diagramatically an injective mapping can be shown as



or

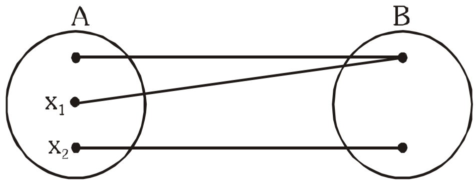


Many-one function (not injective) :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one there exist $x_1, x_2 \in A$, $\mathbf{f(x_1) = f(x_2)}$ **but** $\mathbf{x_1 \neq x_2}$.

Diagramatically a many one mapping can be shown as



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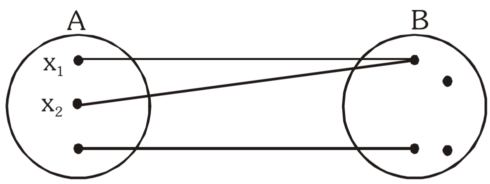


Note :

(i) If a line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

(ii) If any line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(iii) If continuous function $f(x)$ is always increasing or decreasing in whole domain, then $f(x)$ is one-one.

(iv) All linear functions are one-one.

(v) All trigonometric functions in their domain are many one

(vi) All even degree polynomials are many one

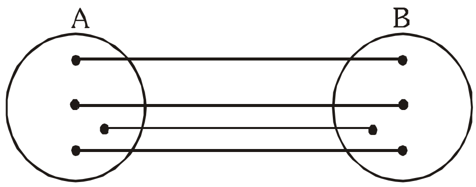
(vii) Linear by Linear is one-one

(viii) Quadratic by quadratic with no common factor is many one.

Onto function (Surjective mapping) :

If the function $f : A \to B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function from A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as



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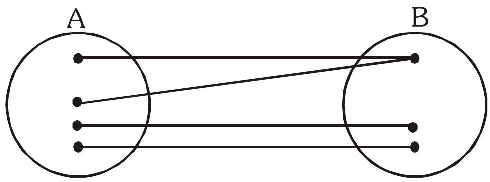
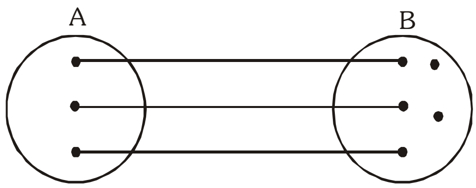


Note that : if range is same as co-domain, then f (x) is onto.

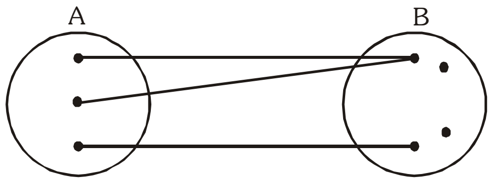
Into function :

If $f: A \rightarrow B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f (x) is into.

Diagramatically into function can be shown as



or



Note :

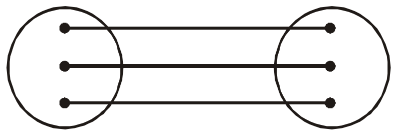
(i) A polynomial function of degree even define from $R \rightarrow R$ will always be into.

(ii) A polynomial function of degree odd defined from $R \rightarrow R$ will always be onto.

(iii) Quadratic by quadratic without any common factor define from $R \rightarrow R$ is always an into function.

Thus a function can be one of these four types :

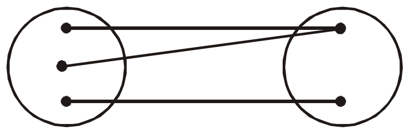
(i) one-one onto (injective & surjective)



(also known as **Bijective** mapping)

(ii) one-one into (injective but not surjective)

(iii) many-one onto (surjective but not injective)



(iv) many-one into (neither surjective nor injective)

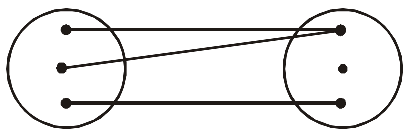
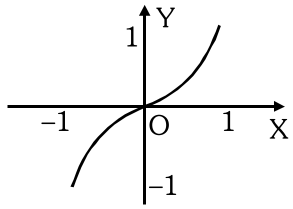


Illustration19 : Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \rightarrow B$. For each of the following functions from A to B, find whether it is bijective or non-bijective.

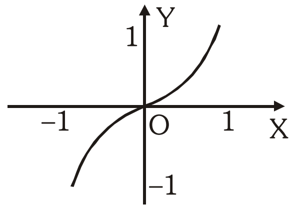
(a) $f(x) = x|x|$ (b) $f(x) = x^3$ (c) $f(x) = \sin\frac{\pi x}{2}$

Solution : (a) $f(x) = x|x| = \begin{cases} -x^2 & , -1 < x < 0 \\ x^2 & , 0 \le x < 1 \end{cases}$,



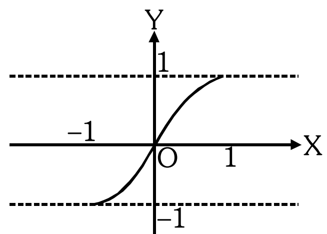
Graphically,
The graph shows f(x) is one-one, as the straight line parallel to x-axis cuts only at one point.
Here, range
$f(x) \in [-1, 1]$
Thus, range = co-domain
Hence, onto.
Therefore, f(x) is one-one onto or (Bijective).

(b) $f(x) = x^3$,



Graphically;
Graph shows f(x) is one-one onto
(i.e. Bijective)
[as explained in above example]

(c) $f(x) = \sin\dfrac{\pi x}{2}$



Graphically;
Which shows f(x) is one-one and onto as range = co-domain.
Therefore, f(x) is bijective.

Illustration 20 : Let $f : N \to I$ be a function defined as $f(x) = x - 1000$. Show that $f(x)$ is an into function.

Solution : Let $f(x) = y = x - 1000 \Rightarrow x = y + 1000 = g(y)$ (say)
here g(y) is defined for each $y \in I$, but $g(y) \notin N$ for $y \le -1000$.
Hence $f(x)$ is into.

Illustration 21 : Let $f : R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is
(A) injective (B) surjective (C) bijective (D) None of these

Solution : We have, $f(x) = x + \sqrt{x^2} = x + |x|$
Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \ne -2$
Also, f is not onto as $f(x) \ge 0 \ \forall \ x \in R$
$\therefore$ range of f = $(0, \infty) \subset R$ **Ans.(D)**

Illustration 22 : Let $f : R \to R$ be a function defined as $f(x) = 2x^3 + 6x^2 + 12x + 3\cos x - 4\sin x$; then f is -
(A) Injective (B) Surjective (C) Bijective (D) Not Surjective

Solution : We have $f(x) = 2x^3 + 6x^2 + 12x + 3\cos x - 4\sin x$

$\Rightarrow f'(x) = 6x^2 - 12x + 12 - 3\sin x - 4\cos x$

$$f'(x) = \underbrace{6(x-1)^2 + 6}_{g(x)} - \underbrace{(3\sin x + 4\cos x)}_{h(x)}$$

range of $g(x) = [6, \infty)$
range of $h(x) = [-5, 5]$
hence $f'(x)$ always lies in the interval $[1, \infty)$
$\Rightarrow f'(x) > 0$

Hence f (x) is increasing i.e. one-one function

Now $x \to \infty \Rightarrow f \to \infty$ & $x \to -\infty \Rightarrow f \to -\infty$ & $f(x)$ is continuous

hence its range is R $\Rightarrow f$ is onto so f is bijective. **Ans. (C)**

Illustration 23 : Let $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$, where $f : R \to R$. Find the value of parameter 'a' so that the given function is one-one.

Solution : $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$

$$f'(x) = \frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2} = \frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x(1-a) + (3-a)$

$g(x)$ will be negative if $4(1-a)^2 + 8(3-a) < 0$

$\Rightarrow \quad 1 + a^2 - 2a + 6 - 2a < 0 \Rightarrow (a-2)^2 + 3 < 0$

which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

Do yourself - 7 :

(i) Is the function $f : N \to N$ (the set of natural numbers) defined by $f(x) = 2x + 3$ surjective ?

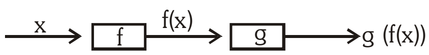
(ii) Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Check whether the functionf(x) is bijective or not.

(iii) A mapping $f : A \to [-1,1]$ defined by $f(x) = \sin x$, $\forall x \in R$, where A is a subset of R (the set of all real numbers) is one-one and onto if A is the interval, then A is belongs to

(A) $[0,2\pi]$ (B) $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ (C) $[-\pi,\pi]$ (D) $[0,\pi]$

13. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION:

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))$ $\forall x \in A$ is called the composite of the two functions f & g.

Diagramatically 

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall x \in A$, f(x) is an element of the domain of 'g' so that we can take its g-image.

Properties of composite functions:

(a) In general composite of functions is not commutative i.e. $gof \neq fog$.

(b) The composition of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composition of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Illustration 24 : If $f(x) = x^2 + 1$, $g(x) = \frac{1}{x-1}$, then find (fog) (x) and (gof) (x).

Solution : Given, $f(x) = x^2 + 1$ .....(1) $\quad g(x) = \frac{1}{x-1}$ ...(2)

Now $(fog)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = f(z)$, where $z = \frac{1}{x-1}$

$= z^2 + 1 \qquad [\because f(x) = x^2 + 1]$

$= \left(\frac{1}{x-1}\right)^2 + 1 = \frac{1}{(x-1)^2} + 1$

Note : Domain of fog(x) is $x \in R - \{1\}$

$(gof)(x) = g(f(x)) = g(x^2 + 1) = g(u)$, where $u = x^2+1$

$= \frac{1}{u-1} = \frac{1}{x^2+1-1} = \frac{1}{x^2}$

Note : Domain of gof(x) is $x \in R - \{0\}$

Illustration 25 : If f be the greatest integer function and g be the modulus function, then

$(gof)\left(-\frac{5}{3}\right) - (fog)\left(-\frac{5}{3}\right) =$

(A) 1 (B) –1 (C) 2 (D) 4

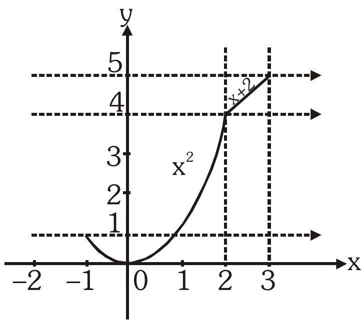
Solution : Given $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right) = g\left\{f\left(\frac{-5}{3}\right)\right\} - f\left\{g\left(\frac{-5}{3}\right)\right\} = g(-2) - f\left(\frac{5}{3}\right) = 2 - 1 = 1$

Ans.(A)

Illustration 26 : Let $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$, find (fog)

Solution : $f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.



It is clear that $g(x) \le 1$; $\forall\ x \in [-1, 1]$

and $1 < g(x) \le 2$; $\forall\ x \in (1, \sqrt{2}]$

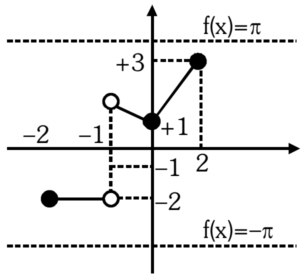
$\Rightarrow \quad f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

Illustration 27: Find the domain and range of h(x) = g(f(x)), where

$$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases} \text{ and } g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$$, [.] denotes the greatest integer function.

Solution :

$$h(x)=g(f(x))=\begin{cases}[f(x)], & -\pi\le f(x)<0\\ \sin(f(x)), & 0\le f(x)\le\pi\end{cases}$$



From graph of f(x), we get

$$h(x)=\begin{cases}[[x]], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$$

$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is $\{-2, -1\}\cup[\sin 3, 1]$

Do yourself - 8 :

(i) $f(x) = x^3 - x$ & g(x) = sin 2x, find

(a) f(f(1)) (b) f(f(–1)) (c) $f\left(g\left(\frac{\pi}{2}\right)\right)$

(d) $f\left(g\left(\frac{\pi}{4}\right)\right)$ (e) g(f(1)) (f) $g\left(g\left(\frac{\pi}{2}\right)\right)$

(ii) If $f(x)=\begin{cases}x+1; & 0\le x<2\\ |x|; & 2\le x<3\end{cases}$, then find fof(x).

14. INVERSE OF A FUNCTION :

Let f : A $\to$ B be a one-one & onto function, then there exists a unique function g : B $\to$ A such that f(x) = y $\Leftrightarrow$ g(y) = x, $\forall$ x $\in$ A & y $\in$ B. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

Properties of inverse function :

(a) The inverse of a bijection is unique.

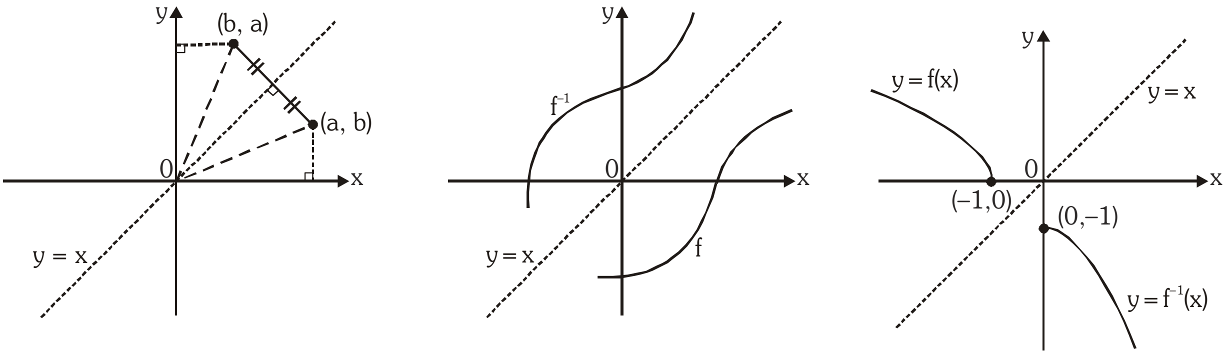
(b) If f : A $\to$ B is a bijection & g : B $\to$ A is the inverse of f, then fog = I_B and gof = I_A, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself.

(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections f : A $\to$ B, g : B $\to$ C then the inverse of gof exists and

$(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(e) Since f(a) = b if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line y = x.



The graph of f^{-1} is obtained by reflecting the graph of f about the line y = x.

Drawing the graph of $y = f^{-1}(x)$ from the known graph of y = f(x)

For drawing the graph of $y = f^{-1}(x)$ take the reflection of y = f(x) about the line y = x. The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph of $y = \sin^{-1}x$. We know that y = f(x) = sinx is invertible if f : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

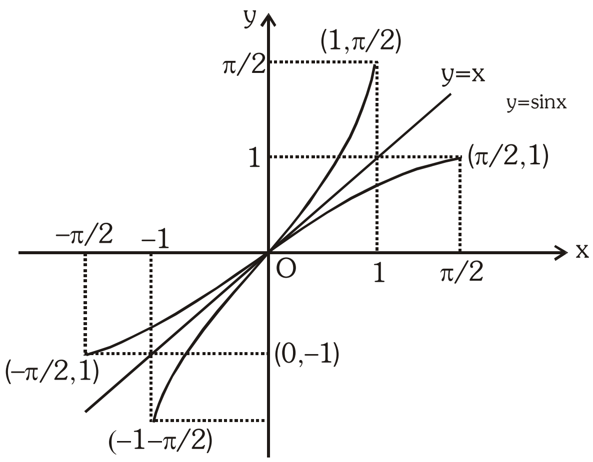


Illustration 28 : Let f : R → R be defined by $f(x) = (e^x - e^{-x})/2$. Is f(x) invertible ? If so, find its inverse.

Solution : Let us check for invertibility of f(x) :

(a) One-One :

$$f(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow f'(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$\Rightarrow f'(x) > 0$, $f(x)$ is increasing function

$\therefore$ $f(x)$ is one-one function.

(b) Onto :

As x tends to larger and larger values so does f(x) and when $x \to \infty$, $f(x) \to \infty$.

Similarly as $x \to -\infty$, $f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

Since f(x) is both one-one and onto, f(x) is invertible.

(c) To find $f^{-1}(x)$: Interchange x & y

$$\frac{e^y - e^y}{2} = x \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow e^y = x \pm \sqrt{1 + x^2}$$

Since $e^y > 0$, hence negative sign is ruled out and

Hence $e^y = x + \sqrt{1 + x^2}$

Taking logarithm, we have $y = \ell n(x + \sqrt{1 + x^2})$ or $f^{-1}(x) = \ell n(x + \sqrt{1 + x^2})$

Illustration 29 : Find the inverse of the function $f(x) = \log_a \left(x + \sqrt{(x^2 + 1)}\right)$; $a > 1$ and assuming it to be an onto function.

Solution : Given $f(x) = \log_a \left(x + \sqrt{x^2 + 1}\right)$

$$\therefore \quad f'(x) = \frac{\log_a e}{\sqrt{(1 + x^2)}} > 0$$

which is a strictly increasing functions.

Thus, f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible.

Interchanging x & y

$$\Rightarrow \log_a \left(y + \sqrt{(y)^2 + 1}\right) = x$$

$$\Rightarrow y + \sqrt{(y)^2 + 1} = a^x \qquad \text{...(i)}$$

$$\text{and } \sqrt{(y)^2 + 1} - y = a^{-x} \qquad \text{...(ii)}$$

From (i) and (ii), we get $y = \frac{1}{2}(a^x - a^{-x})$ or $f^{-1}(x) = \frac{1}{2}(a^x - a^{-x})$

Illustration 30 : Find the inverse of the function $f(x) = \ell n(x^2 + 3x + 1)$; $x \in [1,3]$ and assuming it to be an onto function.

Solution : Given $f(x) = \ell n(x^2 + 3x + 1)$

$$\therefore \quad f'(x) = \frac{2x + 3}{(x^2 + 3x + 1)} > 0 \forall x \in [1,3]$$

Which is a strictly increasing function. Thus f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible.

Interchanging x & y

$\Rightarrow (y)^2 + 3(y) + 1 - e^x = 0$

$$\therefore \; y = \frac{-3 \pm \sqrt{9 - 4.(1 - e^x)}}{2} = \frac{-3 \pm \sqrt{(5 + 4e^x)}}{2}$$

$$\Rightarrow \; y = \frac{-3 + \sqrt{(5 + 4e^x)}}{2} \qquad (\text{as } y \in [1,3])$$

$$\text{Hence } f^{-1}(x) = \frac{-3 + \sqrt{(5 + 4e^x)}}{2}$$

Do yourself - 9 :

(i) Let f : [–1, 1] → [–1, 1] defined by f(x) = x|x|, find $f^{-1}(x)$.

(ii) $f(x) = 1 + \ell n(x + 2)$, find $f^{-1}(x)$.

15. ODD & EVEN FUNCTIONS :

Consider a function $f(x)$ such that both x and –x are in its domain then

$$\text{If } \begin{cases} f(-x) = f(x) & \text{then } f \text{ is said to be an even function} \\ f(-x) = -f(x) & \text{then } f \text{ is said to be an odd function} \end{cases}$$

Note :

(i) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(ii) A function may neither be odd nor even.

(iii) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(iv) Every constant function is even function.

(v) Inverse of an even function is not defined.

(vi) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

Special Note : If a function $f(x)$ is defined as $f(a+x) = f(a-x)$ then this function is symmetric about line x = a

(vii) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{i.e. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(viii) If f (x) is odd and defined at x = 0, then f (0) = 0.

f(x)	g(x)	f (x) + g(x)	f(x) – g(x)	f (x) . g(x)	f (x)/g(x)	(go$f$) (x)	($f$og)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration 31 : Which of the following functions is (are) even, odd or neither :

(i) $f(x) = x^2 \sin x$ (ii) $f(x) = \sin x - \cos x$ (iii) $f(x) = \dfrac{e^x + e^{-x}}{2}$

Solution : (i) $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$. Hence f(x) is odd.

(ii) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$.

Hence f(x) is neither even nor odd.

(iii) $f(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^x}{2} = f(x)$. Hence f(x) is even

Illustration 32: Identify the given functions as odd, even or neither :

(i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$ (ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Solution : (i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

Clearly domain of f(x) is $R \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1 = \frac{-e^x . x}{1 - e^x} - \frac{x}{2} + 1 = \frac{(e^x - 1 + 1)x}{(e^x - 1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Replacing x, y by zero, we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0 \Rightarrow f(x) = -f(-x)$

Hence f(x) is an odd function.

Do yourself - 10 :

(i) Which of the following functions is (are) even, odd or neither :

(a) $f(x) = x^3 \sin 3x$ (b) $f(x) = \dfrac{e^{x^2} + e^{-x^2}}{2x}$

(c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (d) $f(x) = x^2 + 2^x$

16. PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number T such that $f(x+T) = f(x) = f(x-T)$, for all values of x within the domain of f. Smallest value of T is called fundamental period of function $f(x)$.

Note :

(i) Odd powers of sinx, cosx, secx, cosecx are periodic with period 2π.

(ii) None zero integral powers of tanx, cotx are periodic with period π.

(iii) Non zero even powers or modulus of sinx, cosx, secx, cosecx are periodic with period π.

(iv) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(v) if $f(x)$ has a period T then **$f(ax + b)$ has a period $T/|a|$**$(a \neq 0)$.

Proof : Let $f(x + T) = f(x)$ and $f[a(x + T') + b] = f(ax + b)$

$f(ax + b + aT') = f(ax + b)$

$f(y + aT') = f(y) = f(y + T) \Rightarrow T = aT' \Rightarrow T' = \frac{T}{a}$

(vi) If $f(x)$ & $g(x)$ are periodic with period T_1 & T_2 respectively, then a period (need not be fundamental) of $f(x) \pm g(x)$ is L.C.M. of (T_1, T_2).

(a) LCM of T_1 & T_2 is defined when T_1/T_2 is rational.

(b) LCM of $\left\{\frac{a}{b}, \frac{p}{q}\right\} = \frac{\text{LCM of } (a, p)}{\text{HCF of } (b, q)}$

In case if there exists a positive K such that K < LCM of T_1 and T_2 and overall function repeats itself after every K, then fundamental period of the function will be K.

(vii) Every constant function is always periodic, whose fundamental period is undefined.

(viii) Inverse of a periodic function does not exist.

Illustration 33 : Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$ (ii) $f(x) = x - [x - b],\ b \in R$

(iii) $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$ (iv) $f(x) = \tan\frac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$ (vi) $f(x) = \frac{(1 + \sin x)(1 + \sec x)}{(1 + \cos x)(1 + \text{cosec}\, x)}$

(vii) $f(x) = e^{x - [x] + |\cos \pi x| + |\cos 2\pi x| + \ldots\ldots + |\cos n\pi|}$

Solution : (i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$

Period of $e^{\ell n \sin x} = 2\pi$, $\tan^3 x = \pi$

$\text{cosec}\,(3x - 5) = \frac{2\pi}{3}$

$\therefore$ Period $= 2\pi$

(ii) $f(x) = x - [x - b] = b + \{x - b\}$

$\therefore$ Period $= 1$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

$\tan\dfrac{\pi}{2}[x + T] = \tan\dfrac{\pi}{2}[x] \Rightarrow \dfrac{\pi}{2}[x + T] = n\pi + \dfrac{\pi}{2}[x]$

$\therefore$ T = 2

$\therefore$ Period = 2

(v) Let f(x) is periodic then f(x + T) = f(x)

$\Rightarrow$ cos(sin(x + T))+ cos(cos(x + T)) = cos(sinx) + cos(cosx)

If x = 0 then cos(sinT) + cos(cosT)

$= \cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing $T = \dfrac{\pi}{2}$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\mathrm{cosec}\, x)} = \dfrac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)}$

$\Rightarrow$ f(x) = tanx

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos\pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Period of x – [x] = 1

Period of $|\cos\pi x| = 1$

Period of $|\cos 2\pi x| = \dfrac{1}{2}$

....

Period of $|\cos n\pi x| = \dfrac{1}{n}$

So period of f(x) will be L.C.M. of all period = 1

Illustration 34 : Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$ (ii) $f(x) = \sin\dfrac{\pi x}{\sqrt{2}} + \cos\dfrac{\pi x}{\sqrt{3}}$ (iii) $f(x) = \sin\dfrac{\pi x}{\sqrt{3}} + \cos\dfrac{\pi x}{2\sqrt{3}}$

Solution : (i) Period of $e^{x-[x]} = 1$

period of sinx = 2π

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $\sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period = $4\sqrt{3}$ **Ans.**

Do yourself - 11 :

(i) Find the fundamental periods (if periodic) of the following functions.

(a) $f(x) = \ell n(\cos x) + \tan^3 x$.

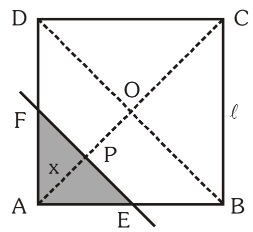
(b) $f(x) = e^{x-[x]}$, [.] denotes greatest integer function

(c) $f(x) = \left|\sin\frac{x}{2}\right| + \left|\cos\frac{x}{2}\right|$

Miscellaneous Illustration :

Illustration 35 : ABCD is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at x = 2, when $\ell = 2$.

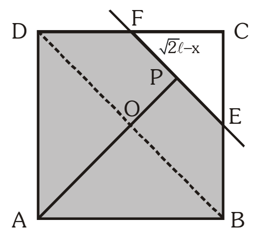
Solution : There are two different situations



Case-I : when x = AP ≤ OA, i.e., $x \le \frac{\ell}{\sqrt{2}}$

$$ar(\Delta AEF) = \frac{1}{2}x.2x = x^2 \quad (\because\ PE = PF = AP = x)$$

Case-II : when x = AP > OA, i.e., $x > \frac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$



$$ar(ABEFDA) = ar(ABCD) - ar(\Delta CFE)$$

$$= \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \quad [\because\ CP = \sqrt{2}\ell - x]$$

$$= \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

$\therefore$ the required function s(x) is as follows :

$$s(x)=\begin{cases} x^2, & 0\le x\le \dfrac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2-\ell^2, & \dfrac{\ell}{\sqrt{2}}< x\le \sqrt{2}\ell \end{cases} \quad ; \text{ area of } s(x)=\begin{cases} \dfrac{1}{2} \text{ at } x=\dfrac{1}{\sqrt{2}} \\ 8(\sqrt{2}-1) \text{ at } x=2 \end{cases}$$

Ans.

Illustration 36 : If the function f(x) satisfies the functional rule, f(x + y) = f(x) + f(y) $\forall$ x,y $\in$ R & f(1) = 5, then find $\sum_{n=1}^{m} f(n)$.

Solution : Here, f(x + y) = f(x) + f(y); put x = t – 1, y = 1

f(t) = f(t – 1) + f(1) ....(1)

$\therefore$ f(t) = f(t – 1) + 5

$\Rightarrow$ f(t) = {f(t – 2) + 5} + 5

$\Rightarrow$ f(t) = f(t – 2) + 2(5)

$\Rightarrow$ f(t) = f(t – 3) + 3(5)

....

....

$\Rightarrow$ f(t) = f{t – (t –1)} + (t – 1)5

$\Rightarrow$ f(t) = f(1)+ (t – 1)5

$\Rightarrow$ f(t) = 5 + (t – 1)5

$\Rightarrow$ f(t) = 5t

$$\therefore \quad \sum_{n=1}^{m} f(n)=\sum_{n=1}^{m}(5n)=5[1+2+3+.....+m]=\frac{5m(m+1)}{2}$$

Hence, $\sum_{n=1}^{m} f(n)=\dfrac{5m(m+1)}{2}$.

ANSWERS FOR DO YOURSELF

1 : **(i)** **(a)** $x \in (0, \infty)$ **(b)** $x \in (-\infty, 0) \cup (4, \infty)$

(ii) **(a)** $\left[\frac{1}{2}, \infty\right)$ **(b)** $[-1, 1]$ **(c)** $[-2, 2]$ **(d)** $[-1, 1]$

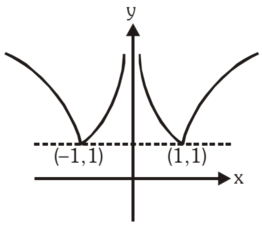
2 : **(i)** **(A)** $\rightarrow$ (p,q,r,s), **(B)** $\rightarrow$ (p), **(C)** $\rightarrow$ (t), **(D)** $\rightarrow$ (r)

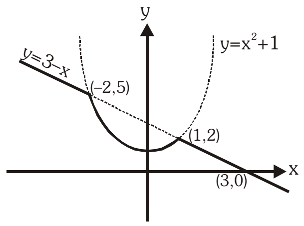
3 : **(i)** **(a)** $(-1,0) \cup (0,1]$ **(b)** $\left(-\infty, -\frac{1}{2}\right] \cup (0,1) \cup (2,\infty)$

(ii) **(a)** $(-\infty,\infty)$ **(b)** $\left[\frac{1}{3}, 1\right]$

4 : **(i)** **(a)** no **(b)** no **(c)** yes

5 : **(i)** $\left[0, \frac{1}{2}\right]$ **(ii)** A **(iii)** **(a)** $y = \frac{1}{x}$ **(b)** $y = \frac{1}{x^2}$

6 : **(i)** 

(ii) 

7 : **(i)** not onto **(ii)** yes **(iii)** B

8 : **(i)** **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0 **(e)** 0 **(f)** 0 **(ii)** $\begin{cases} x+2, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ x, & 2 \le x < 3 \end{cases}$

9 : **(i)** $f^{-1}(x) = \begin{cases} -\sqrt{-x}, & -1 \le x \le 0 \\ \sqrt{x}, & 0 \le x \le 1 \end{cases}$ **(ii)** $y = -2 + e^{x-1}]$

10: **(i)** **(a)** even **(b)** odd **(c)** odd **(d)** neither even nor odd

11: **(i)** **(a)** 2π **(b)** 1 **(c)** π

EXERCISE (O-1)

1. Find the domain of definition of the given functions :

(i) $y = \sqrt{-px}\,(p > 0)$ **FN0001**

(ii) $y = \dfrac{1}{x^2+1}$ **FN0002**

(iii) $y = \dfrac{1}{x^3-x}$ **FN0003**

(iv) $y = \dfrac{1}{\sqrt{x^2-4x}}$ **FN0004**

(v) $y = \sqrt{x^2-4x+3}$ **FN0005**

(vi) $y = \dfrac{x}{\sqrt{x^2-3x+2}}$ **FN0006**

(vii) $y = \sqrt{1-|x|}$ **FN0007**

(viii) $y = \log_x 2$. **FN0008**

(ix) $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ **FN0009**

(x) $y = \dfrac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x}$ **FN0012**

2. Find the range of the following functions :

(i) $f(x) = \dfrac{x-1}{x+2}$ **FN0015**

(ii) $f(x) = \dfrac{2}{x}$ **FN0016**

(iii) $f(x) = \dfrac{1}{x^2-x+1}$ **FN0017**

(iv) $f(x) = e^{(x-1)^2}$ **FN0019**

(v) $f(x) = x^3 - x^2 + x + 1$ **FN0020**

(vi) $f(x) = \sin^2 x - 2\sin x + 4$ **FN0022**

(vii) $f(x) = \sin(\log_2 x)$ **FN0023**

(viii) $f(x) = 2^{x^2} + 1$ **FN0024**

3. Graph the function $F(x) = \begin{cases} 3-x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

FN0041

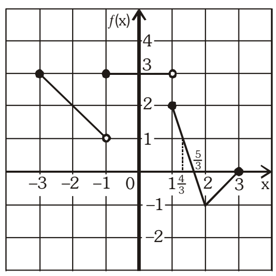
4. Solve the following inequalities using graph of f(x) :

(a) $0 \le f(x) \le 1$

(b) $-1 \le f(x) \le 2$

(c) $2 \le f(x) \le 3$

(d) $f(x) > -1$ & $f(x) < 0$



FN0043

Straight Objective Type

5. If [x] and {x} denotes the greatest integer function less than or equal to x and fractional part function respectively, then the number of real x, satisfying the equation (x–2)[x] = {x} – 1, is-

(A) 0 (B) 1 (C) 2 (D) infinite

FN0036

6. The range of the function $f(x) = \text{sgn}\left(\dfrac{\sin^2 x + 2\sin x + 4}{\sin^2 x + 2\sin x + 3}\right)$ is (where sgn(.) denotes signum function)-

(A) {–1,0,1} (B) {–1,0} (C) {1} (D) {0,1}

FN0034

7. If $2f(x) - 3f\left(\dfrac{1}{x}\right) = x^2$, x is not equal to zero, then f(2) is equal to-

(A) $-\dfrac{7}{4}$ (B) $\dfrac{5}{2}$ (C) –1 (D) none of these

FN0037

8. The number of integers lying in the domain of the function $f(x) = \sqrt{\log_{0.5}\left(\dfrac{5-2x}{x}\right)}$ is -

(A) 3 (B) 2 (C) 1 (D) 0

FN0027

9. The range of the function $f : N \to I;\ f(x) = (-1)^{x-1}$, is -

(A) [–1, 1] (B) {– 1, 1} (C) {0, 1} (D) {0, 1, – 1}

FN0028

10. The range of the function $f(x) = e^{-x} + e^{x}$, is -

(A) $f(x) \geq 1$ (B) $f(x) \leq 1$ (C) $f(x) \geq 2$ (D) $f(x) \leq 2$

FN0029

11. If $f(x) = \dfrac{4^x}{4^x + 2}$, then f(x) + f(1 – x) is equal to-

(A) 0 (B) –1 (C) 1 (D) 4

FN0038

12. A function f has domain [– 1, 2] and range [0, 1]. The domain and range respectively of the function g defined by g (x) = 1 – f (x + 1) is

(A) [–1, 1] ; [–1, 0] (B) [–2, 1]; [0, 1] (C) [0, 2] ; [–1, 0] (D) [1, 3] ; [–1, 0]

FN0030

13. If $f : R \to R$ & $f(x) = \dfrac{\sin([x]\pi)}{x^2 + 2x + 3} + 2x - 1 + \sqrt{x(x-1) + \dfrac{1}{4}}$ (where [x] denotes integral part of x), then f(x) is -

(A) one-one but not onto

(B) one-one & onto

(C) onto but not one-one

(D) neither one-one nor onto

FN0048

14. Which of the following function is surjective but not injective

(A) $f : R \to R \;\; f(x) = x^4 + 2x^3 - x^2 + 1$ (B) $f : R \to R \;\; f(x) = x^3 + x + 1$

(C) $f : R \to R^+ \;\; f(x) = \sqrt{1+x^2}$ (D) $f : R \to R \;\; f(x) = x^3 + 2x^2 - x + 1$

FN0049

15. If $f(x) = x|x|$ then $f^{-1}(x)$ equals-

(A) $\sqrt{|x|}$ (B) $(\text{sgn}\, x).\sqrt{|x|}$ (C) $-\sqrt{|x|}$ (D) Does not exist

(where sgn(x) denotes signum function of x)

FN0052

16. If $f : (-\infty,3] \to [7,\infty)$; $f(x) = x^2 - 6x + 16$, then which of the following is true -

(A) $f^{-1}(x) = 3 + \sqrt{x-7}$ (B) $f^{-1}(x) = 3 - \sqrt{x-7}$

(C) $f^{-1}(x) = \dfrac{1}{x^2 - 6x + 16}$ (D) f is many-one

FN0050

17. $f : R \to R$ such that $f(x) = \ell n\left(x + \sqrt{x^2+1}\right)$. Another function g(x) is defined such that $gof(x) = x \;\forall\; x \in R$. Then g(2) is -

(A) $\dfrac{e^2 + e^{-2}}{2}$ (B) e^2 (C) $\dfrac{e^2 - e^{-2}}{2}$ (D) e^{-2}

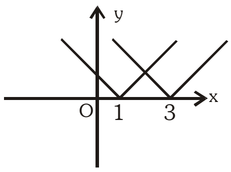
FN0051

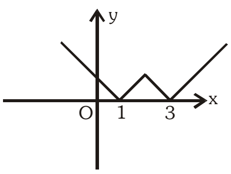
18. Let $P(x) = kx^3 + 2k^2x^2 + k^3$. The sum of all real numbers k for which $(x - 2)$ is a factor of P(x), is

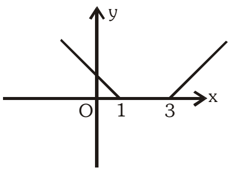
(A) 4 (B) 8 (C) – 4 (D) –8

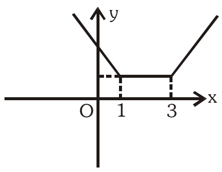
FN0053

19. Which of the following is the graph of $y = |x - 1| + |x - 3|$?

(A) 

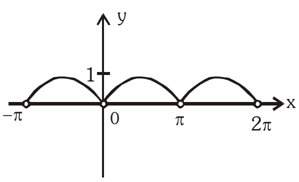
(B) 

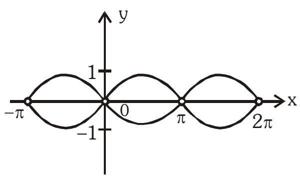
(C) 

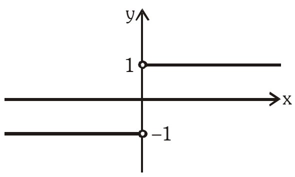
(D) 

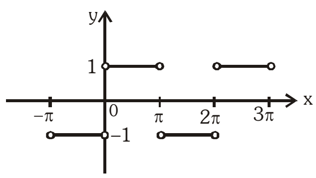
FN0044

20. Which of the following is the graph of $y = \dfrac{|\sin x|}{\sin x}$?

(A) 

(B) 

(C) 

(D) 

FN0045

Multiple Correct Answer

21. Let $f(x) = x^2 + 3x + 2$, then number of solutions to -

(A) $f(|x|) = 2$ is 1 (B) $f(|x|) = 2$ is 3 (C) $|f(x)| = 0.125$ is 4 (D) $|f(|x|)| = 0.125$ is 8

FN0060

22. Which of the following pair(s) of function have same graphs?

(A) $f(x) = \frac{\sec x}{\cos x} - \frac{\tan x}{\cot x}$, $g(x) = \frac{\cos x}{\sec x} + \frac{\sin x}{\operatorname{cosec} x}$

(B) $f(x) = \operatorname{sgn}(x^2 - 4x + 5)$, $g(x) = \operatorname{sgn}\left(\cos^2 x + \sin^2\left(x + \frac{\pi}{3}\right)\right)$ where sgn denotes signum function.

(C) $f(x) = e^{\ln(x^2+3x+3)}$, $g(x) = x^2 + 3x + 3$

(D) $f(x) = \frac{\sin x}{\sec x} + \frac{\cos x}{\operatorname{cosec} x}$, $g(x) = \frac{2\cos^2 x}{\cot x}$

FN0056

23. For each real x, let $f(x) = \max\{x, x^2, x^3, x^4\}$, then $f(x)$ is -

(A) x^4 for $x \le -1$ (B) x^2 for $-1 < x \le 0$ (C) $f\left(\frac{1}{2}\right) = \frac{1}{2}$ (D) $f\left(\frac{1}{2}\right) = \frac{1}{4}$

FN0061

24. Let $f(x) = \sin^6 x + \cos^6 x$, then -

(A) $f(x) \in [0, 1]\ \forall\, x \in R$ (B) $f(x) = 0$ has no solution

(C) $f(x) \in \left[\frac{1}{4}, 1\right] \forall x \in R$ (D) $f(x)$ is an injective function

FN0057

25. Let $f(x) = \begin{cases} x^2 - 3x + 4 & ; \ x < 3 \\ x + 7 & ; \ x \ge 3 \end{cases}$ and $g(x) = \begin{cases} x + 6 & ; \ x < 4 \\ x^2 + x + 2 & ; \ x \ge 4 \end{cases}$, then which of the following is/are true -

(A) $(f + g)(1) = 9$ (B) $(f - g)(3.5) = 1$ (C) $(f\,g)(0) = 24$ (D) $\left(\frac{f}{g}\right)(5) = \frac{8}{3}$

FN0058

Matrix Match Type

26. Match the functions given in column-I correctly with mappings given in column-II.

Column-I	Column-II
(A) $f : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[\frac{4}{7}, \frac{4}{3}\right]$	(P) Injective mapping
$f(x) = \frac{1}{x^2 + x + 1}$	(Q) Non-injective mapping
(B) $f : [-2, 2] \to [-1, 1]$	(R) Surjective mapping
$f(x) = \sin x$	(S) Non-surjective mapping
(C) $f : R - I \to R$	(T) Bijective mapping
$f(x) = \ell n\{x\}$, (where {.} represents fractional part function)	
(D) $f : (-\infty, 0] \to [1, \infty)$, $f(x) = (1 + \sqrt{-x}) + (\sqrt{-x} - x)$	

FN0062

Linked Comprehension Type

Paragraph for Question 27 & 28

Let $f(x)=\begin{cases} x & ; & x<0 \\ 1-x & ; & x\ge 0 \end{cases}$ & $g(x)=\begin{cases} x^2 & ; & x<-1 \\ 2x+3 & ; & -1\le x\le 1 \\ x & ; & x>1 \end{cases}$

On the basis of above information, answer the following questions :

27. Range of $f(x)$ is -

(A) $(-\infty,1]$ (B) $(-\infty,\infty)$ (C) $(-\infty,0]$ (D) $(-\infty,2]$

FN0063

28. Range of $g(f(x))$ is -

(A) $(-\infty,\infty)$ (B) $[1,3)\cup(3,\infty)$ (C) $[1,\infty)$ (D) $[0,\infty)$

FN0063

EXERCISE (O-2)

Straight Objective Type

1. The number of integral values of x satisfying the inequality $[x-5]\,[x-3]+2<[x-5]+2[x-3]$ (where [.] represents greatest integer function) is -

(A) 0 (B) 1 (C) 2 (D) 3

FN0067

2. Range of $f(x)=\dfrac{\sec x+\tan x-1}{\tan x-\sec x+1};\ x\in\left(0,\dfrac{\pi}{2}\right)$ is-

(A) (0,1) (B) $(1,\infty)$ (C) (–1,0) (D) $(-\infty,-1)$

FN0065

3. If $f(x,y)=\max(x,y)+\min(x,y)$ and $g(x,y)=\max(x,y)-\min(x,y)$, then the value of $f\left(g\left(-\dfrac{2}{3},-\dfrac{3}{2}\right),g(-3,-4)\right)$ is greater than -

(A) 1 (B) 2 (C) 3 (D) 4

FN0066

4. If functions $f(x)$ and $g(x)$ are defined on $R\to R$ such that $f(x)=\begin{cases} x+3 & , & x\in \text{rational} \\ 4x & , & x\in\text{irrational}\end{cases}$,

$g(x)=\begin{cases} x+\sqrt{5} & , & x\in\text{irrational} \\ -x & , & x\in\text{rational}\end{cases}$ then $(f-g)(x)$ is -

(A) one-one & onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one

FN0072

5. Let $f: A\to B$ be an onto function such that $f(x)=\sqrt{x-2-2\sqrt{x-3}}-\sqrt{x-2+2\sqrt{x-3}}$, then set 'B' is-

(A) [–2,0] (B) [0,2] (C) [–3,0] (D) [–1,0]

FN0073

6. Let $f: R\to R$ and $f(x)=x^3+ax^2+bx-8$. If $f(x)=0$ has three real roots & $f(x)$ is a bijective function, then (a + b) is equal to

(A) 0 (B) 6 (C) –6 (D) 12

FN0074

7. Which of the following functions is an odd function ?

(A) $|x-2| + (x+2)\ \text{sgn}(x+2)$

(B) $\dfrac{1}{x(e^x-1)} + \dfrac{1}{2x}$

(C) $\log\left(\sin x + \sqrt{1+\sin^2 x}\right)$

(D) $e^{-4x}(e^{2x}-1)^4$

(where sgn(x) denotes signum function of x)

FN0077

8. Period of $f(x) = \{x\} + \left\{x+\dfrac{1}{3}\right\} + \left\{x+\dfrac{2}{3}\right\}$ is equal to (where {.} denotes fractional part function)

(A) 1 (B) $\dfrac{2}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{3}$

FN0078

9. Let $f(x) = 2x - \left\{\dfrac{x}{\pi}\right\}$ and $g(x) = \cos x$, where {.} denotes fractional part function, then period of $gof(x)$ is -

(A) $\dfrac{\pi}{2}$ (B) π (C) $\dfrac{3\pi}{2}$ (D) $\dfrac{\pi}{4}$

FN0079

Multiple Correct Answer Type

10. Let $f(x) = \begin{cases} x^2 & ; \quad 0 < x < 2 \\ 2x-3 & ; \quad 2 \le x < 3 \\ x+2 & ; \quad x \ge 3 \end{cases}$. Then :-

(A) $f\left\{f\left(f\left(\dfrac{3}{2}\right)\right)\right\} = f\left(\dfrac{3}{2}\right)$

(B) $1 + f\left\{f\left(f\left(\dfrac{5}{2}\right)\right)\right\} = f\left(\dfrac{5}{2}\right)$

(C) $f\{f(1)\} = f(1) = 1$

(D) none of these

FN0087

11. The range of the function $f(\theta) = \sqrt{8\sin^2\theta + 4\cos^2\theta - 8\sin\theta\cos\theta}$ is -

(A) $\left[\sqrt{5}-1, \sqrt{5}+1\right]$

(B) $\left[0, \sqrt{5}+1\right]$

(C) $\left[\sqrt{6-\sqrt{20}}, \sqrt{6+\sqrt{20}}\right]$

(D) none of these

FN0082

12. For the function $f(x) = |x+3| - |x+1| - |x-1| + |x-3|$, identify correct option(s)

(A) Range of $f(x)$ is $(-\infty, 4]$

(B) maximum value of $f(x)$ is 4

(C) $f(x) = 4$ has infinite solutions

(D) $f(x) = 0$ has infinite solutions

FN0083

13. Which of the following statement(s) is(are) correct ?

(A) If f is a one-one mapping from set A to A, then f is onto.

(B) If f is an onto mapping from set A to A, then f is one-one

(C) Let f and g be two functions defined from $R \to R$ such that gof is injective, then f must be injective.

(D) If set A contains 3 elements while set B contains 2 elements, then total number of functions from A to B is 8.

FN0085

14. If $f(x) = ax + b$ and $f(f(f(x))) = 27x + 13$ where a and b are real numbers, then-

(A) $a + b = 3$ (B) $a + b = 4$ (C) $f'(x) = 3$ (D) $f'(x) = -3$

FN0093

15. Let $f(x)=\begin{cases} x^2-4 & \text{if } |x|\le 3 \\ 5\,\text{sgn}|x-3| & \text{if } |x|>3 \end{cases}$

and $g(x)=2\tan^{-1}(e^x)-\dfrac{\pi}{2}$ for all $x \in R$, then which of the following is wrong ?

(where sgn(x) denotes signum function of x)

(A) $f(x)$ is an even function
(B) $go f(x)$ is an even function
(C) $g(x)$ is an odd function
(D) $f o f(x)$ is an odd function

FN0196

Matrix Match Type

16. **Column-I** **Column-II**

Number of integers in

(A) Domain of $f(x)=\ell n\{x\}$ (P) 0

(B) Domain of $f(x)=\sqrt{\sec(\sin x)}+\sqrt{\left[x+\dfrac{1}{x}\right]}+\sqrt{10-[x]^2}$ (Q) 2

(C) Range of $f(x) = x^2 - 2x + 2,\ x \in [0,2]$ (R) 3

(D) Range of $f(x)=\sqrt{25-[x]^2}$ (S) less than 3

(T) more than 3

(where [.] and {.} denote greatest integer function and fractional part function respectively)

FN0095

17. Match the function mentioned in column-I with the respective classification given in column-II. (where [.] and {.} denote greatest integer function and fractional part function respectively)

Column-I **Column-II**

(A) $f: R \to R^+\ f(x)=\left(e^{[x]}\right)\left(e^{\{x\}}\right)$ (P) one-one

(B) $f:(-\infty,-2)\cup(0,\infty)\to R\ f(x) = \ell n(x^2 + 2x)$ (Q) many-one

(C) $f : [-2,2] \to [-1,1]\ f(x) = \sin x$ (R) onto

(D) $f : R \to R\quad f(x) = x^3 - 3x^2 + 3x - 7$ (S) periodic

(T) aperiodic

FN0096

EXERCISE (S-1)

1. Find the domains of definitions of the following functions :

(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x)=\sqrt{\cos 2x}+\sqrt{16-x^2}$ **FN0097**

(ii) $f(x) = \log_7 \log_5 \log_3 \log_2(2x^3 + 5x^2 - 14x)$ **FN0098**

(iii) $f(x) = \ln\left(\sqrt{x^2-5x-24}-x-2\right)$ **FN0099**

(iv) $y=\log_{10}\sin(x-3)+\sqrt{16-x^2}$ **FN0101**

(v) $f(x)=\sqrt{x^2-|x|}+\dfrac{1}{\sqrt{9-x^2}}$ **FN0103**

2. Find the domain & range of the following functions.

(i) $y = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$ **FN0107**

(ii) $y = \dfrac{2x}{1+x^2}$ **FN0108**

(iii) $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$ **FN0109**

(iv) $f(x) = \dfrac{x}{1+|x|}$ **FN0110**

(v) $f(x) = \dfrac{\sqrt{x+4} - 3}{x - 5}$ **FN0112**

3. The sum of integral values of the elements in the domain of $f(x) = \sqrt{\log_{\frac{1}{2}}|3-x|}$ is

FN0116

4. Number of integers in range of $f(x) = x(x+2)(x+4)(x+6) + 7,\ x \in [-4,2]$ is

FN0117

5. Identify the pair(s) of functions which are identical ?
(where [x] denotes greatest integer and {x} denotes fractional part function)

(i) $f(x) = \text{sgn}(x^2 - 3x + 4)$ and $g(x) = e^{[\{x\}]}$

(ii) $f(x) = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$ and $g(x) = \tan x$

(iii) $f(x) = ln(1+x) + ln(1-x)$ and $g(x) = ln(1-x^2)$

(iv) $f(x) = \dfrac{\cos x}{1-\sin x}$ and $g(x) = \dfrac{1+\sin x}{\cos x}$

FN0120

6. Classify the following functions f(x) defined in $R \to R$ as injective, surjective, both or none.

(a) $f(x) = \dfrac{x^2 + 4x + 30}{x^2 - 8x + 18}$

FN0125

(b) $f(x) = x^3 - 6x^2 + 11x - 6$

FN0126

(c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$

FN0127

7. Solve the following problems from (a) to (d) on functional equation :

(a) The function f(x) defined on the real numbers has the property that $f(f(x)).(1+f(x)) = -f(x)$ for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of f(3).
FN0121

(b) Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of f(21).
FN0122

(c) Let f be function defined from $R^+ \to R^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and $f(2) = 6$, find the value of f(50).
FN0123

(d) Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x. Then find the value of f(300).
FN0124

8. Suppose f(x) = sinx and $g(x) = 1 - \sqrt{x}$. Then find the domain and range of the following functions.
(a) fog (b) gof (c) fof (d) gog

FN0128

9. A function $f : R \to R$ is such that $f\left(\frac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Prove the following.

(a) $f(f(x)) = x$ (b) $f(1/x) = -f(x), x \neq 0$ (c) $f(-x-2) = -f(x) - 2$.

FN0129

10. $f(x) = \begin{cases} 1-x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} -x & \text{if } x < 1 \\ 1-x & \text{if } x \ge 1 \end{cases}$ find (fog)(x) and (gof)(x).

FN0132

11. Find whether the following functions are even or odd or none :

(a) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$ **FN0139**

(b) $f(x) = \frac{x(a^x + 1)}{a^x - 1}$ **FN0140**

(c) f(x) = sinx + cosx **FN0141**

(d) $f(x) = x\sin^2 x - x^3$ **FN0142**

(e) f(x) = sinx – cosx **FN0143**

(f) $f(x) = \frac{(1+2^x)^2}{2^x}$ **FN0144**

(g) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$ **FN0146**

12. (i) Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :

(a) $10^x + 10^y = 10$ (b) $x + |y| = 2y$

FN0113

(ii) The function f(x) is defined on the interval [0, 1]. Find the domain of definition of the functions.

(a) f(sinx) (b) f(2x + 3)

FN0114

(iii) Given that $y = f(x)$ is a function whose domain is [4,7] and range is [–1, 9]. Find the range and domain of

(a) $g(x) = \frac{1}{3}f(x)$ (b) $h(x) = f(x - 7)$

FN0115

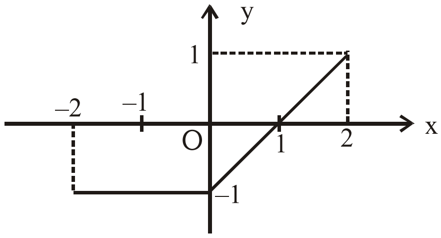
13. Compute the inverse of the functions :

(a) $f(x) = \ell n\left(x + \sqrt{x^2+1}\right)$ **FN0133**

(b) $f(x) = 2^{\frac{x}{x-1}}$ **FN0134**

(c) $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ **FN0135**

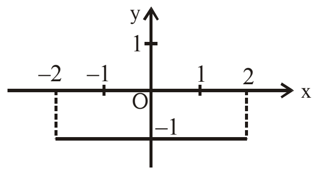
14. The graph of the function $y = f(x)$ is as follows :



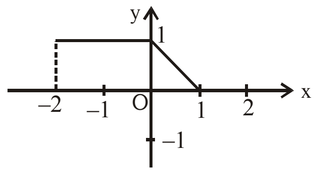
Match the function mentioned in **Column-I** with the respective graph given in **Column-II**.

Column-I | **Column-II**

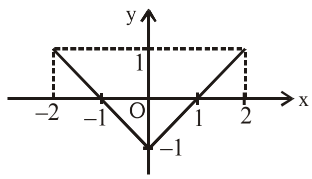
(A) $y = |f(x)|$ | (P)



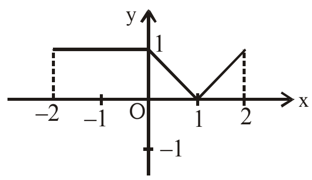
(B) $y = f(|x|)$ | (Q)



(C) $y = f(-|x|)$ | (R)



(D) $y = \frac{1}{2}(|f(x)| - f(x))$ | (S)



FN0154

15. If $f(x) = a \log\left(\frac{1+x}{1-x}\right) + bx^3 + c \sin x + 5$ and $f(\log_3 2) = 4$, then $f\left(\log_3\left(\frac{1}{2}\right)\right)$ is equal to

FN0155

16. If $f(x) = \begin{cases} 0 & x < 1 \\ 2x - 2 & x \geq 1 \end{cases}$; then the number of solutions of the equation $f(f(f(x))) = x$ is

FN0151

17. Let 'f' be an even periodic function with period '4' such that $f(x) = 2^x - 1,\ 0 \leq x \leq 2$.
The number of solutions of the equation $f(x) = 1$ in $[-10, 20]$ are

FN0150

18. Let $f(x) = \frac{2x-1}{x+3}$. If $f^{-1} = \frac{ax+b}{c-x}$, then $a + b + c$ is

FN0137

19. Let $f(x)$ be a periodic function with period 'p' satisfying $f(x) + f(x+3) + f(x+6) + ... + f(x+42)$ = constant $\forall x \in R$, then sum of digits of 'p' is

FN0149

EXERCISE (S-2)

1. (a) Let $P(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ be a polynomial such that $P(1) = 1$; $P(2) = 2$; $P(3) = 3$; $P(4) = 4$; $P(5) = 5$ and $P(6) = 6$, then find the value of $P(7)$.

FN0158

(b) Let a and b be real numbers and let $f(x) = a\sin x + b\sqrt[3]{x} + 4, \forall x \in R$. If $f(\log_{10}(\log_3 10)) = 5$ then find the value of $f(\log_{10}(\log_{10} 3))$.

FN0159

2. Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by x – 1 is 1 and the remainder when p(x) is divided by x – 4 is 10. If r(x) is the remainder when p(x) is divided by (x –1)(x – 4), find the value of r(2006).

FN0160

3. A function f, defined for all $x, y \in R$ is such that $f(1) = 2$; $f(2) = 8$ & $f(x+y) - kxy = f(x) + 2y^2$, where k is some constant. Find f(x) & show that :

$$f(x+y)f\left(\frac{1}{x+y}\right) = k \text{ for } x + y \neq 0.$$

FN0161

4. If $f(x) = -1 + |x - 2|, 0 \le x \le 4$

$g(x) = 2 - |x|, -1 \le x \le 3$

Then find fog(x) & gof(x). Draw rough sketch of the graphs of fog(x) & gof(x).

FN0162

5. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If $f(x)$ is divided by $x^3 - x$ then the remainder is some function of x say g(x). Find the value of g(10).

FN0163

6. Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$.

FN0164

7. Let $f(x) = \dfrac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\frac{1}{2006}\right) + f\left(\frac{2}{2006}\right) + f\left(\frac{3}{2006}\right) + ...f\left(\frac{2005}{2006}\right)$

FN0165

8. Let $f(x) = (x+1)(x+2)(x+3)(x+4) + 5$ where $x \in [-6, 6]$. If the range of the function is [a, b] where $a, b \in N$ then find the value of (a + b).

FN0166

9. The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of a + b + c + abc.

FN0167

10. $f(x)$ and g(x) are linear function such that for all x, $f(g(x))$ and $g(f(x))$ are Identity functions. If $f(0) = 4$ and $g(5) = 17$, compute f(2006).

FN0168

11. The function $f(x)$ has the property that for each real number x in its domain, 1/x is also in its domain and $f(x) + f(1/x) = x$. Find the largest set of real numbers that can be in the domain of $f(x)$?

FN0170

EXERCISE (JM)

1. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$, and **[JEE(Main)-2016]**

$S = \{x \in R : f(x) = f(-x)\}$; then S :

(1) contains more than two elements. (2) is an empty set.

(3) contains exactly one element (4) contains exactly two elements

FN0174

2. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is : **[JEE(Main)-2017]**

(1) neither injective nor surjective. (2) invertible.

(3) injective but not surjective. (4) surjective but not injective

FN0175

3. Let $S = \{x \in R : x \geq 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S : **[JEE(Main)-2018]**

(1) contains exactly one element. (2) contains exactly two elements.

(3) contains exactly four elements. (4) is an empty set.

FN0176

4. For $x \in R - \{0, 1\}$, let $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \frac{1}{1-x}$ be three given functions. If a function, J(x) satisfies $(f_2 o J o f_1)(x) = f_3(x)$ then J(x) is equal to :- **[JEE(Main)-Jan 19]**

(1) $f_3(x)$ (2) $f_1(x)$ (3) $f_2(x)$ (4) $\frac{1}{x}f_3(x)$

FN0177

5. Let N be the set of natural numbers and two functions f and g be defined as $f,g : N \to N$ such that : $f(n) = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$ and $g(n) = n-(-1)^n$. The fog is : **[JEE(Main)-Jan 19]**

(1) Both one-one and onto (2) One-one but not onto

(3) Neither one-one nor onto (4) onto but not one-one

FN0178

6. Let $f : R \to R$ be defined by $f(x) = \frac{x}{1+x^2}$, $x \in R$. Then the range of f is : **[JEE(Main)-Jan 19]**

(1) $(-1, 1) - \{0\}$ (2) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (3) $R - \left[-\frac{1}{2}, \frac{1}{2}\right]$ (4) $R - [-1, 1]$

FN0179

7. The number of functions f from {1, 2, 3, ..., 20} onto {1, 2, 3, ..., 20} such that f(k) is a multiple of 3, whenever k is a multiple of 4, is :- **[JEE(Main)-Jan 19]**

(1) $(15)! \times 6!$ (2) $5^6 \times 15$ (3) $5! \times 6!$ (4) $6^5 \times (15)!$

FN0180

8. If $f(x)=\log_e\left(\frac{1-x}{1+x}\right), |x|<1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : **[JEE(Main)-Apr 19]**

(1) $2f(x)$ (2) $2f(x^2)$ (3) $(f(x))^2$ (4) $-2f(x)$

FN0181

9. Let $\sum_{k=1}^{10} f(a+k)=16\left(2^{10}-1\right)$, where the function f satisfies $f(x+y)=f(x)f(y)$ for all natural numbers x, y and $f(1)=2$. then the natural number 'a' is **[JEE(Main)-Apr 19]**

(1) 4 (2) 3 (3) 16 (4) 2

FN0182

10. Le $f(x) = x^2$, $x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R, f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true ? **[JEE(Main)-Apr 19]**

(1) $f(g(S)) \neq f(S)$ (2) $f(g(S)) = S$ (3) $g(f(S)) = g(S)$ (4) $g(f(S)) \neq S$

FN0183

11. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is : **[JEE(Main)-Apr 19]**

(1) 2 (2) 3 (3) 4 (4) 1

FN0184

12. For $x\in\left(0,\frac{3}{2}\right)$, let $f(x)=\sqrt{x}$, $g(x)=\tan x$ and $h(x)=\frac{1-x^2}{1+x^2}$. If $\phi(x)=((hof)og)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to : **[JEE(Main)-Apr 19]**

(1) $\tan\frac{\pi}{12}$ (2) $\tan\frac{7\pi}{12}$ (3) $\tan\frac{11\pi}{12}$ (4) $\tan\frac{5\pi}{12}$

FN0185

13. For $x \in R$, let $[x]$ denote the greatest integer $\leq x$, then the sum of the series

$\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+.....+\left[-\frac{1}{3}-\frac{99}{100}\right]$ is **[JEE(Main)-Apr 19]**

(1) –153 (2) –133 (3) –131 (4) –135

FN0186

EXERCISE (JA)

1. If functions f(x) and g(x) are defined on $R \to R$ such that

$$f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases},\quad g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$$, then $(f-g)(x)$ is -

(A) one-one and onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one **[JEE 2005 (Scr.)]**

FN0191

2. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is- **[JEE 2011, 3, (–1)]**

(A) $\pm\sqrt{n\pi}, n\in\{0,1,2,....\}$ (B) $\pm\sqrt{n\pi}, n\in\{1,2,...\}$

(C) $\frac{\pi}{2}+2n\pi, n\in\{....,-2,-1,0,1,2,.....\}$ (D) $2n\pi, n\in\{....,-2,-1,0,1,2,.....\}$

FN0192

3. The function $f : [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[JEE 2012, 3, (–1)]**

(A) one-one and onto
(B) onto but not one-one
(C) one-one but not onto
(D) neither one-one nor onto

FN0193

4. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$ Then - **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ is an odd function
(B) $f(x)$ is a one-one function
(C) $f(x)$ is an onto function
(D) $f(x)$ is an even function

FN0195

ANSWER KEY
FUNCTIONS
EXERCISE (O-1)

1. (i) $-\infty < x \le 0$ (ii) $x \in R$ (iii) $x \in R - \{-1,0,1\}$

(iv) $-\infty < x < 0$ & $4 < x < \infty$ (v) $-\infty < x \le 1$ and $3 \le x < \infty$ (vi) $-\infty < x < 1$ and $2 < x < \infty$

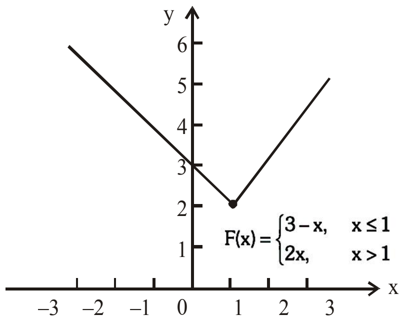
(vii) $-1 \le x \le 1$ (viii) $0 < x < 1$ and $1 < x < \infty$. (ix) $-2 \le x < 0$ and $0 < x < 1$

(x) $2k\pi < x < (2k + 1)\pi$, where k is an integer.

2. (i) $R - \{1\}$ (ii) $R - \{0\}$ (iii) $(0,4/3]$ (iv) $[1,\infty)$ (v) R

(vi) $[3,7]$ (vii) $[-1,1]$ (viii) $[2,\infty)$

3.



4. (a) $\left[\frac{4}{3}, \frac{5}{3}\right] \cup \{3\}$ (b) $[-2, -1) \cup [1, 3]$ (c) $[-3, -2] \cup [-1, 1]$ (d) $\left(\frac{5}{3}, 3\right) - \{2\}$

5. D **6.** C **7.** A **8.** C **9.** B **10.** C **11.** C **12.** B **13.** B

14. D **15.** B **16.** B **17.** C **18.** D **19.** D **20.** D **21.** A,C

22. A,B,C,D **23.** A,B,C **24.** A,B,C **25.** A,B,C

26. (A)→(P,R,T); (B)→(Q,R); (C)→(Q,S); (D)→(P,R,T) **27.** A **28.** C

EXERCISE (O-2)

1. C **2.** B **3.** A **4.** B **5.** A **6.** B **7.** C **8.** D **9.** B

10. A,B,C **11.** A,C **12.** B,C,D **13.** C,D **14.** B,C **15.** D

16. (A)→(P,S); (B)→(R); (C)→(Q,S); (D)→(T)

17. (A)→(P,R,T); (B)→(Q,R,T); (C)→(Q,R,T); (D)→(P,R,T)

EXERCISE (S-1)

1. **(i)** $\left[-\frac{5\pi}{4}, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ **(ii)** $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$ **(iii)** $(-\infty, -3]$

(iv) $(3 - 2\pi < x < 3 - \pi) \cup (3 < x \le 4)$ **(v)** $(-3, -1] \cup \{0\} \cup [1, 3)$

2. **(i)** D : $x \in R$, R : $[0, 2]$ **(ii)** D = R; range $[-1, 1]$

(iii) D : $\{x \mid x \in R;\ x \ne -3;\ x \ne 2\}$ R : $\{f(x) \mid f(x) \in R,\ f(x) \ne 1/5;\ f(x) \ne 1\}$

(iv) D : R, R : $(-1, 1)$ **(v)** D : $[-4, \infty) - \{5\}$; R : $\left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{3}\right]$

3. 6 **4.** 401 **5.** (i), (iii) are identical

6. **(a)** neither surjective nor injective; **(b)** surjective but not injective; **(c)** neither injective nor surjective

7. (a) –3/4; (b) 64; (c) 30, (d) 5050

8. **(a)** domain is $x \geq 0$; range $[-1, 1]$; **(b)** domain $2k\pi \leq x \leq 2k\pi + \pi$; range $[0, 1]$

(c) domain $x \in R$; range $[-\sin 1, \sin 1]$ **(d)** domain is $0 \leq x \leq 1$; range is $[0, 1]$

10. $(gof)(x) = \begin{cases} x & \text{if } x \leq 0 \\ -x^2 & \text{if } 0 < x < 1 \\ 1-x^2 & \text{if } x \geq 1 \end{cases}$; $(fog)(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1+x & \text{if } 0 \leq x < 1 \\ x & \text{if } x \geq 1 \end{cases}$

11. (a) odd, (b) even, (c) neither odd nor even, (d) odd, (e) neither odd nor even, (f) even, (g) even

12. (i) **(a)** $y = \log(10 - 10^x), -\infty < x < 1$; **(b)** $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \leq x < +\infty$

(ii) **(a)** $2K\pi \leq x \leq 2K\pi + \pi$ where $K \in I$; **(b)** $[-3/2, -1]$

(iii) **(a)** Range : $[-1/3, 3]$, Domain = $[4, 7]$; **(b)** Range $[-1, 9]$ and domain $[11, 14]$

13. **(a)** $\frac{e^x - e^{-x}}{2}$; **(b)** $\frac{\log_2 x}{\log_2 x - 1}$; **(c)** $\frac{1}{2}\log\frac{1+x}{1-x}$ **14.** (A) S; (B) R; (C) P; (D) Q

15. 6 **16.** 2 **17.** 15 **18.** 6 **19.** 9

EXERCISE (S-2)

1. **(a)** 727, **(b)** 3 **2.** 6016 **3.** $f(x) = 2x^2$

4. $fog(x) = \begin{cases} -(1+x), & -1 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$; $gof(x) = \begin{cases} x+1, & 0 \leq x < 1 \\ 3-x, & 1 \leq x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases}$

$fof(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 4-x, & 3 \leq x \leq 4 \end{cases}$; $gog(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ x, & 0 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$

5. 21 **6.** x = 0 or 5/3 **7.** 1002.5 **8.** 5049 **9.** 20

10. 122 **11.** {–1, 1}

EXERCISE (JM)

1. 4 **2.** 4 **3.** 2 **4.** 1 **5.** 4 **6.** 2 **7.** 1 **8.** 1

9. 2 **10.** 3 **11.** 4 **12.** 3 **13.** 2

EXERCISE (JA)

1. A **2.** A **3.** B **4.** A,B,C

INVERSE TRIGONOMETRIC FUNCTION

1. INTRODUCTION :

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or (arc sinx), $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x. The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$ and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions sinx, cosx etc are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

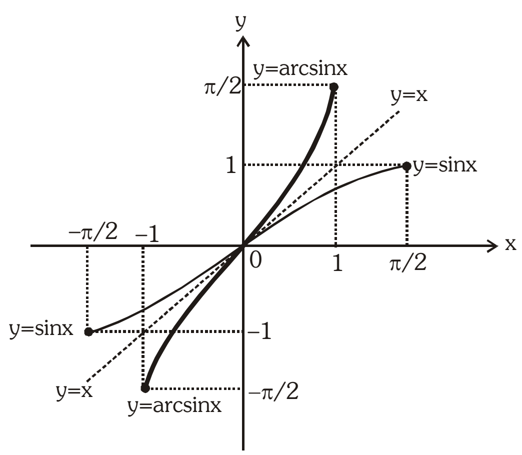
2. DOMAIN & RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS :

S.No	*f* (x)	Domain	Range
(1)	$\sin^{-1}x$	$\lvert x\rvert \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\lvert x\rvert \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in R$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\lvert x\rvert \ge 1$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$
(5)	$\mathrm{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
(6)	$\cot^{-1}x$	$x \in R$	$(0, \pi)$

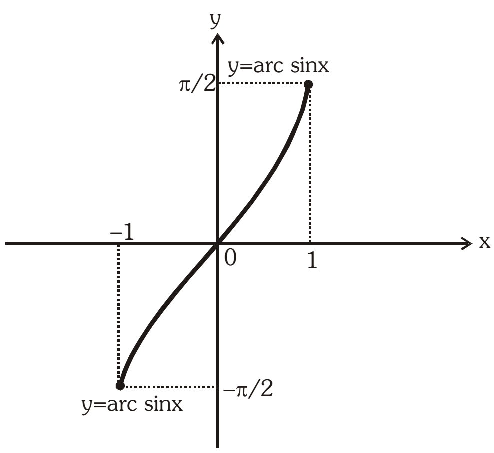
3. GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f:\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ $\qquad$ $f^{-1}:[-1, 1]\to[-\pi/2, \pi/2]$

$f(x) = \sin x$ $\qquad$ $f^{-1}(x) = \sin^{-1}(x)$



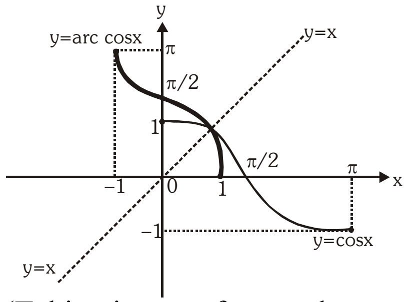
(Taking image of sin x about y = x to get $\sin^{-1}x$)



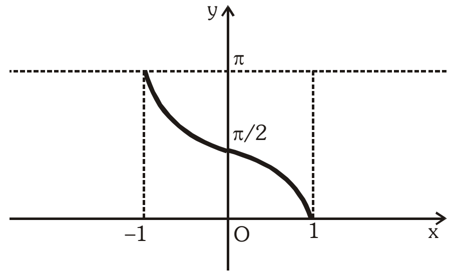
($y = \sin^{-1}x$)

(b) $f : [0, \pi] \rightarrow [-1, 1]$ $\quad$ $f^{-1} : [-1, 1] \rightarrow [0, \pi]$

$f(x) = \cos x$ $\quad$ $f^{-1}(x) = \cos^{-1} x$



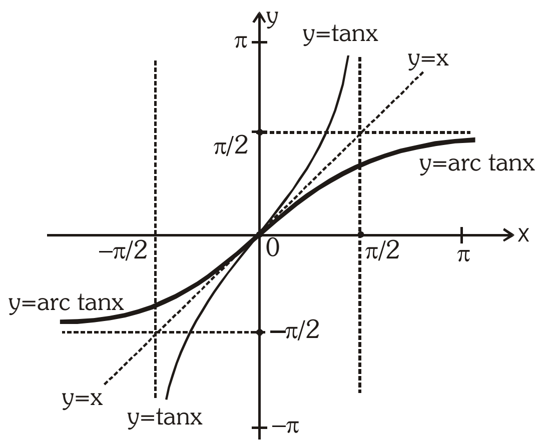
(Taking image of cos x about y = x)



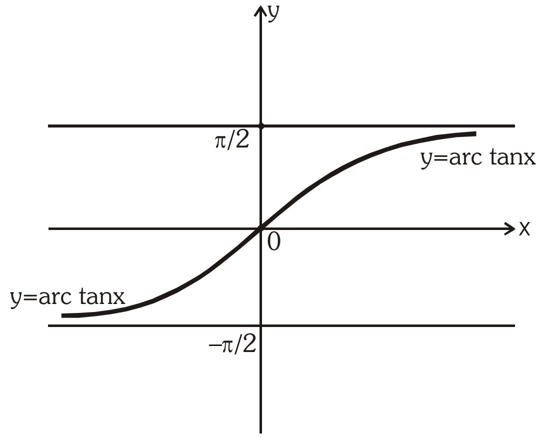
($y = \cos^{-1}x$)

(c) $f : (-\pi/2, \pi/2) \rightarrow R$ $\quad$ $f^{-1} : R \rightarrow (-\pi/2, \pi/2)$

$f(x) = \tan x$ $\quad$ $f^{-1}(x) = \tan^{-1} x$



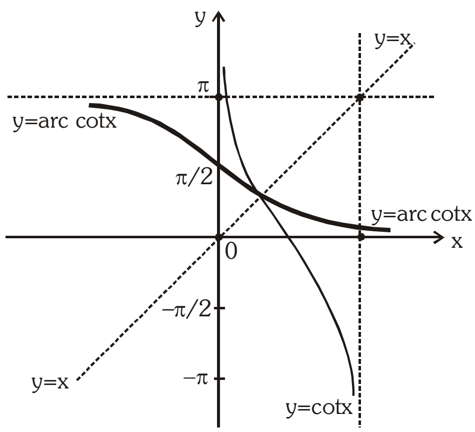
(Taking image of tan x about y = x)



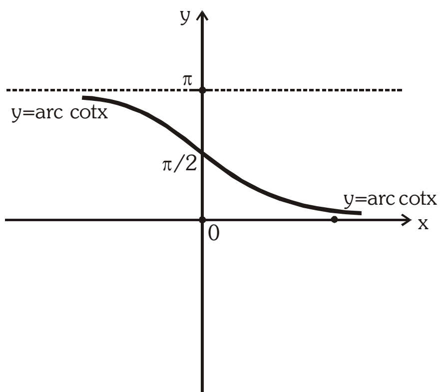
($y = \tan^{-1}x$)

(d) $f : (0, \pi) \rightarrow R$ $\quad$ $f^{-1} : R \rightarrow (0, \pi)$

$f(x) = \cot x$ $\quad$ $f^{-1}(x) = \cot^{-1} x$



(Taking image of cot x about y = x)



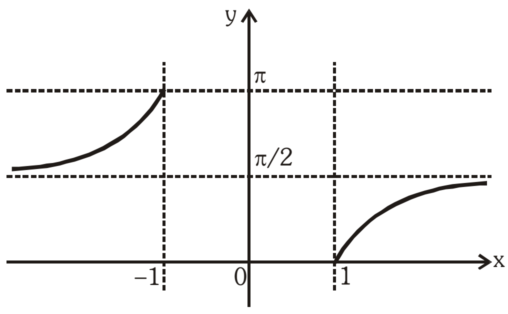
($y = \cot^{-1}x$)

(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

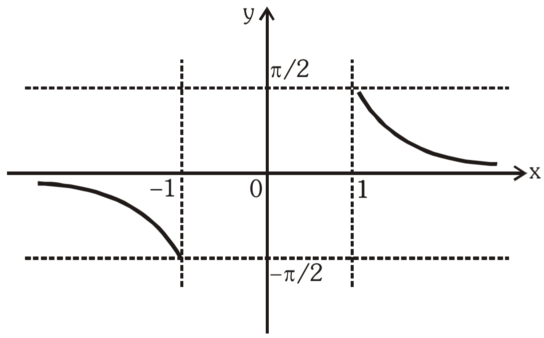


(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



From the above discussions following IMPORTANT points can be concluded :

(i) All the inverse trigonometric functions represent an angle.

(ii) If x > 0, then all six inverse trigonometric functions viz $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ represent an acute angle.

(iii) If x < 0, then $\sin^{-1}x$, $\tan^{-1}x$ & $\text{cosec}^{-1}x$ represent an angle from $-\pi/2$ to 0 (IV^{th} quadrant)

(iv) If x < 0, then $\cos^{-1} x$, $\cot^{-1}x$ & $\sec^{-1}x$ represent an obtuse angle. (II^{nd} quadrant)

(v) III^{rd} quadrant is never used in range of inverse trigonometric function.

Illustration 1 : The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Solution : $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ **Ans.(C)**

Illustration 2 : If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$ then find the value of $\sum_{i=1}^{2n} x_i$

Solution : We know, $0 \le \cos^{-1} x \le \pi$

Hence, each value $\cos^{-1}x_1, \cos^{-1}x_2, \cos^{-1}x_3, ..., \cos^{-1}x_{2n}$ are non-negative their sum is zero only when each value is zero.

i.e., $\cos^{-1}x_i = 0$ for all i

$\Rightarrow$ $x_i = 1$ for all i

$\therefore$ $\sum_{i=1}^{2n} x_i = x_1 + x_2 + x_3 + ... + x_{2n} = \underbrace{\{1 + 1 + 1 .... + 1\}}_{2n \text{ times}} = 2n$ {using (i)}

$\Rightarrow$ $\sum_{i=1}^{2n} x_i = 2n$ **Ans.**

Do yourself - 1 :

(i) If α, β are roots of the equation $6x^2 + 11x + 3 = 0$, then

(A) both $\cos^{-1}\alpha$ and $\cos^{-1}\beta$ are real (B) both $\text{cosec}^{-1}\alpha$ and $\text{cosec}^{-1}\beta$ are real

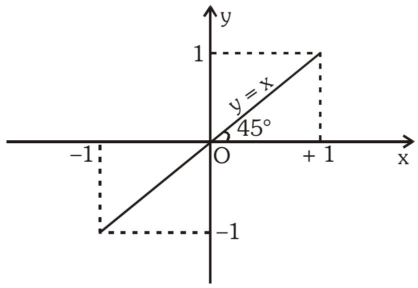
(C) both $\cot^{-1}\alpha$ and $\cot^{-1}\beta$ are real (D) none of these

(ii) If $\sin^{-1}x + \sin^{-1}y = \pi$ and $x = ky$, then find the value of $39^{2k} + 5^k$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

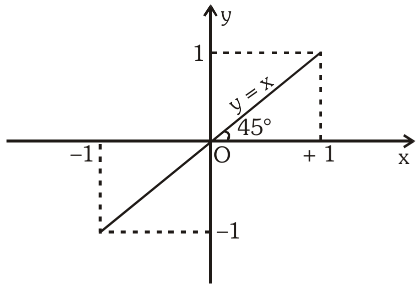
P-1 (i) $y = \sin(\sin^{-1}x) = x$

$x \in [-1,1], y \in [-1,1]$



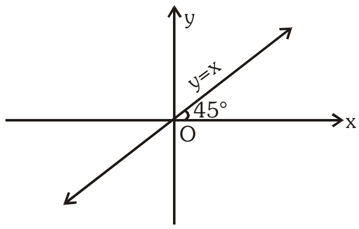
(ii) $y = \cos(\cos^{-1} x) = x$

$x \in [-1,1], y \in [-1,1]$



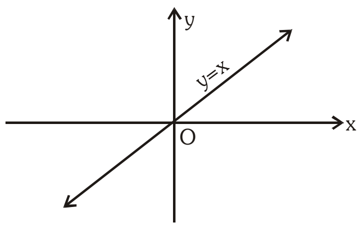
(iii) $y = \tan(\tan^{-1} x) = x$

$x \in R, y \in R$



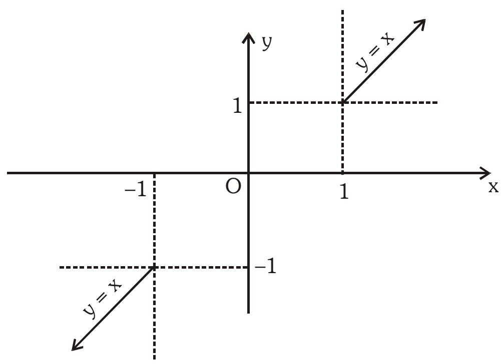
(iv) $y = \cot(\cot^{-1} x) = x$,

$x \in R; y \in R$



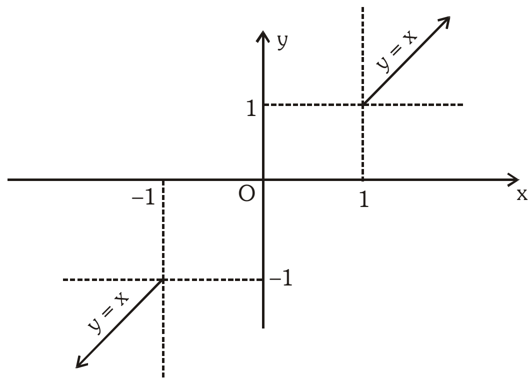
(v) $y = \text{cosec}(\text{cosec}^{-1} x) = x$,

$|x| \geq 1, |y| \geq 1$



(vi) $y = \sec(\sec^{-1} x) = x$

$|x| \geq 1$; $|y| \geq 1$



Note : All the above functions are aperiodic.

Illustration 3 : Evaluate the following :

(i) $\sin(\cos^{-1}3/5)$ (ii) $\cos(\tan^{-1} 3/4)$ (iii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$

(iv) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

Solution : (i) Let $\cos^{-1} 3/5 = \theta$. Then,

$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,

$\tan\theta = 3/4$

$\Rightarrow \cos\theta = \frac{4}{5}$ $\left\{\because \text{ as } \cos^2\theta = \frac{1}{1+\tan^2\theta}\right\}$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

(iv) Let $\tan^{-1}\left(\frac{1}{5}\right) = \theta \Rightarrow \tan\theta = \frac{1}{5}$

$\tan\left(2\theta-\frac{\pi}{4}\right) = \frac{\tan(2\theta)-1}{1+\tan 2\theta}$ and $\tan 2\theta = \frac{2\times\frac{1}{5}}{1-\frac{1}{25}} = \frac{5}{12}$ $(\because \tan\theta = \frac{1}{5})$

$\Rightarrow \tan\left(2\theta-\frac{\pi}{4}\right) = \frac{\frac{5}{12}-1}{1+\frac{5}{12}} = -\frac{7}{17}$ **Ans.**

Illustration 4 : Prove that $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1} 3) = 15$

Solution : We have,

$\sec^2 (\tan^{-1}2) + \text{cosec}^2 (\cot^{-1}3)$

$$=\left\{\sec\left(\tan^{-1}2\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}3\right)\right\}^2=\left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$=\left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2+\left\{\text{cosec}\left(\text{cosec}^{-1}\sqrt{10}\right)\right\}^2=\left(\sqrt{5}\right)^2+\left(\sqrt{10}\right)^2=15$$

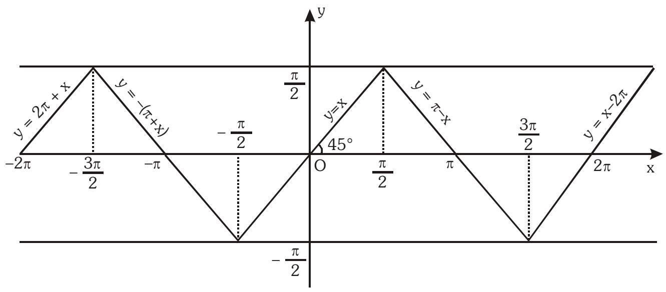
Do yourself - 2 :

Evaluate the following :

(i) $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$ **(ii)** $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right)$ **(iii)** $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)$ **(iv)** $\sin\left(\cos^{-1}\frac{3}{5}\right)$

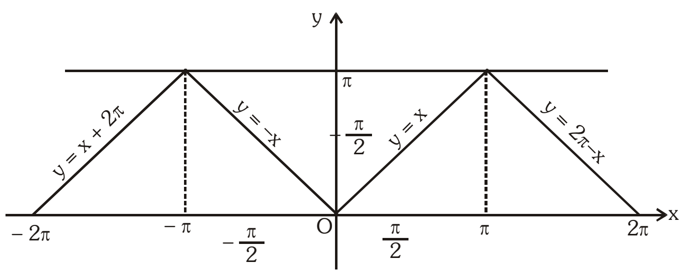
P-2 (i) $y = \sin^{-1} (\sin x), x \in R, y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x & ,-\pi\le x\le-\frac{\pi}{2}\\ x & ,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x & ,\ \frac{\pi}{2}\le x\le\pi\end{cases}$$



(ii) $y = \cos^{-1} (\cos x), x \in R, y \in [0,\pi]$, periodic with period 2π and it is an even function.

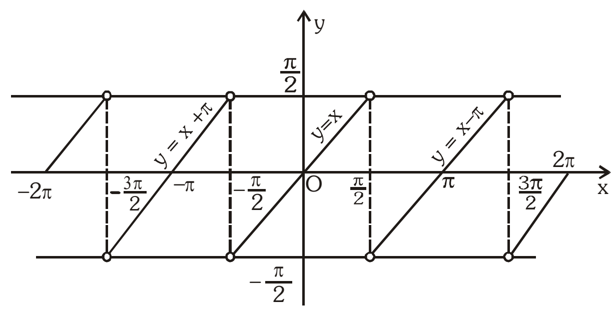
$$\cos^{-1}(\cos x)=\begin{cases}-x & ,\ -\pi\le x\le 0\\ x & ,\ 0<x\le\pi\end{cases}$$



(iii) $y = \tan^{-1} (\tan x)$

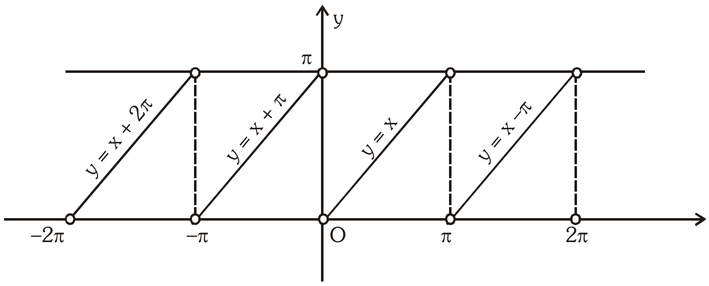
$x \in R-\left\{(2n-1)\frac{\pi}{2}, n\in I\right\};\ \ y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x)=\begin{cases}x+\pi & ,\ -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ x & ,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\ x-\pi & ,\ \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$



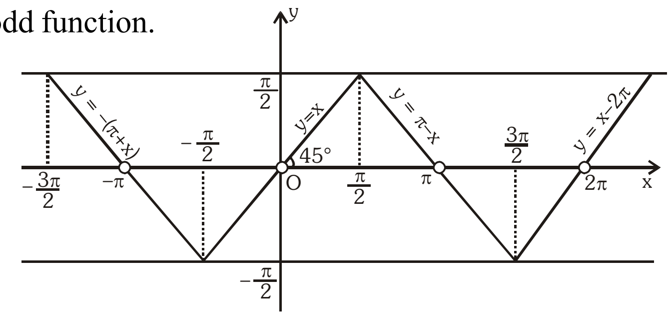
(iv) $y = \cot^{-1}(\cot x), x \in R - \{n\pi, n \in I\}, y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x-\pi, & \pi < x < 2\pi \end{cases}$$



(v) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x), x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π

and it is an odd function.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

and it is an even function.

$$x \in R - \left\{(2n-1)\frac{\pi}{2}\, n \in I\right\},\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

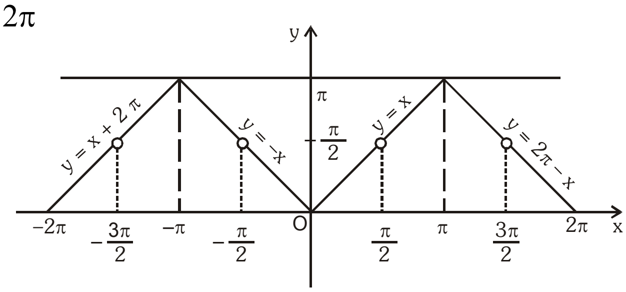


Illustration 5 : The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

(A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Solution : $\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$ **Ans.(B)**

Illustration 6 : Evaluate the following :

(i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$ (iii) $\cot^{-1}(\cot 4)$

Solution : (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore \quad \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

(ii) We know that,

$\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

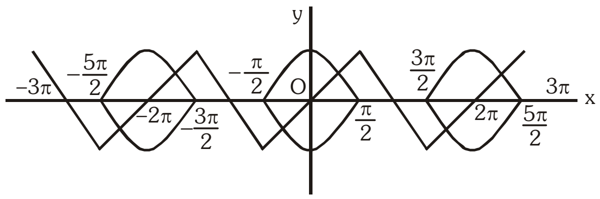
$\therefore \quad \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$

(iii) $\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$

Ans.

Illustration 7 : Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \le 3\pi$.

Solution : Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



Do yourself - 3 :

Evaluate the following :

(i) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ **(ii)** $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ **(iii)** $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

P-3 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\quad |x| \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ $\quad x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$ $\quad |x| \ge 1$

P-4 (i) $\sin^{-1}(-x) = -\sin^{-1}x$, $|x| \le 1$

(ii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \ge 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \le 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \ge 1$

P-5 (i) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $|x| \ge 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $|x| \ge 1$

(iii) $$\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \ x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \ x < 0 \end{cases}$$

Illustration 8 : Find value of x if $\cos^{-1}(-x) + \tan^{-1}(-x) - 2\sin^{-1}(x) + \sec^{-1}\left(-\frac{1}{x}\right) = \frac{\pi}{4}$ for $|x| \le 1$.

Solution : $\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \cos^{-1}(-x) = \frac{\pi}{4}$

$\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \pi - \cos^{-1}(x) = \frac{\pi}{4}$

$2\pi - 2(\sin^{-1}x + \cos^{-1}x) - \frac{\pi}{4} = \tan^{-1}x$

$2\pi - \pi - \frac{\pi}{4} = \tan^{-1}x \Rightarrow \tan^{-1}x = \frac{3\pi}{4}$ Hence no soluton **Ans.**

Do yourself - 4 :

(i) Prove the following :

(a) $\cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ (b) $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

(ii) Find the value of $\sin(\tan^{-1}a + \tan^{-1}\frac{1}{a})$; $a \ne 0$

P-6 (i) (a) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \ \& \ xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy} & \text{where } x > 0, y > 0 \ \& \ xy > 1 \\ \frac{\pi}{2}, & \text{where } x > 0, y > 0 \ \& \ xy = 1 \end{cases}$$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$ where $x > 0, y > 0$

(c) $\tan^{-1}x + \tan^{-1}y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if $x>0, y>0, z>0$ & $xy+yz+zx<1$

(ii) (a) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}] & \text{where } x>0, y>0 \ \& \ (x^2+y^2) \le 1 \\ \pi - \sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}] & \text{where } x>0, y>0 \ \& \ x^2+y^2>1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(iii) (a) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0, y > 0$

(b) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right) ; & x<y, \quad x, y>0 \\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right); & x>y, \quad x, y>0 \end{cases}$

Note : In the above results x & y are taken positive. In case if these are given as negative, we first apply **P-4** and then use above results.

Illustration 9 : Prove that

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution : (i) L.H.S. $= \tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$

$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\}$ $\left\{\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{ if } xy<1\right\}$

$= \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right) = \text{R.H.S.}$

(ii) $\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$

$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ **Ans.**

Illustration 10 : Prove that $\sin^{-1}\frac{12}{13}+\cot^{-1}\frac{4}{3}+\tan^{-1}\frac{63}{16}=\pi$

Solution : We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \qquad \left[\because \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5} \text{ and } \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]$$

$$=\pi$$

Illustration 11 : Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution : We have, L.H.S. = $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{3}{4}\right)$

$$\left[\because \cos^{-1}\left(\frac{12}{13}\right)=\tan^{-1}\left(\frac{5}{12}\right) \ \& \ \sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{3}{4}\right)\right]$$

$$\text{L.H.S.} =\tan^{-1}\left(\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right)=\tan^{-1}\left(\frac{56}{33}\right)$$

$$\text{R.H.S.} = \sin^{-1}\left(\frac{56}{65}\right)=\tan^{-1}\left(\frac{56}{33}\right)$$

L.H.S. = R.H.S. Hence Proved

Do yourself - 5 :

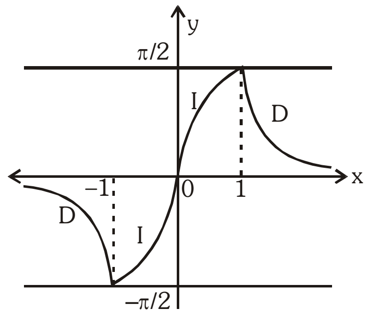
Prove the following :

(i) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{36}{85}\right)$ **(ii)** $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$

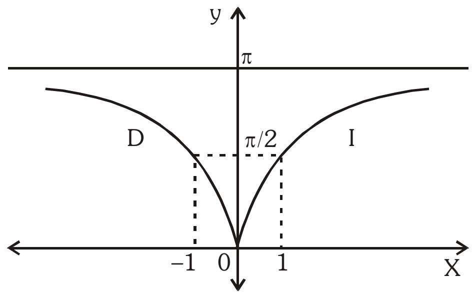
(iii) $\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{7}{24}\right)=\tan^{-1}\left(\frac{1}{2}\right)$

4. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

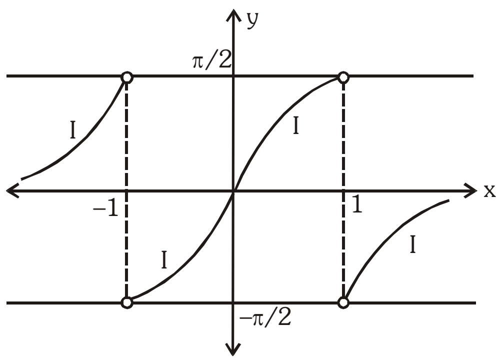
(a) $$y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & \text{if } |x|\le 1\\ \pi-2\tan^{-1}x & \text{if } x>1\\ -(\pi+2\tan^{-1}x) & \text{if } x<-1\end{cases}$$



(b) $$y=f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & \text{if } x\ge 0\\ -2\tan^{-1}x & \text{if } x<0\end{cases}$$

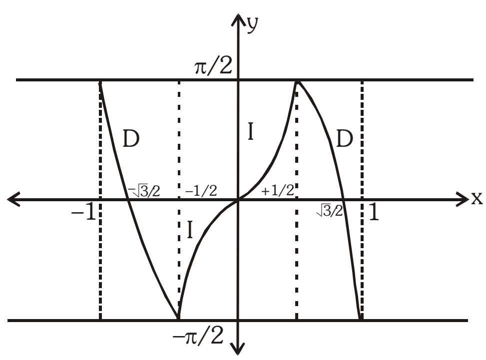


(c) $$y=f(x)=\tan^{-1}\frac{2x}{1-x^2}=\begin{cases}2\tan^{-1}x & \text{if } |x|<1\\ \pi+2\tan^{-1}x & \text{if } x<-1\\ -(\pi-2\tan^{-1}x) & \text{if } x>1\end{cases}$$



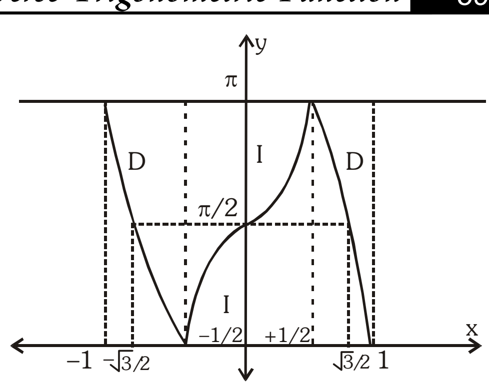
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$=\begin{cases}-(\pi+3\sin^{-1}x) & \text{if } -1\le x\le -\frac{1}{2}\\ 3\sin^{-1}x & \text{if } -\frac{1}{2}\le x\le \frac{1}{2}\\ \pi-3\sin^{-1}x & \text{if } \frac{1}{2}\le x\le 1\end{cases}$$

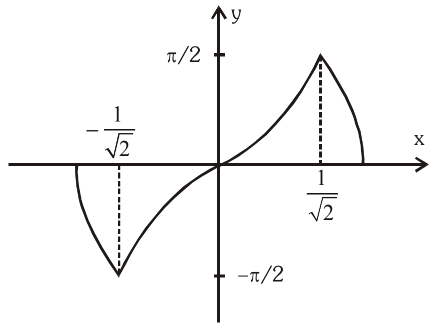


(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$



(f) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -\left(\pi + 2\sin^{-1}x\right) & -1 \le x \le -\frac{1}{\sqrt{2}} \\ 2\sin^{-1}x & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$



(g) $$\cos^{-1}\left(2x^2 - 1\right) = \begin{cases} 2\cos^{-1}x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & -1 \le x \le 0 \end{cases}$$

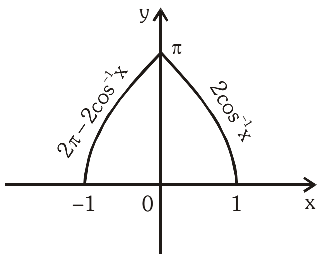


Illustration 12 : Prove that : $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Solution : We have, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

Illustration 13 : Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$

Solution : We have, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left\{\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right\}=\frac{1}{2}\times 2\tan^{-1}\sqrt{x}=\tan^{-1}\sqrt{x}.$

Alter : Putting $\sqrt{x}=\tan\theta$, we have $\Rightarrow \theta\in\left[0,\frac{\pi}{4}\right]$

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\theta \quad \because \left(2\theta\in\left[0,\frac{\pi}{2}\right]\right)$$

$$=\tan^{-1}\sqrt{x}=\text{LHS}$$

Illustration 14 : Prove that : (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution : (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left\{2\tan^{-1}\frac{1}{5}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}, \text{if } |x|<1\right]$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}=\tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\}-\tan^{-1}.\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\}+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1} \qquad \left[\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\right]$$

$$= 2\tan^{-1}\frac{13}{39}+\tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if}\ |x|<1\right]$$

$$= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

Do yourself - 6 :

Prove the following results :

(i) $2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\left(\frac{4}{7}\right)$ **(ii)** $2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$

6. EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration 15 : The equation $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$ has

(A) no solution (B) only one solution (C) two solutions (D) three solutions

Solution : Given equation is $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x + (\cos^{-1}x + \sin^{-1}x) = \frac{11\pi}{6} \quad \Rightarrow \cos^{-1}x + \frac{\pi}{2} = \frac{11\pi}{6} \Rightarrow \cos^{-1}x = 4\pi/3$

which is not possible as $\cos^{-1}x \in [0, \pi]$ **Ans.(A)**

Illustration 16 : If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = 5\pi^2/8$, then x is equal to-

(A) –1 (B) 0 (C) 1 (D) none of these

Solution : The given equation can be written as $(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x = 5\pi^2/8$

Since $\tan^{-1}x + \cot^{-1}x = \pi/2$ we have

$(\pi/2)^2 - 2\tan^{-1}x(\pi/2 - \tan^{-1}x) = 5\pi^2/8$

$\Rightarrow 2(\tan^{-1}x)^2 - 2(\pi/2)\tan^{-1}x - 3\pi^2/8 = 0 \Rightarrow \tan^{-1}x = -\pi/4 \Rightarrow x = -1$ **Ans. (A)**

Illustration 17 : Solve the equation : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

Solution : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

taking tangent on both sides

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)=1 \;\Rightarrow\; \frac{\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)+\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}=1$$

$$\Rightarrow \quad \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}.\frac{x+1}{x+2}}=1 \;\Rightarrow\; \frac{(x-1)(x+2)+(x-2)(x+1)}{x^2-4-(x^2-1)}=1 \;\Rightarrow\; 2x^2-4=-3 \;\Rightarrow\; x=\pm\frac{1}{\sqrt{2}}$$

Now verify $x=\frac{1}{\sqrt{2}}$

$$=\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}-2}\right)+\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}+1}{\frac{1}{\sqrt{2}}+2}\right)=\tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right)+\tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)$$

$$=\tan^{-1}\left(\frac{(2\sqrt{2}+1)(\sqrt{2}-1)+(2\sqrt{2}-1)(\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)-(\sqrt{2}-1)(\sqrt{2}+1)}\right)=\tan^{-1}\left(\frac{6}{6}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$x=-\frac{1}{\sqrt{2}}$

$$=\tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}-1}{-\frac{1}{\sqrt{2}}-2}\right)+\tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}+1}{-\frac{1}{\sqrt{2}}+2}\right)=\tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)+\tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right)\ \{\text{same as above}\}$$

$$=\tan^{-1}(1)=\frac{\pi}{4}$$

$\therefore \quad x=\pm\frac{1}{\sqrt{2}}$ are solutions **Ans.**

Illustration 18 : Solve the equation : $2\tan^{-1}(2x+1)=\cos^{-1}x$.

Solution : Here, $2\tan^{-1}(2x+1)=\cos^{-1}x$

or $\cos(2\tan^{-1}(2x+1))=x$ $\qquad \left\{\text{We know } \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$

$\therefore \quad \frac{1-(2x+1)^2}{1+(2x+1)^2}=x \;\Rightarrow\; (1-2x-1)(1+2x+1)=x(4x^2+4x+2)$

$\Rightarrow \quad -2x \cdot 2(x+1) = 2x(2x^2+2x+1) \quad \Rightarrow \quad 2x(2x^2+2x+1+2x+2) = 0$

$\Rightarrow \quad 2x(2x^2+4x+3) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad 2x^2+4x+3 = 0 \quad$ {No solution}

Verify $\quad x = 0$

$2\tan^{-1}(1) = \cos^{-1}(1) \quad \Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore \quad x = 0$ is only the solution **Ans.**

Do yourself - 7 :

Solve the following equation for x :

(i) $\sin\left[\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1} x\right] = 1$ **(ii)** $\cos^{-1} x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

7. INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTION :

Illustration 19 : Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution : $\sin^{-1}(\sin 5) > x^2 - 4x \quad \Rightarrow \quad \sin^{-1}[\sin(5-2\pi)] > x^2 - 4x$

$\Rightarrow \quad x^2 - 4x < 5 - 2\pi \Rightarrow \quad x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow \quad 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi} \quad \Rightarrow \quad x \in (2-\sqrt{9-2\pi},\ 2+\sqrt{9-2\pi})$

Ans.

Illustration 20 : Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Solution : $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow \quad ([\cot^{-1}x] - 3)^2 \le 0 \Rightarrow [\cot^{-1}x] = 3 \Rightarrow 3 \le \cot^{-1}x < 4 \Rightarrow x \in (-\infty, \cot 3]$

Illustration 21 : If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of n is -

(A) 1 (B) 5 (C) 9 (D) none of these

Solution : $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$

$\Rightarrow \quad \cot\left(\cot^{-1}\left(\frac{n}{\pi}\right)\right) < \cot\left(\frac{\pi}{6}\right) \Rightarrow \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \pi\sqrt{3} \quad \Rightarrow \quad n < 5.5$ (approx)

$\Rightarrow \quad n = 5 \quad \because (n \in N)$ **Ans. (B)**

Do yourself - 8 :

(i) Solve the inequality $\tan^{-1}x > \cot^{-1}x$.

(ii) Complete solution set of inequation $(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$, is

(A) $\left[0, \frac{1}{\sqrt{2}}\right)$ (B) $\left[-1, \frac{1}{\sqrt{2}}\right)$ (C) $(-1, \sqrt{2})$ (D) none of these

8. SUMMATION OF SERIES :

Illustration 22 : Prove that :

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

Solution : L.H.S.

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\left(\tan^{-1}c_2-\tan^{-1}c_1\right)+\left(\tan^{-1}c_3-\tan^{-1}c_2\right)+\ldots.+\left(\tan^{-1}c_n-\tan^{-1}c_{n-1}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{1}{c_1}\right)-\tan^{-1}c_1+\tan^{-1}c_n+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\left(\cot^{-1}c_1+\tan^{-1}c_1\right)+\left(\tan^{-1}c_n+\cot^{-1}c_n\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\frac{\pi}{2}+\frac{\pi}{2}=\tan^{-1}\left(\frac{x}{y}\right)=\text{R.H.S.}$$

Do yourself - 9 :

(i) Evaluate : $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$

Miscellaneous Illustrations :

Illustration 23 : If $\tan^{-1} y = 4\tan^{-1} x, \left(|x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution : We have $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1} y = 2\tan^{-1}\frac{2x}{1-x^2}$ (as |x| < 1)

$$= \tan^{-1}\frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\frac{4x(1-x^2)}{x^4-6x^2+1} \qquad \left(\text{as}\left|\frac{2x}{1-x^2}\right| < 1\right)$$

$\Rightarrow \quad y = \frac{4x(1-x^2)}{x^4-6x^2+1}$

If $\quad x = \tan\frac{\pi}{8}$

$\Rightarrow \quad \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2} \Rightarrow$ y is not defined $\Rightarrow x^4 - 6x^2 + 1 = 0$ **Ans.**

Illustration 24 : If $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show A > B.

Solution : We have, $A = 2\tan^{-1}\left(2\sqrt{2}-1\right) = 2\tan^{-1}(1.828)$

$\Rightarrow \quad A > 2\tan^{-1}\left(\sqrt{3}\right) \Rightarrow A > \frac{2\pi}{3}$ .... (i)

also we have, $\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$

$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$

also, $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3.\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) = \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow \quad 3\sin^{-1}(1/3) < \sin^{-1}\left(\sqrt{3}/2\right) \Rightarrow 3\sin^{-1}(1/3) < \pi/3$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}\left(\sqrt{3}/2\right) \Rightarrow \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \frac{2\pi}{3}$ .... (ii)

From (i) and (ii), we have A > B.

Illustration 25 : If $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ then find the sum of all possible values of $\tan\theta$.

Solution : $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) \Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$

$$\Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}\right] \Rightarrow \theta = \tan^{-1}(2\tan^2\theta) - \frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right) \qquad \text{........ (i)}$$

taking tangent on both sides

$$\Rightarrow \quad \tan\theta = \frac{6\tan^2\theta - \tan\theta}{3+2\tan^3\theta} \Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$$

$$\Rightarrow \quad 2\tan\theta(\tan^3\theta - 3\tan\theta + 2) = 0 \quad \Rightarrow \quad 2\tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$$

$\Rightarrow \quad \tan\theta = 0, 1, -2$ which satisfy equation (i)

$\therefore \quad$ sum = 0 + 1 – 2 = –1 **Ans.**

ANSWERS FOR DO YOURSELF

1 :	**(i)**	C	**(ii)**	1526				
2 :	**(i)**	$\frac{15}{8}$	**(ii)**	$\frac{1}{\sqrt{10}}$	**(iii)**	1	**(iv)**	$\frac{4}{5}$
3 :	**(i)**	$\frac{\pi}{6}$	**(ii)**	$\frac{\pi}{6}$	**(iii)**	$\frac{\pi}{6}$		
4 :	**(ii)**	$\begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$						
7 :	**(i)**	$\frac{1}{5}$	**(ii)**	1	**(iii)**	$\sqrt{3}$		
8 :	**(i)**	$(1, \infty)$	**(ii)**	B				
9 :	**(i)**	$\pi/4$						

EXERCISE (O-1)

Straight Objective Type

1. The domain of the function $\sin^{-1}\left(\log_2\left(\frac{x}{3}\right)\right)$ is-

(A) $\left[\frac{1}{2},3\right]$ (B) $\left[\frac{1}{2},3\right]$ (C) $\left[\frac{3}{2},6\right]$ (D) $\left[\frac{1}{2},2\right]$

IT0001

2. Domain of the function $f(x) = \log_e \cos^{-1}\{\sqrt{x}\}$ is, where {.} represents fractional part function -

(A) $x \in R$ (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) $x \in R - \{x \mid x \in I\}$

IT0002

3. The value of $\tan^2(\sec^{-1}3) + \cot^2(\text{cosec}^{-1}4)$ is -

(A) 9 (B) 16 (C) 25 (D) 23

IT0003

4. If $x > 0$ $\cos^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{16}{x}\right)$ then x is -

(A) 12 (B) 16 (C) 20 (D) 320

IT0006

5. If $\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1}x$, then -

(A) $x \in [-1, 0]$ (B) $x \in [0, 1]$ (C) $x \in \left[0, \frac{1}{\sqrt{2}}\right]$ (D) $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

IT0008

6. Number of integral ordered pairs (a,b) for which $\sin^{-1}(1 + b + b^2 + \ldots \infty) + \cos^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} - \ldots\ldots\infty\right) = \frac{\pi}{2}$ is-

(A) 0 (B) 4 (C) 9 (D) infinitely many

IT0007

7. The value of $\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right\}$ is -

(A) 5 (B) 6 (C) 0 (D) 10

IT0018

Multiple Correct Answer Type

8. Consider the function $f(x) = e^x$ and $g(x) = \sin^{-1}x$, then which of the following is/are necessarily true.

(A) Domain of gof = Domain of f (B) Range of $gof \subset$ Range of g

(C) Domain of gof is $(-\infty, 0]$ (D) Range of gof is $\left[-\frac{\pi}{2}, 0\right)$

IT0013

9. Let $f(x) = \sin^{-1}(\tan x) + \cos^{-1}(\cot x)$ then

(A) $f(x) = \frac{\pi}{2}$ wherever defined (B) domain of $f(x)$ is $x = n\pi \pm \frac{\pi}{4}$, $n \in I$

(C) period of $f(x)$ is $\frac{\pi}{2}$ (D) $f(x)$ is many one function

IT0011

10. Let $f(x) = e^{x^3 - x^2 + x}$ be an invertible function such that $f^{-1} = g$, then -

(A) $g(e) = 0$ (B) Domain of 'g' is R^+ (C) Range of 'g' is R (D) $f(g(e)) = e$

IT0014

11. Value of $3\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ is greater than

(A) $\frac{\pi}{2}$ (B) $\frac{2\pi}{3}$ (C) $\frac{3\pi}{4}$ (D) $\frac{5\pi}{6}$

IT0015

12. Which of the following is/are correct ?

(A) $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) \forall\ x \in R - \{0\}$

(B) If $f : R \to R$ such that $f(x) = \text{sgn}(e^x)$ then $f(x)$ is an into function.

(C) If $f : R^+ \to R$ such that $f(x) = \sin x + x$ then $f(x)$ is an odd function.

(D) If $f : R \to R$ such that $f(x) = \frac{e^x}{e^{[x]}}$ then $f(x)$ is a periodic function .

(where [.] represents greatest integer function)

IT0012

EXERCISE (O-2)

Straight Objective Type

1. The range of the function $f(x) = \sin^{-1}(\log_2(-x^2 + 2x + 3))$ is -

(A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left[-\frac{\pi}{2}, 0\right]$ (C) $\left[0, \frac{\pi}{2}\right]$ (D) $[-1, 1]$

IT0016

2. Range of $f(x) = \cot^{-1}(\log_e(1 - x^2))$ is -

(A) $(0,\pi)$ (B) $\left(0, \frac{\pi}{2}\right)$ (C) $\left[\frac{\pi}{2}, \pi\right)$ (D) $\left(0, \frac{\pi}{2}\right]$

IT0017

3. $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \tan^{-1}\frac{2r+1}{r^4 + 2r^3 + r^2 + 1}$ is equal to -

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{8}$

IT0009

4. Number of solution(s) of the equation $\cos^{-1}\sqrt{x} - \sin^{-1}\sqrt{x-1} + \cos^{-1}\sqrt{1-x} - \sin^{-1}\frac{1}{\sqrt{x}} = \frac{\pi}{2}$ is -

(A) 0 (B) 1 (C) 2 (D) 4

IT0019

5. $\sum\limits_{r=0}^{\infty} \tan^{-1}\left(\frac{r((r+1)!)}{(r+1) + ((r+1)!)^2}\right)$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\cot^{-1}3$ (D) $\tan^{-1}2$

IT0020

Multiple Correct Answer Type

6. Let $f(x)=\sqrt{\cos^{-1}\sqrt{1-x^2}-\sin^{-1}x}$ then which of the following statement/s is/are correct -

(A) Domain of $f(x)$ is $[-1,1]$ (B) Domain of $f(x)$ is $[0,1]$

(C) Range of $f(x)$ is $\{0\}$ (D) Range of $f(x)$ is $\left[0,\sqrt{\pi}\right]$

IT0021

7. If $\alpha=2\tan^{-1}(\sqrt{3-2\sqrt{2}})+\sin^{-1}\left(\frac{1}{\sqrt{6}-\sqrt{2}}\right)$, $\beta=\cot^{-1}\left(\sqrt{3}-2\right)+\frac{1}{8}\sec^{-1}(-2)$ and $\gamma=\tan^{-1}\frac{1}{\sqrt{2}}+\cos^{-1}\frac{1}{\sqrt{3}}$, then

(A) $\alpha=\beta$ (B) $\alpha+\beta=3\gamma$ (C) $4(\beta-\gamma)=\alpha$ (D) $\beta=\gamma$

IT0022

8. If α is only real root of the equation $x^3+(\cos 1)x^2+(\sin 1)x+1=0$, then $\left(\tan^{-1}\alpha+\tan^{-1}\frac{1}{\alpha}\right)$ **cannot** be equal to-

(A) 0 (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{2}$ (D) π

IT0023

EXERCISE (S-1)

1. (a) Find the following :

(i) $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iv) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

IT0031

(b) Find the following :

(i) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$ (ii) $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

(iii) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ **IT0032**

2. Find the domain of definition the following functions.
(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(i) $f(x)=\arccos\frac{2x}{1+x}$ **IT0033**

(ii) $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\frac{1+x^2}{2x}$ **IT0034**

(iii) $f(x)=\sin^{-1}\left(\frac{x-3}{2}\right)-\log_{10}(4-x)$ **IT0035**

(iv) $f(x)=\sin^{-1}(2x+x^2)$ **IT0036**

(v) $f(x)=\sqrt{3-x}+\cos^{-1}\left(\frac{3-2x}{5}\right)+\log_6(2|x|-3)+\sin^{-1}(\log_2 x)$ **IT0038**

3. Identify the pair(s) of functions which are identical. Also plot the graphs in each case.

(a) $y=\tan(\cos^{-1}x);\ y=\dfrac{\sqrt{1-x^2}}{x}$ (b) $y=\tan(\cot^{-1}x); y=\dfrac{1}{x}$

(c) $y=\sin(\text{arc}\tan x); y=\dfrac{x}{\sqrt{1+x^2}}$ (d) $y=\cos(\text{arc}\tan x); y=\sin(\text{arc}\cot x)$

IT0040

4. Let y = $\sin^{-1}(\sin 8)-\tan^{-1}(\tan 10)+\cos^{-1}(\cos 12)-\sec^{-1}(\sec 9)+\cot^{-1}(\cot 6)-\text{cosec}^{-1}(\text{cosec } 7)$. If y simplifies to $a\pi+b$, then find $(a-b)$.

IT0041

5. If α and β are the roots of the equation $x^2+5x-49=0$, then find the value of $\cot(\cot^{-1}\alpha+\cot^{-1}\beta)$.

IT0044

6. If $a>b>c>0$, then find the value of : $\cot^{-1}\left(\dfrac{1+ab}{a-b}\right)+\cot^{-1}\left(\dfrac{1+bc}{b-c}\right)+\cot^{-1}\left(\dfrac{1+ca}{c-a}\right)$.

IT0045

7. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\dfrac{1}{2}\right), \tan^{-1}\left(\dfrac{1}{2}+k\right)$ and $\tan^{-1}\left(\dfrac{1}{2}+2k\right)$.

IT0046

8. Find the simplest value of

(a) $f(x)=\text{arc}\cos x+\text{arc}\cos\left(\dfrac{x}{2}+\dfrac{1}{2}\sqrt{3-3x^2}\right),\ x\in\left(\dfrac{1}{2},1\right)$ **IT0047**

(b) $f(x)=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right),\ x\in R-\{0\}$ **IT0048**

9. Least integral value of x for which inequality $\sin^{-1}\left(\sin\left(\dfrac{2e^x+3}{e^x+1}\right)\right)>\pi-\dfrac{5}{2}$ holds, is

IT0029

10. Number of integral solutions of the equation $2\sin^{-1}\sqrt{x^2-x+1}+\cos^{-1}\sqrt{x^2-x}=\dfrac{3\pi}{2}$ is

IT0027

11. If $\sin^{-1}\sin\left(\dfrac{10\pi}{3}\right)+\cos^{-1}\cos\dfrac{22\pi}{3}+\tan^{-1}\tan 10=a\pi+b$, then $\left|\dfrac{3ab}{80}\right|$ is equal to

IT0026

12. Solve the following :

(a) $\sin^{-1}x+\sin^{-1}2x=\dfrac{\pi}{3}$ **IT0053**

(b) $\tan^{-1}\dfrac{1}{1+2x}+\tan^{-1}\dfrac{1}{1+4x}=\tan^{-1}\dfrac{2}{x^2}$ **IT0054**

(c) $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}(3x)$ **IT0055**

(d) $\sin^{-1}x+\sin^{-1}y=\dfrac{2\pi}{3}$ & $\cos^{-1}x-\cos^{-1}y=\dfrac{\pi}{3}$ **IT0057**

13. Find the sum of the series :

(a) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + ...$ to n terms. **IT0058**

(b) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + ... + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + ... \infty$ **IT0059**

(c) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13}$ to n terms. **IT0060**

(d) $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{65}} + \sin^{-1}\frac{1}{\sqrt{325}} + ... + \sin^{-1}\frac{1}{\sqrt{4n^4+1}} + ...\infty$ terms. **IT0061**

EXERCISE (S-2)

1. If $\alpha = 2 \text{ arc} \tan\left(\frac{1+x}{1-x}\right)$ & $\beta = \text{arc} \sin\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$, what the value of $\alpha + \beta$ will be if $x > 1$.
IT0062

2. Solve the following :

(a) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$ **IT0063**

(b) $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$ $(a > 0,\ b > 0)$ **IT0064**

3. Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.
IT0066

4. Prove that the equation, $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \alpha\pi^3$ has no roots for $\alpha < \frac{1}{32}$ and $\alpha > \frac{7}{8}$.
IT0067

5. Solve the following inequalities :

(a) $\text{arc} \cot^2 x - 5 \text{ arc} \cot x + 6 > 0$ **IT0068**

(b) $\text{arc} \sin x > \text{arc} \cos x$ **IT0069**

(c) $\tan^2(\text{arc} \sin x) > 1$ **IT0070**

6. If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10} \tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.
IT0072

7. Solve for x : $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.
IT0074

8. Find the set of values of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution.
IT0075

EXERCISE (JM)

1. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is : **[JEE (Main)-2015]**

(1) $\frac{3x-x^3}{1+3x^2}$ (2) $\frac{3x+x^3}{1+3x^2}$ (3) $\frac{3x-x^3}{1-3x^2}$ (4) $\frac{3x+x^3}{1-3x^2}$

IT0081

2. If $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}\left(x>\frac{3}{4}\right)$, then x is equal to : **[JEE (Main)-Jan 19]**

(1) $\frac{\sqrt{145}}{12}$ (2) $\frac{\sqrt{145}}{10}$ (3) $\frac{\sqrt{146}}{12}$ (4) $\frac{\sqrt{145}}{11}$

IT0082

3. If $x = \sin^{-1}(\sin 10)$ and $y=\cos^{-1}(\cos 10)$, then $y - x$ is equal to: **[JEE (Main)-Jan 19]**

(1) π (2) 7π (3) 0 (4) 10

IT0083

4. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is : **[JEE (Main)-Jan 19]**

(1) $\frac{22}{23}$ (2) $\frac{23}{22}$ (3) $\frac{21}{19}$ (4) $\frac{19}{21}$

IT0084

5. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval :- **[JEE (Main)-Jan 19]**

(1) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$ (2) $(\cot 5, \cot 4)$

(3) $(\cot 2, \infty)$ (4) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

IT0085

6. Considering only the principal values of inverse functions, the set $A=\left\{x\geq 0:\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$ **[JEE (Main)-Jan 19]**

(1) is an empty set (2) Contains more than two elements

(3) Contains two elements (4) is a singleton

IT0086

7. If $\alpha=\cos^{-1}\left(\frac{3}{5}\right), \beta=\tan^{-1}\left(\frac{1}{3}\right)$, where $0<\alpha, \beta<\frac{\pi}{2}$, then $\alpha-\beta$ is equal to : **[JEE (Main)-Apr 19]**

(1) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (2) $\tan^{-1}\left(\frac{9}{14}\right)$ (3) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

IT0087

8. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \le x \le 1, -2 \le y \le 2, x \le \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[JEE (Main)-Apr 19]**

(1) $4\sin^2\alpha - 2x^2y^2$ (2) $4\cos^2\alpha + 2x^2y^2$ (3) $4\sin^2\alpha$ (4) $2\sin^2\alpha$

IT0088

9. The value of $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$ is equal to : **[JEE (Main)-Apr 19]**

(1) $\pi-\sin^{-1}\left(\frac{63}{65}\right)$ (2) $\pi-\cos^{-1}\left(\frac{33}{65}\right)$ (3) $\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$ (4) $\frac{\pi}{2}-\cos^{-1}\left(\frac{9}{65}\right)$

IT0089

EXERCISE (JA)

1. If 0 < x< 1, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[JEE 2008, 3]**

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

IT0091

2. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE(Advanced) 2013, 2]**

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

IT0092

3. Let $f : [0,4\pi] \to [0,\pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0,4\pi]$ satisfying the equation $f(x)=\frac{10-x}{10}$ is **[JEE(Advanced)-2014, 3]**

IT0094

4. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE(Advanced)-2015, 4]**

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha+\beta) > 0$ (D) $\cos\alpha < 0$

IT0095

5. The number of real solutions of the equation

$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$ lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is____

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.) **[JEE(Advanced)-2018]**

IT0096

6. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.)

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST-I	LIST-II
P. The range of f is	**1.** $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	**2.** $(0, 1)$
R. The domain of f contains	**3.** $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :
(A) **P → 4; Q → 2; R → 1; S → 1**
(B) **P → 3; Q → 3; R → 6; S → 5**
(C) **P → 4; Q → 2; R → 1; S → 6**
(D) **P → 4; Q → 3; R → 6; S → 5**

[JEE(Advanced)-2018]
IT0097

7. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals

[JEE(Advanced)-2019, 3(0)]
IT0098

ANSWER KEY

INVERSE TRIGONOMETRIC FUNCTION

EXERCISE (O-1)

1. C **2.** B **3.** D **4.** C **5.** A **6.** A **7.** C **8.** B,C

9. A,B,C,D **10.** B,C,D **11.** A,B **12.** B,D

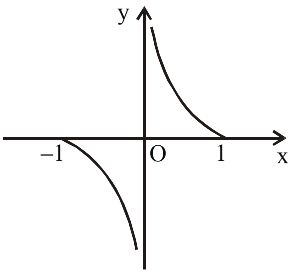
EXERCISE (O-2)

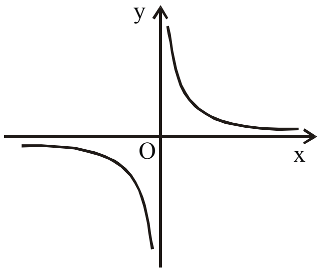
1. A **2.** C **3.** A **4.** A **5.** B **6.** A,D **7.** A,C **8.** A,B,D

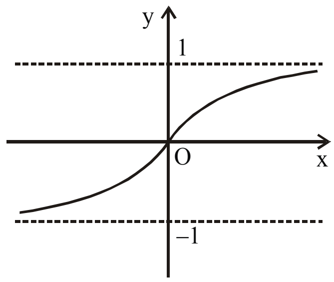
EXERCISE (S-1)

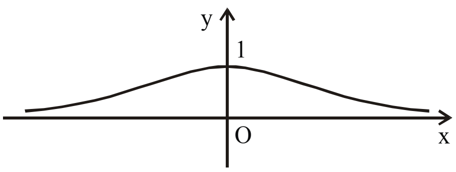
1. **(a)** (i) $\frac{1}{\sqrt{3}}$, (ii) $\frac{5\pi}{6}$, (iii) $\frac{4}{5}$, (iv) $\frac{17}{6}$; **(b)** (i) $\frac{1}{2}$, (ii) -1, (iii) $-\frac{\pi}{4}$

2. **(i)** $-1/3 \le x \le 1$, **(ii)** $\{1, -1\}$, **(iii)** $1 \le x < 4$, **(iv)** $[-(1+\sqrt{2}), (\sqrt{2}-1)]$, **(v)** $(3/2,\ 2]$

3. (a) 

(b) 

(c) 

and (d) 

all are identical.

4. 53 **5.** 10 **6.** π **7.** $k = \frac{11}{4}$ **8.** **(a)** $\frac{\pi}{3}$; **(b)** $\frac{1}{2}\tan^{-1} x$

9. 1 **10.** 2 **11.** 1 **12.** **(a)** $x = \frac{1}{2}\sqrt{\frac{3}{7}}$; **(b)** $x = 3$; **(c)** $x = 0, \frac{1}{2}, -\frac{1}{2}$; **(d)** $x = \frac{1}{2}, y = 1$

13. **(a)** $\text{arc}\cot\left[\frac{2n+5}{n}\right]$, **(b)** $\frac{\pi}{4}$, **(c)** $\text{arc}\tan(x+n) - \text{arc}\tan x$, **(d)** $\frac{\pi}{4}$

EXERCISE (S-2)

1. $-\pi$ **2.** **(a)** $x = \frac{4}{3}$ **(b)** $x = \frac{a-b}{1+ab}$ **3.** $x = 1; y = 2$ & $x = 2; y = 7$

5. **(a)** $(\cot 2, \infty) \cup (-\infty, \cot 3)$ **(b)** $\left(\frac{\sqrt{2}}{2}, 1\right]$ **(c)** $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$

6. $k = 25$ **7.** $x \in (-1, 1)$ **8.** $a \in [-2\pi, \pi] - \{0\}$

EXERCISE (JM)

1. 3 **2.** 1 **3.** 1 **4.** 3 **5.** 3 **6.** 4 **7.** 1 **8.** 3 **9.** 3

EXERCISE (JA)

1. C **2.** B **3.** 3 **4.** B,C,D **5.** 2 **6.** A **7.** 0.00