DIFFERENTIAL EQUATION

1. **DIFFERENTIAL EQUATION:**

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation. We are

concerned with ordinary differential equations only.

2. ORDER OF DIFFERENTIAL EQUATION:

The order of a differential equation is the order of the highest differential coefficient occurring in it.

3. DEGREE OF DIFFERENTIAL EQUATION:

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation:

$$f\left(x,y\right)\!\!\left\lceil\frac{d^{m}y}{dx^{m}}\right\rceil^{\!p} + \varphi\!\left(x,y\right)\!\!\left\lceil\frac{d^{m-l}\left(y\right)}{dx^{m-l}}\right\rceil^{\!q} + \dots = 0 \text{ is of order } m \text{ \& degree p.}$$

Note:

- (i) The exponents of all the differential coefficient should be free from radicals and fraction.
- (ii) The degree is always positive natural number.
- (iii) The degree of differential equation may or may not exist.

Illustration 1: Find the order and degree of the following differential equation:

(i)
$$\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$$
 (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx} = \sqrt{3x + 5}$

Solution:

- (i) The given differential equation can be re-written as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$
 - Hence order is 2 and degree is 3.
- (ii) The given differential equation has the order 2. Since the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.
- (iii) Its order is 1 and degree 1.

Ans.

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The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are -Illustration 2:

Solution: Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations). Ans. (A)

Do yourself - 1:

Find the order and degree of following differential equations

(i)
$$[1 + (y')^2]^{1/2} = x^2 + y$$
 (ii) $(1 + y')^{1/2} = y''$

(ii)
$$(1 + y')^{1/2} = y'$$

(iii)
$$y' = \sin y$$

FORMATION OF A DIFFERENTIAL EQUATION: 4.

In order to obtain a differential equation whose solution is

$$f(x_1, y_1, c_1, c_2, c_3, \dots, c_n) = 0$$

where c_1, c_2, \dots, c_n are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require (n+1) equations.

A differential equation is obtained as follows:

- Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.
- (b) Eliminate the arbitrary constants.
- The eliminant is the required differential equation. (c)

Note:

- A differential equation represents a family of curves all satisfying some common properties. (i) This can be considered as the geometrical interpretation of the differential equation.
- For there being n differentiation, the resulting equation must contain a derivative of nth order i.e. equal to number of independent arbitrary constant.
- Illustration 3: Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.
- Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is Solution: $(y-\beta)^2 = a(x-\alpha)$

Differentiating both sides, we get $2(y-\beta) \frac{dy}{dx} = a$

Again differentiating, we get

$$\Rightarrow 2(y - \beta) \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \qquad \Rightarrow \quad a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$$
 Ans

Illustration 4: Find the differential equation whose solution represents the family: $c(y+c)^2 = x^3$

Solution: $c (y + c)^2 = x^3$

Differentiating, we get, c. $[2(y+c)]\frac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have

$$\frac{2x^{3}}{(y+c)^{2}}(y+c)\frac{dy}{dx} = 3x^{2} \implies \frac{2x^{3}}{y+c}\frac{2x^{3}}{y+c} = 3x^{2}$$

i.e.
$$\frac{2x}{y+c} \frac{dy}{dx} = 3 \implies \frac{2x}{3} \left[\frac{dy}{dx} \right] = y+c$$

Hence
$$c = \frac{2x}{3} \left[\frac{dy}{dx} \right] - y$$

Substituting value of c in equation (i), we get $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right] \left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$,

which is the required differential equation.

Ans.

Illustration 5: Find the differential equation whose solution represents the family : $y = a \cos\theta x + b \sin\theta x$, where $\theta = \text{fixed constant}$

Solution: $y = a \cos\theta x + b \sin\theta x$, $\theta = fixed constant(i)$

Differentiating, we get $\frac{dy}{dx} = -\theta a \sin\theta x + \theta b \cos\theta x$

Again differentiating, we get $\frac{d^2y}{dx^2} = -\theta^2 a \cos\theta x - \theta^2 b \sin\theta x$

using equation (i), we get $\frac{d^2y}{dx^2} = -\theta^2 y$ Ans.

Do yourself - 2

Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

(i)
$$y = 2x + ce^x$$
 (ii) $y = \left(\frac{a}{x^2}\right) + bx$ (iii) $y = ae^{2x} + be^{-2x} + c$

5. SOLUTION OF DIFFERENTIAL EQUATION:

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$

and that of, $\frac{dy}{dx} = px + q$ is $y = \frac{px^2}{2} + qx + c$, where c is arbitrary constant.

(i) A general solution or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is $x = A\cos 2t + B\sin 2t$ where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, $x = 10 \cos 2t + 5 \sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Note:

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$$\log x - \log (y + 2) = k$$
(i)

where k is an arbitrary constant is the general solution of the differential equation xy' = y + 2. The solution given by equation (i) can also be re-written as

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1$$
 ...(ii)

$$x = c_1(y+2) \qquad ...(iii)$$

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as $y + 2 = c_{\gamma}x$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions.

For example, the differential equation $\left| \frac{dy}{dx} \right| + |y| = 0$ has only the trivial solution, i.e. y = 0.

The differential equation $\left| \frac{dy}{dx} \right| + |y| + c = 0$, c > 0 has no solution.

6. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Separation of Variables:

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form A(x)dx + B(y) dy = 0

where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x) dx + \int B(y) dy = c$$

where 'c' is the arbitrary constant.

Solution : Differential equation can be rewritten as

$$xy \frac{dy}{dx} = (1 + y^2) \left(1 + \frac{x}{1 + x^2} \right) \Rightarrow \frac{y}{1 + y^2} dy = \left(\frac{1}{x} + \frac{1}{1 + x^2} \right) dx$$

Integrating, we get

$$\frac{1}{2} \ln(1 + y^2) = \ln x + \tan^{-1} x + \ln c \Rightarrow \sqrt{1 + y^2} = \csc^{\tan^{-1} x}.$$
 Ans.

Illustration 7: Solve the differential equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution: The given equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

Case-I : y = 0 is one solution of differential equation

Case-II : If $y \neq 0$, then

$$\Rightarrow \frac{1-y^2}{y^3} dy + \frac{1+x^2}{x^3} dx = 0 \Rightarrow \int \left(\frac{1}{y^3} - \frac{1}{y}\right) dy + \int \left(\frac{1}{x^3} + \frac{1}{x}\right) dx = 0$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2} \left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

$$y = 0 \text{ or } \log\left(\frac{x}{y}\right) = \frac{1}{2} \left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

Illustration 8: Solve: $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$

Solution: Case-I: y = -1 is one solution of differential equation

Case-II : If $y \neq -1$, then

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3) dx$$

Integrate and solve for y: $3(y+1)^{1/3} = \frac{1}{2}(x-3)^2 + C$

$$(y+1)^{1/3} = \frac{1}{6}(x-3)^2 + C_0 \implies y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$

$$y = -1$$
 or $y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$

Ans.

Ans.

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Do yourself - 3:

Solve the following differential equations:

(i)
$$\frac{2dy}{dx} = \frac{y(x+1)}{x}$$
 (ii)

$$(ii) \qquad \sqrt{1+4x^2} \, \mathrm{d}y = y^3 x \, \mathrm{d}x$$

(ii)
$$\sqrt{1+4x^2} \, dy = y^3 x dx$$
 (iii) $(tany) \frac{dy}{dx} = \sin(x+y) + \sin(x-y)$

Equation of the form: (i)

$$\Rightarrow$$
 y' = f (ax + by + c), b \neq 0

To solve this, substitute t = ax + by + c. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration 9: Solve
$$\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$$
.

Solution:
$$\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$$

Substituting,
$$x + y = t$$
, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore
$$\frac{dt}{dx} - 1 = \cos t - \sin t$$

$$\Rightarrow \int \frac{dt}{1 + \cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2 \frac{t}{2} dt}{2\left(1 - \tan \frac{t}{2}\right)} = \int dx \Rightarrow -\ell n \left|1 - \tan \frac{x + y}{2}\right| = x + c.$$

....(i)

Ans.

Illustration 10: Solve:
$$y' = (x + y + 1)^2$$

Solution:
$$y' = (x + y + 1)^2$$

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1 + t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x + C)$$
$$x + y + 1 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 1$$

Do yourself - 4:

Solve the following differential equations:

(i)
$$\frac{dy}{dx} = (y - 4x)^2$$

(ii)
$$\tan^2(x + y)dx - dy = 0$$

(ii) Equation of the form :
$$\Rightarrow \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case I: If
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$
 then

Let
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$$
 then $a_1 = \lambda a_2$ (i); $b_1 = \lambda b_2$ (ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2 x + \lambda b_2 y + c_1}{a_2 x + b_2 y + c_2} \implies \frac{dy}{dx} = \frac{\lambda (a_2 x + b_2 y) + c_1}{a_2 x + b_2 y + c_2}$$

or we can say,
$$\frac{dy}{dx} = f(a_2x + b_2y)$$

which can be solved by substituting $t = a_2x + b_2y$

Illustration 11: Solve:
$$(x + y)dx + (3x + 3y - 4) dy = 0$$

Solution:

Let
$$t = x + y$$

$$\Rightarrow$$
 dy = dt - dx

So we get, tdx + (3t - 4)(dt - dx) = 0

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0 \quad \Rightarrow \quad 2dx - 3dt + \frac{2}{2-t}dt = 0$$

Integrating and replacing t by x + y, we get

$$2x-3t-2[\ell n|(2-t)|]=c_1$$

$$\Rightarrow$$
 2x - 3(x + y) - 2[ℓn |(2 - x - y)|] = c_1

$$\Rightarrow$$
 $x + 3y + 2\ell n |(2 - x - y)| = c$

Ans.

Case II: If $\mathbf{a}_2 + \mathbf{b}_1 = \mathbf{0}$, then a simple cross multiplication and substituting d(xy) for xdy + ydx and integrating term by term, yield the results easily.

Illustration 12: Solve
$$\frac{dy}{dx} = \frac{x-2y+1}{2x+2y+3}$$

$$\frac{dy}{dx} = \frac{x-2y+1}{2x+2y+3}$$

$$\Rightarrow$$
 2xdy + 2y dy + 3dy = xdx - 2y dx + dx

$$\Rightarrow$$
 $(2y + 3) dy = (x + 1) dx - 2(xdy + ydx)$

On integrating, we get

$$\Rightarrow \int (2y+3)dy = \int (x+1)dx - \int 2d(xy)$$

Solving:
$$y^2 + 3y = \frac{x^2}{2} + x - 2xy + c$$

Ans.

Do yourself - 5:

Solve the following differential equations:

(i)
$$\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$$

(ii)
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{3x - 5y}{5x + y + 3}$$

Equation of the form: (iii)

$$\Rightarrow$$
 $vf(xv)dx + xg(xv)dv = 0$

The substitution xy = z, reduces differential equation of this form to the form in which the variables are separable.

Let
$$xy = z$$

$$dy = \left[\frac{xdz - zdx}{x^2}\right]$$

using equation (ii) & (iii), equation (i) becomes $\frac{z}{x} f(z) dx + xg(z) \left| \frac{x dz - z dx}{x^2} \right| = 0$

$$\Rightarrow \frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0 \qquad \Rightarrow \qquad \frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0$$

$$\Rightarrow \frac{z}{x} \{f(z) - g(z)\} dx + g(z) dz = 0$$

$$\Rightarrow \frac{1}{x} dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$$

Illustration 13: Solve $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

Solution:

Let
$$xy = y$$

$$y = \frac{v}{x} \implies dy = \frac{xdv - vdx}{x^2}$$

Now, differential equation becomes
$$\Rightarrow \frac{v}{x}(v+1)dx + x(1+v+v^2)\left(\frac{xdv-vdx}{x^2}\right) = 0$$

On solving, we get

$$v^3 dx - x(1 + v + v^2) dv = 0$$

separating the variables & integrating we get

$$\Rightarrow \int \frac{dx}{x} - \int \left(\frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\right) dv = 0 \quad \Rightarrow \quad \ell nx + \frac{1}{2v^2} + \frac{1}{v} - \ell nv = c$$

$$\Rightarrow 2v^2 \ell n \left(\frac{v}{x}\right) - 2v - 1 = -2cv^2 \quad \Rightarrow \quad 2x^2 y^2 \ell n y - 2xy - 1 = Kx^2 y^2 \qquad \text{where } K = -2c$$

Do yourself - 6:

Solve the following differential equations:

(i)
$$(y - xy^2)dx - (x + x^2y)dy = 0$$

(ii)
$$y(1 + 2xy)dx + x(1 - xy)dy = 0$$

(iv) Transformation to polar-co-ordinates:

Sometimes conversion of cartesian co-ordinates into polar coordinates helps us in separating the variables.

(1)
$$x = r \cos \theta$$
, $y = r \sin \theta$
then $x^2 + y^2 = r^2$
 $xdx + ydy = rdr$
 $xdy - ydx = r^2d\theta$

(2)
$$x = r \sec \theta$$
, $y = r \tan \theta$
then $x^2 - y^2 = r^2$
 $xdx - ydy = rdr$
 $xdy - ydx = r^2 \sec \theta d\theta$

Illustration 14: Solve:
$$\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$$

Solution: The given equation can be reduced to

$$\frac{xdx + ydy}{ydx - xdy} = \frac{(x^2 + y^2)^2}{x^2}$$

Substituting $x = r \cos \theta$

$$y = r \sin \theta$$

we get,
$$\frac{rdr}{r^2d\theta} = \frac{-(r^2)^2}{r^2\cos^2\theta} \implies \int \frac{dr}{r^3} = -\int \sec^2\theta d\theta \implies -\frac{1}{2r^2} = -\tan\theta + c$$

Substituting, $\frac{1}{2(x^2 + y^2)} = \frac{y}{x} + K$ Ans.

Do yourself - 7:

Solve the following differential equations:

(i)
$$xdx + ydy = xdy - ydx$$

(ii)
$$ydx - xdy = xy dy - x^2dx$$

(b) Homogeneous equations:

A function f(x,y) is said to be a homogeneous function of degree n, if the substitution $x = \lambda x$, $y = \lambda y$, $\lambda > 0$ produces the equality

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

The degree of homogeneity 'n' can be any real number.

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Illustration 15: Find the degree of homogeneity of function

(i)
$$f(x,y) = x^2 + y^2$$
 (ii) $f(x,y) = (x^{3/2} + y^{3/2})/(x + y)$ (iii) $f(x,y) = \sin\left(\frac{x}{y}\right)$

Solution:

(i)
$$f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 (x^2 + y^2) = \lambda^2 f(x,y)$$

degree of homogeneity $\rightarrow 2$

(ii)
$$f(\lambda x, \lambda y) = \frac{\lambda^{3/2} x^{3/2} + \lambda^{3/2} y^{3/2}}{\lambda x + \lambda y}$$
$$f(\lambda x, \lambda y) = \lambda^{1/2} f(x, y)$$
degree of homogeneity $\rightarrow 1/2$

(iii)
$$f(\lambda x, \lambda y) = \sin\left(\frac{\lambda x}{\lambda y}\right) = \lambda^{\circ} \sin\left(\frac{x}{y}\right) = \lambda^{\circ} f(x,y)$$

degree of homogeneity $\rightarrow 0$

Illustration 16: Determine whether or not each of the following functions is homogeneous.

(i)
$$f(x,y) = x^2 - xy$$
 (ii) $f(x,y) = \frac{xy}{x + y^2}$ (iii) $f(x,y) = \sin xy$

Solution:

(i)
$$f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 xy = \lambda^2 (x^2 - xy) = \lambda^2 f(x,y)$$
 homogeneous.

(ii)
$$f(\lambda x, \lambda y) = \frac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x, y)$$

not homogeneous.

(iii)
$$f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$$

not homogeneous.

Do yourself - 8:

(i) Find the degree of homogeneity of function
$$f(x,y) = x^3 \ln \left[\sqrt{x+y} / \sqrt{x-y} \right]$$

(ii) Find the degree of homogeneity of function
$$f(x,y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$$

(iii) Determine whether or not each of the following functions is homogeneous.

(a)
$$f(x,y) = \sqrt{x^2 + 2xy + 3y^2}$$
 (b) $f(x,y) = x + y \cos \frac{y}{x}$ (c) $f(x,y) = x \sin y + y \sin x$.

(i) Homogeneous first order differential equation

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where f(x,y) and g(x,y) are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$$y = vx$$
,

so that the dependent variable y is changed to another variable v.

Since f(x,y) and g(x,y) are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right)$$
 and $g(x,y) = x^n g_1\left(\frac{y}{x}\right)$.

As
$$y = vx$$
, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x \frac{\mathrm{d}v}{\mathrm{d}x}.$$

The given differential equation, therefore, becomes

$$v + x \frac{dv}{dx} = \frac{f_1(v)}{g_1(v)} \Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note: Sometimes homogeneous equation can be solved by substituting x = vy or by using polar coordinate substitution.

Illustration 17: The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x \cos y}$ is -

(A)
$$\sin^2 y = x \sin y + \frac{x^2}{2} + c$$

(B)
$$\sin^2 y = x \sin y - \frac{x^2}{2} + c$$

(C)
$$\sin^2 y = x + \sin y + \frac{x^2}{2} + c$$

(D)
$$\sin^2 y = x - \sin y + \frac{x^2}{2} + c$$

Solution:

Here,
$$\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x \cos y}$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x},$$

(put
$$\sin y = t$$
)

$$\Rightarrow \frac{dt}{dx} = \frac{t+x}{2t-x}$$

(put
$$t = vx$$
)

$$\frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v + 1}{2v - 1}$$

$$\therefore x \frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v+1-2v^2+v}{2v-1}$$

or
$$\frac{2v-1}{-2v^2+2v+1}$$
 dv = $\frac{dx}{x}$ on solving, we get

$$\sin^2 y = x \sin y + \frac{x^2}{2} + c.$$

Ans. (A)

Solve the differential equation $(1 + 2e^{x/y}) dx + 2e^{x/y} (1 - x/y) dy = 0$. Illustration 18:

Solution:

The equation is homogeneous of degree 0.

Put
$$x = vy$$
, $dx = v dy + y dv$,

Then, differential equation becomes

$$(1 + 2e^{v})(v dy + y dv) + 2e^{v}(1 - v) dy = 0 \Rightarrow (v + 2e^{v}) dy + y(1 + 2e^{v}) dv = 0$$

$$\frac{dy}{v} + \frac{1 + 2e^{v}}{v + 2e^{v}} dv = 0$$

Integrating and replacing v by x/y, we get $\ell n \ y + \ell n \ (v + 2e^v) = \ell nc$ and $x + 2 \ ye^{x/y} = c$

Do yourself - 9:

Solve the following differential equations:

$$(i) y' = \frac{3x - y}{x + y}$$

(ii)
$$(x - y \ell ny + y \ell nx) dx + x(\ell ny - \ell nx) dy = 0$$

(iii)
$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$
, $y(1) = 1$

(ii) Equations reducible to homgeneous form

The equation of the form
$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$
 where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to u,v as

$$x = u + h$$
, $y = v + k$

where h,k are the constants to be chosen so as to make the given equation homgeneous.

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}u}$$

:. The equation becomes,
$$\frac{dv}{du} = \frac{a_1u + b_1v + (a_1h + b_1k + c_1)}{a_2u + b_2v + (a_2h + b_2k + c_2)}$$

Let h and k be chosen so as to satisfy the equation

$$a_1 h + b_1 k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

...(ii)

Solve for h and k from (i) and (ii)

Now
$$\frac{du}{dv} = \frac{a_1 u + b_1 v}{a_2 u + b_2 v}$$

is a homgeneous equation and can be solved by substituting v = ut.

Illustration 19: Solve $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Solution: Put
$$x = X + h$$
, $y = Y + k$

We have
$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k + 3)}{2X + 3Y + (2h + 3k + 4)}$$

To determine h and k, we write

$$h + 2k + 3 = 0$$
, $2h + 3k + 4 = 0 \Rightarrow h = 1$, $k = -2$

So that
$$\frac{dY}{dX} = \frac{X + 2Y}{2X + 3Y}$$

Putting Y = VX, we get

$$V + X \frac{dV}{dX} = \frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1} dV = -\frac{dX}{X}$$

$$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right] dV = -\frac{dX}{X}$$

$$\Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}} \log\left(\sqrt{3}V-1\right) - \frac{2-\sqrt{3}}{2\sqrt{3}} \log\left(\sqrt{3}V+1\right) = (-\log X + c)$$

$$\frac{2+\sqrt{3}}{2\sqrt{3}} \log\left(\sqrt{3}Y-X\right) - \frac{2-\sqrt{3}}{2\sqrt{3}} \log\left(\sqrt{3}Y+X\right) = A \text{ where A is another constant and } X = x-1, Y = y+2.$$
Ans.

Do yourself - 10:

(i) Solve the differential equation : $\frac{dy}{dx} = \frac{x + 2y - 5}{2x + y - 4}$

(c) Linear differential equations:

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form;

 $a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n(x)y = \phi(x)$, where $a_0(x)$, $a_1(x)$ $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear . e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0 \text{ is not}$ linear, though its degree is 1.

Illustration 20: Which of the following equation is linear?

(A)
$$\frac{dy}{dx} + xy^2 = 1$$
 (B) $x^2 \frac{dy}{dx} + y = e^x$ (C) $\frac{dy}{dx} + 3y = xy^2$ (D) $x \frac{dy}{dx} + y^2 = \sin x$

Solution : Clearly answer is (B)

Illustration 21: Which of the following equation is non-linear?

(A)
$$\frac{dy}{dx} = \cos x$$
 (B) $\frac{d^2y}{dx^2} + y = 0$ (C) $dx + dy = 0$ (D) $x\left(\frac{dy}{dx}\right) + \frac{3}{\left(\frac{dy}{dx}\right)} = y^2$

Solution: Clearly answer is (D)

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(i) Linear differential equations of first order:

The most general form of a linear differential equation of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

....(i)

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $e^{\int Pdx} \left[\frac{dy}{dx} + Py \right] = Qe^{\int Pdx}$

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right) = Qe^{\int Pdx} \qquad ...(ii)$$

On integrating equation (ii), we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

This is the required general solution.

Note:

- (i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.
- (ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation; $(x+y+1)\frac{dy}{dx} = y^2 + 3 \text{ can be written as } (y^2+3)\frac{dx}{dy} = x+y+1 \text{ which is a linear differential equation.}$

Illustration 22: Solve
$$(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$$
.

Solution: Differential equation can be rewritten as $(1 + y^2) \frac{dx}{dy} + x = e^{\tan^{-1} y}$

or
$$\frac{dx}{dy} + \frac{1}{1+y^2} . x = \frac{e^{\tan^{-1} y}}{1+y^2}$$
 ...(i)

I.
$$F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

so solution is $xe^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2} dy$

Let
$$e^{\tan^{-1} y} = t \Rightarrow \frac{e^{\tan^{-1} y}}{1 + v^2} dy = dt$$

$$xe^{tan^{-1}y} = \int t dt$$
 [Putting $e^{tan^{-1}y} = t$]

or
$$x e^{\tan^{-1} y} = \frac{t^2}{2} + \frac{c}{2} \implies 2x e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c.$$

Ans.

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Illustration 23: The solution of differential equation $(x^2 - 1) \frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$ is -

(A)
$$y(x^2-1) = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + C$$

(B)
$$y(x^2 + 1) = \frac{1}{2} \log \left| \frac{x - 1}{x + 1} \right| - C$$

(C)
$$y(x^2-1) = \frac{5}{2} \log \left| \frac{x-1}{x+1} \right| + C$$

(D) none of these

Solution: The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2 \quad xy = \frac{1}{x^2-1}$$
 \Rightarrow $\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2}$...(i)

This is linear differential equation of the form

$$\frac{dy}{dx}$$
 + Py = Q, where P = $\frac{2x}{x^2 - 1}$ and Q = $\frac{1}{(x^2 - 1)^2}$

$$\therefore \text{ I.F.} = e^{\int P dx} = e^{\int 2x/(x^2 - 1)dx} = e^{\log(x^2 - 1)} = (x^2 - 1)$$

multiplying both sides of (i) by I.F. = $(x^2 - 1)$, we get

$$(x^2-1) \frac{dy}{dx} + 2 xy = \frac{1}{x^2-1}$$

integrating both sides we get

$$y(x^2 - 1) = \int \frac{1}{x^2 - 1} dx + C$$

[Using:
$$y(I.F.) = \int Q.(I.F.) dx + C$$
]

$$\Rightarrow y(x^2-1) = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + C.$$

This is the required solution.

Ans.(A)

Do yourself - 11:

Solve the following differential equations:

(i)
$$\frac{xdy}{dx} = 2y + x^4 + 6x^2 + 2x, x \neq 0$$

(ii)
$$(x-a)\frac{dy}{dx} + 3y = 12(x-a)^3, x > a > 0$$

(iii)
$$y \ln y dx + (x - \ln y) dy = 0$$

(ii) Equation reducible to linear form:

The equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x,

is called Bernoulli's equation.

On dividing by
$$y^n$$
, we get $y^{-n} \frac{dy}{dx} + Py^{-n+1} = Q$

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Let
$$y^{-n+1} = t$$
, so that $(-n+1)y^{-n} \frac{dy}{dx} = \frac{dt}{dx}$

then equation becomes
$$\frac{dt}{dx} + P(1-n)t = Q(1-n)$$

which is linear with t as a dependent variable.

Illustration 24: Solve the differential equation $x \frac{dy}{dx} + y = x^3y^6$.

Solution: The given differential equation can be written as

$$\frac{1}{y^{6}} \frac{dy}{dx} + \frac{1}{xy^{5}} = x^{2}$$

Putting $y^{-5} = v$ so that

$$-5 \text{ y}^{-6} \frac{\text{dy}}{\text{dx}} = \frac{\text{dv}}{\text{dx}} \text{ or } \text{y}^{-6} \frac{\text{dy}}{\text{dx}} = -\frac{1}{5} \frac{\text{dv}}{\text{dx}} \text{ we get}$$

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2 \qquad(i)$$

This is the standard form of the linear deferential equation having integrating factor

I.F =
$$e^{\int -\frac{5}{x} dx} = e^{-5 \log x} = \frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get v.
$$\frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5} dx$$

$$\Rightarrow \frac{\mathbf{v}}{\mathbf{x}^5} = \frac{5}{2}\mathbf{x}^{-2} + \mathbf{c}$$

$$\Rightarrow$$
 y⁻⁵ x⁻⁵ = $\frac{5}{2}$ x⁻² + c which is the required solution.

Ans.

Illustration 25: Find the solution of differential equation $\frac{dy}{dx} - y \tan x = -y^2 \sec x$.

Solution:
$$\frac{1}{v^2} \frac{dy}{dx} - \frac{1}{v} \tan x = -\sec x$$

$$\frac{1}{v} = v$$
; $\frac{-1}{v^2} \frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \frac{-dv}{dx} - v \tan x = -\sec x$$

$$\frac{dv}{dx} + v \tan x = \sec x,$$

Here
$$P = \tan x$$
, $Q = \sec x$

$$I.F. = e^{\int \tan x dx} = |\sec x|$$

$$v |secx| = \int sec^2 x \, dx + c$$

Hence the solution is $y^{-1} |\sec x| = \tan x + c$

Ans.

Do yourself - 12:

Solve the following differential equations:

(i)
$$y' + 3y = e^{3x} y^2$$
 (ii) $xdy - \{y + xy^3 (1 + \ell nx)\}dx = 0$ (iii) $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

7. TRAJECTORIES:

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

Working rule for finding orthogonal trajectory

- 1. Form the differential equation of family of curves
- 2. Write $-\frac{1}{dy/dx}$ for $\frac{dy}{dx}$ or $-\frac{r^2d\theta}{dr}$ for $\frac{dr}{d\theta}$ if differential equation is in the polar form.
- 3. Solve the new differential equation to get the equation of orthogonal trajectories.

Note: A family of curves is self-orthogonal if it is its own orthogonal family.

Illustration 26: Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Solution: Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, We get

$$2x + 4y \frac{dy}{dx} - \frac{dy}{dx} = 0$$

or $2x + (4y - 1) \frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories,

we get

$$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$$

$$\Rightarrow \int \frac{dy}{4y-1} = \int \frac{dx}{2x} \Rightarrow \frac{1}{4} \ln (4y-1) = \frac{1}{2} \ln x + \frac{1}{2} \ln a, \text{ where a is any constant.}$$

$$\Rightarrow \ell n(4y-1) = 2 \ \ell n \ x + 2 \ \ell n \ a \ or \ , \ 4y-1 = a^2 x^2$$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory, which is of the form

$$y = cx^2 + k$$
 where $c = \frac{a^2}{4}$, $k = \frac{1}{4}$.

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Illustration 27: Prove that $\frac{x^2}{1+\lambda} + \frac{y^2}{4+\lambda} = 1$ are selforthogonal family of curves.

Solution:

$$\frac{x^2}{1+\lambda} + \frac{y^2}{4+\lambda} = 1 \qquad \dots (i)$$

Differentiating (i) with respect to x, we have

$$\frac{x}{1+\lambda} + \frac{y}{4+\lambda} \frac{dy}{dx} = 0 \qquad ...(ii)$$

From (i) and (ii), we have to eliminate λ .

Now, (ii) gives

$$\lambda = \frac{-\left[4x + y\frac{dy}{dx}\right]}{x + y\frac{dy}{dx}}$$

$$\Rightarrow 1 + \lambda = \frac{(1-4)x}{x + y(dy/dx)}, \quad 4 + \lambda = \frac{-(1-4)y(dy/dx)}{x + y(dy/dx)}$$

Substituting these values in (i), we ge

$$\left(x + y\frac{dy}{dx}\right)\left(x - y\frac{dx}{dy}\right) = -3 \qquad \dots(iii)$$

as the differential equation of the given family.

Changing dy/dx to -dx/dy in (iii), we obtain

$$\left(x - y\frac{dx}{dy}\right)\left(x + y\frac{dy}{dx}\right) = -3 \qquad ...(iv)$$

which is the same as (iii). Thus we see that the family (i) as self-orthogonal, i.e., every member of the family (i) cuts every other member of the same family orthogonally.

Do yourself - 13:

- Find the orthogonal trajectories of the following families of curves:
 - x + 2y = C

(b) $y = Ce^{-2x}$

Note:

Following exact differentials must be remembered:

(i)
$$xdy + y dx = d(xy)$$

(ii)
$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$

(ii)
$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$
 (iii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(iv)
$$\frac{xdy + ydx}{xy} = d(\ell nxy)$$

$$(iv) \quad \frac{xdy + ydx}{xy} = d\left(\ell nxy\right) \qquad \qquad (v) \quad \frac{dx + dy}{x + y} = d\left(\ell n(x + y)\right)(vi) \quad \frac{xdy - ydx}{xy} = d\left(\ell n\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{xy} = d\left(\ell n \frac{y}{x}\right)$$

(vii)
$$\frac{ydx - xdy}{xy} = d\left(\ell n \frac{x}{y}\right)$$

$$(vii) \qquad \frac{ydx - xdy}{xy} = d \left(\ell n \frac{x}{y} \right) \qquad (viii) \quad \frac{xdy - ydx}{x^2 + y^2} = d \left(tan^{-1} \frac{y}{x} \right) \quad (ix) \quad \frac{ydx - xdy}{x^2 + y^2} = d \left(tan^{-1} \frac{x}{y} \right)$$

$$(x) \qquad \frac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right] (xi) \qquad d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2} \quad (xii) \qquad d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$$

$$d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$$

(xii)
$$d\left(\frac{e^x}{v}\right) = \frac{ye^x dx - e^x dy}{v^2}$$

(xiii)
$$d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$$

Solution: The given differential equation can be written as;

$$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$$

$$\Rightarrow$$
 sec² (xy) d (xy) + sin x dx + sin y dy = 0

$$\Rightarrow$$
 d (tan (xy)) + d (- cos x) + d (- cos y) = 0

$$\Rightarrow$$
 tan (xy) - cos x - cos y = c.

Ans.

Do yourself - 14:

Solve the following differential equations:

(i)
$$x dx + y dy + 4y^3(x^2 + y^2)dy = 0.$$

(ii)
$$x dy - ydx - (1 - x^2)dx = 0$$
.

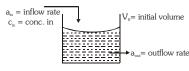
8. APPLICATION OF DIFFERENTIAL EQUATIONS:

(a) Mixing Problems

A chemical in a liquid solution with given concentration c_{in} gm/lit. (or dispersed in a gas) runs into a container with a rate of a_{in} lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate (a_{out} litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

Rate of change of amount =
$$\begin{pmatrix} rate & at & which \\ chemical \\ in & container \end{pmatrix} - \begin{pmatrix} rate & at & which \\ chemical \\ departs \end{pmatrix}$$
(i)

Arrival rate = (conc. in) × (inflow rate) = $c_{in} \times a_{in}$ $a_{in} = inflow rate$ If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time



Departure rate =
$$\begin{pmatrix} \text{concentration in} \\ \text{container at time t} \end{pmatrix}$$
. (outflow rate)
= $\frac{y(t)}{V(t)}$. (a_{out})

where volume of mixture at time t, V(t) = initial volume + (inflow rate – outflow rate) × t = $V_0 + (a_{in} - a_{out})t$

Accordingly, Equation (i) becomes

$$\frac{dy(t)}{dt} = (chemical's given arrival rate) - \frac{y(t)}{V(t)} \cdot (out flow rate) \qquad(ii)$$

$$\frac{d(y(t))}{dt} = c_{in}a_{in} - \frac{y(t)}{V_0 + (a_{in} - a_{out})t}.a_{out}$$

This leads to a first order linear D.E. which can be solved to obtain y(t) i.e. amount of chemical at time 't'.

Illustration 29: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution: Let y(t) be the amount of salt after t min.

Given y(0) = 20 kg

rate in =
$$\left(\frac{0.03\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min.}}\right) = \frac{0.75\text{kg}}{\text{min.}}$$

As $a_{in} = a_{out}$, so the tank always contains 5000 L of liquid so the conc. at time 't' is $\left(\frac{y(t)}{5000}\right) \frac{kg}{L}$

so rate out =
$$\left(\frac{y(t)}{5000} \frac{kg}{L}\right) \left(\frac{25L}{min}\right) = \frac{y(t)}{200} \frac{kg}{min}$$

$$\frac{dy(t)}{dt} = 0.75 - \frac{y(t)}{200}$$

by solving as linear D.E. or variable separable and using initial condition, we get

$$y(t) = 150 - 130 e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130 e^{-30/100} = 38.1 \text{ kg}$$

Do yourself - 15:

- (i) A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$ kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank
 - at the same rate. Find
 (a) the amount and
 - **(b)** the concentration of salt in the tank at any time t.

(b) Exponential Growth and Decay:

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t)$$
, where k is a constant(i)

$$\int \frac{\mathrm{d}y(t)}{y(t)} = \int k\mathrm{d}t$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if k > 0) or law of natural decay (if k < 0).

In the context of population growth, we can write

$$\frac{dP}{dt} = kP$$
 or $\frac{1}{P}\frac{dP}{dt} = k$

where k is growth rate divided by the population size; it is called **the relative growth rate.**

- Illustration 30: A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find (a) an expression for the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.
- **Solution:** (a) Let N denote the amount of material present at time t.

So,
$$\frac{dN}{dt} - kN = 0$$

This differential equation is separable and linear, its solution is

$$N = ce^{kt} \qquad(i)$$

At t=0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}$ or c = 50. Thus,

$$N = 50e^{kt}$$
(ii)

At t = 2, 10 percent of the original mass of 50kg or 5kg has decayed. Hence, at t = 2, N = 50 - 5 = 45. Substituting these values into (ii) and solving for k, we have

$$45 = 50e^{2k} \text{ or } k = \frac{1}{2} \ln \frac{45}{50}$$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{\frac{1}{2}(\ln 0.9)t}$$
(iii)

where t is measured in hours.

- **(b)** We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find $N = 50e^{-2 \ln{(0.9)}}$ kg
- (c) We require when N = 50/2 = 25. Substituting N = 25 into (iii) and solving for t, we find

$$25 = 50e^{\frac{1}{2}(\ln 0.9)t} \implies t = \ln \left(\frac{1}{2}\right) / \left[\frac{1}{2}\ln (0.9)\right] \text{hours}$$

(c) Geometrical applications:

Let $P(x_1, y_1)$ be any point on the curve y = f(x), then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent =
$$x_1 - y_1 \left(\frac{dx}{dy} \right)$$

y-intercept of the tangent =
$$y_1 - x_1 \left(\frac{dy}{dx} \right)$$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are;
$$x_1 + y_1 \frac{dy}{dx}$$
 and $y_1 + x_1 \frac{dx}{dy}$

- (iii) Length of tangent = PT = $|y_1| \sqrt{1 + (dx/dy)_{(x_1, y_1)}^2}$
- (iv) Length of normal = PN = $|y_1| \sqrt{1 + (dy/dx)_{(x_1, y_1)}^2}$
- (v) Length of sub-tangent = $ST = \left| y_1 \left(\frac{dx}{dy} \right)_{(x_1, y_1)} \right|$
- (vi) Length of sub-normal = $SN = \left| y_1 \left(\frac{dy}{dx} \right)_{(x_1, y_1)} \right|$
- (vii) Length of radius vector = $\sqrt{x_1^2 + y_1^2}$

Do yourself - 16:

- (i) At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.
- (ii) Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

Miscellaneous Illustrations :

Illustration 31: Solve $(y \log x - 1) ydx = xdy$.

Solution: The given differential equation can be written as

$$x \frac{dy}{dx} + y = y^2 \log x \qquad \dots (i)$$

Divide by
$$xy^2$$
. Hence $\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x} \log x$

Let
$$\frac{1}{v} = v \Rightarrow -\frac{1}{v^2} \frac{dy}{dx} = \frac{dv}{dx}$$
 so that $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x$ (ii)

(ii) is the standard linear differential equation with $P = -\frac{1}{x}$, $Q = -\frac{1}{x} \log x$

I.F. =
$$e^{\int pdx} = e^{\int -1/x dx} = 1/x$$

The solution is given by

$$v \cdot \frac{1}{x} = \int \frac{1}{x} \left(-\frac{1}{x} \log x \right) dx = -\int \frac{\log x}{x^2} \, dx = \frac{\log x}{x} - \int \frac{1}{x} \cdot \frac{1}{x} \, dx = \frac{\log x}{x} + \frac{1}{x} + C$$

$$\Rightarrow$$
 v = 1 + log x + cx = log ex + cx

or
$$\frac{1}{y} = \log ex + cx$$
 or $y (\log ex + cx) = 1$.

Illustration 32: For a certain curve y = f(x) satisfying $\frac{d^2y}{dx^2} = 6x - 4$, f(x) has a local minimum value 5

when x = 1. Find the equation of the curve and also the global maximum and global minimum values of f(x) given that $0 \le x \le 2$.

Solution: Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When x = 1, $\frac{dy}{dx} = 0$, so that A = 1. Hence

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$
 ...(i)

Integrating, we get $y = x^3 - 2x^2 + x + B$

When x = 1, y = 5, so that B = 5.

Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points x = 1/3, x = 1

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at x = 1/3, y has a local maximum.

At
$$x = 1$$
, $\frac{d^2y}{dx^2}$ is positive.

Therefore at x = 1, y has a local minimum.

Also
$$f(1) = 5$$
, $f(\frac{1}{3}) = \frac{139}{27}$. $f(0) = 5$, $f(2) = 7$

Hence the global maximum value = 7, and the global minimum value = 5.

Ans.

Illustration 33: Solve $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

Solution: $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x.$

Rearrange it:

 $(\sin x - \sin y)\cos x dx + \sin x \cos y dy = 0.$

Put $u = \sin y$, So, $du = \cos y dy$:

Substituting, we get

$$(\sin x - u)\cos x dx + \sin x du = 0$$
, $\frac{du}{dx} - u \frac{\cos x}{\sin x} = -\cos x$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp \int -\frac{\cos x}{\sin x} dx = \exp \{-\ln(\sin x)\} = \frac{1}{\sin x}.$$

Hence,
$$u \frac{1}{\sin x} = -\int \frac{\cos x}{\sin x} dx = -\ln |\sin x| + C$$
.

Solve for $u : u = -\sin x \ln |\sin x| + C \sin x$.

Put y back : $\sin y = -\sin x \ln |\sin x| + C \sin x$.

Ans.

Illustration 34: Solve the equation $x \int_{0}^{x} y(t) dt = (x+1) \int_{0}^{x} t y(t) dt, x > 0$

Solution: Differentiating the equation with respect to x, we get

$$xy(x) + 1 \cdot \int_{0}^{x} y(t)dt = (x+1)xy(x) + 1 \cdot \int_{0}^{x} ty(t)dt$$

i.e.,
$$\int_{0}^{x} y(t)dt = x^{2}y(x) + \int_{0}^{x} ty(t)dt$$

Differentiating again with respect to x, we get $y(x) = x^2 y'(x) + 2xy(x) + xy(x)$

i.e.,
$$(1-3x)y(x) = \frac{x^2dy(x)}{dx}$$

i.e.,
$$\frac{(1-3x)dx}{x^2} = \frac{dy(x)}{y(x)}$$
, integrating we get

i.e.,
$$y = \frac{c}{x^3} e^{-1/x}$$

Ans.

ANSWERS FOR DO YOURSELF

: (i) one, two

(ii) two, two

(iii) one, one

- 2 : (i) y' y = 2(1 x)
- (ii) $x^2y'' + 2xy' 2y = 0$
- (iii) y''' = 4y'

- 3 : (i) $\ell ny^2 = x + \ell n|x| + k$
- (ii) $-\frac{1}{2v^2} = \frac{1}{4}\sqrt{1+4x^2} + k$ (iii) $\sec y = -2 \cos x + C$

- 4: (i) $\frac{y-4x+2}{v-4x-2} = ce^{-4x}$
- (ii) $2(x-y) = c + \sin 2(x+y)$
- **5** : (i) $x + 2y + \ln|2x y| + c = 0$ (ii) $\frac{y^2}{2} + 3y \frac{3x^2}{2} + 5xy = 0$
- **6** : (i) $x = cye^{xy}$
- (ii) $y = cx^2e^{-1/xy}$
- 7 : (i) $\ln(x^2 + y^2) = 2 \tan^{-1} \left(\frac{y}{x}\right) + c$ (ii) $\sqrt{x^2 y^2} = \sin^{-1} \left(\frac{y}{x}\right) + c$

- (iii) (a) homogeneous (b) homogeneous (c) not homogeneous

- **9**: (i) $(3x + y)(x y) = c_0$ (ii) $y \ln \left(\frac{y}{x}\right) y + x \ln x + cx = 0$ (iii) $x^2y(2x + y) = 3$
- **10**: (i) $x + y 3 = C(x y + 1)^3$
- **11**: (i) $y = \frac{x^4}{2} + 6x^2 \ln |x| 2x + cx^2$ (ii) $y = 2(x-a)^3 + \frac{c}{(x-a)^3}$ (iii) $2x \ln y = \ln^2 y + C$.

- 12: (i) $y = \frac{1}{(c-x)e^{3x}}$ (ii) $\frac{x^2}{v^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \ln x\right) + c$ (iii) $\frac{1}{v} = -\sin x + ce^x$.

- **13**: (i) (a) y 2x = K (b) $y^2 = x + K$
- **14**: (i) $\frac{1}{2} \ell n(x^2 + y^2) + y^4 = C$ (ii) $y + x^2 + 1 = Cx$
- **15**: (i) (a) $-5e^{-0.2t} + 5 \text{ kg}$ (b) $\frac{1}{2}(-e^{0.2t} + 1) \text{ kg}/\ell$
- **16**: (i) $\frac{X}{V} = X^2 + C$
- (ii) $x^2 + y^2 = C$

EXERCISE (O-1)

- Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation 1. $D^3y - 3D^2y - 4Dy + 12y = 0$, is **DE0002**
 - (A) 0

(B) 1

(C)2

- (D) more than 2
- The value of the constant 'm' and 'c' for which y = mx + c is a solution of the differential equation 2. $D^2v - 3Dv - 4v = -4x$.
 - (A) is m = -1; c = 3/4 (B) is m = 1; c = -3/4
- (C) no such real m, c
- (D) is m = 1; c = 3/4

3. Consider the two statements

Statement-1: $y = \sin kt$ satisfies the differential equation y'' + 9y = 0.

Statement-2: $y = e^{kt}$ satisfy the differential equation y'' + y' - 6y = 0

The value of k for which both the statements are correct is

- (A) 3
- (B) 0

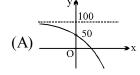
(C) 2

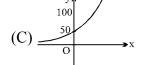
- (D)3
- **DE0005**
- 4. The differential equation corresponding to the family of curves $y = e^x (ax + b)$ is
 - (A) $\frac{d^2y}{dy} + 2\frac{dy}{dy} y = 0$

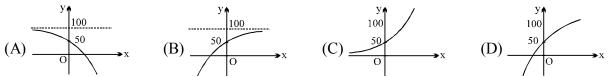
(B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

(C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

- (D) $\frac{d^2y}{dx^2} 2\frac{dy}{dx} y = 0$
- **DE0001**
- 5. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is K > 0, is
 - (A) $\frac{dr}{dt} + K = 0$
- (B) $\frac{d\mathbf{r}}{dt} \mathbf{K} = 0$ (C) $\frac{d\mathbf{r}}{dt} = \mathbf{K}\mathbf{r}$
- (D) none
- **DE0009**
- 6. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is
 - (A) $ve^{\overline{y}} = e$
- (B) $x e^{\frac{\hat{x}}{y}} = e$
- (C) $xe^{\frac{y}{x}} = e$
- **DE0010**
- Which one of the following curves represents the solution of the initial value problem 7.
 - Dy = 100 y, where y(0) = 50







- 8. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

- (A) x = 0
- (B) y = 0
- (C) y = x + 1
- (D) x + y + 1 = 0
- A function y = f(x) satisfies $(x + 1) \cdot f'(x) 2(x^2 + x) f(x) = \frac{e^{x^2}}{(x + 1)}, \forall x > -1$ 9. If f(0) = 5, then f(x) is
- (A) $\left(\frac{3x+5}{x+1}\right) \cdot e^{x^2}$ (B) $\left(\frac{6x+5}{x+1}\right) \cdot e^{x^2}$ (C) $\left(\frac{6x+5}{(x+1)^2}\right) \cdot e^{x^2}$ (D) $\left(\frac{5-6x}{x+1}\right) \cdot e^{x^2}$

(A)
$$y = 2 - (2 + a^2) e^{\frac{x^2 - a^2}{2}}$$

(B)
$$y = 1 - (2 + a^2) e^{\frac{x^2 - a^2}{2}}$$

(C)
$$y = 2 - (1 + a^2)e^{\frac{x^2 - a^2}{2}}$$

(D) none

DE0017

EXERCISE (O-2)

One or more than one correct

1. The differential equation, $x \frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$:

DE0019

- (A) is of order 1
- (B) is of degree 2
- (C) is linear
- (D) is non linear
- 2. A curve y = f(x) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve
 - (A) is homogeneous.
 - (B) can be converted into linear differential equation with some suitable substitution.
 - (C) is the family of circles touching the x-axis at the origin.
 - (D) the family of circles touching the y-axis at the origin.

DE0022

- 3. The function f(x) satisfying the equation, $f^2(x) + 4 f'(x) \cdot f(x) + [f'(x)]^2 = 0$.
 - (A) $f(x) = c \cdot e^{(2-\sqrt{3})x}$

(B) $f(x) = c \cdot e^{(2+\sqrt{3})x}$

(C) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$

(D) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$

DE0021

- where c is an arbitrary constant.
- **4.** If y = f(x) is solution of the differential equation, $x^2 \frac{dy}{dx} \cdot \cos \frac{1}{x} y \sin \frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$, then
 - (A) $f(x) = \sin \frac{1}{x} \cos \frac{1}{x}$

(B) $\lim_{x \to \infty} \left(-f(x) \right)^x = \frac{1}{e}$

DE0027

(C) $\lim_{x\to\infty} \left(-f(x)\right)^x = e$

- (D) $f'(x) + 2xf'(x) + x^2f''(x) = -\frac{2}{x^2}\sin\frac{1}{x}$
- 5. Consider the differential equation $\frac{dy}{dx} + y \tan x = x \tan x + 1$. Then
 - (A) The integral curves satisfying the differential equation and given by $y = x + c \cos x$.
 - (B) The angle at which the integral curves cut the y-axis is $\frac{\pi}{4}$.
 - (C) Tangents to all the integral curves at their point of intersection with y-axis are parallel.
 - (D) None of these

EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. State the order and degree of the following differential equations:

(i)
$$\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$$
 (ii)
$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

(ii)
$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

DE0029

- Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where 2. a & b are arbitrary constants. **DE0030**
 - Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g, f& c (b) are arbitrary constants. **DE0031**
 - Obtain the differential equation associated with the primitive, (c) $y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x$, where c_1 , c_2 , c_3 are arbitrary constants.

DE0032

Solve the following differential equation for Q.3 to Q.9.

3.
$$\frac{\ln(\sec x + \tan x)}{\cos x} dx = \frac{\ln(\sec y + \tan y)}{\cos y} dy$$
 DE0033 4. $(1 - x^2) (1 - y) dx = xy (1 + y) dy$

4.
$$(1-x^2)(1-y) dx = xy(1+y) dy$$
 DE0034

5.
$$\frac{dy}{dx} + \frac{\sqrt{(x^2 - 1)(y^2 - 1)}}{xy} = 0$$

DE0035 6.
$$y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right)$$

DE0036

7.
$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

DE0037 8.
$$\frac{dy}{dx} = \frac{x(2 \ln x + 1)}{\sin y + y \cos y}$$

DE0038

9. (a)
$$\frac{dy}{dx} + \sin \frac{x+y}{2} = \sin \frac{x-y}{2}$$

DE0039

(b)
$$\sin x \cdot \frac{dy}{dx} = y \cdot lny \text{ if } y = e, \text{ when } x = \frac{\pi}{2}$$

DE0040

- **10.** The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being k > 0. Find
 - Population at any time t, given initial population of the town being 2500. (a)
 - If 10 years later the population has fallen to 1900, find the time when the population will (b) be 1500.
 - Predict about the population of the town in the long run.

DE0041

A normal is drawn at a point P(x,y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then 11. show that the differential equation describing such curves is, $y\frac{dy}{dy}=\pm\sqrt{k^2-y^2}$. Find the equation of such **DE0043** a curve passing through (0,k).

- 12. Given y(0) = 2000 and $\frac{dy}{dx} = 32000 20y^2$, then find the value of $\lim_{x \to \infty} y(x)$.
- 13. Let f(x) is a continuous function which takes positive values for $x \ge 0$ and satisfy $\int_0^x f(t)dt = x\sqrt{f(x)}$ with

 $f(1) = \frac{1}{2}$. Find the value of $f(\sqrt{2} + 1)$.

(E-2)

[HOMOGENEOUS]

1. $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$

2. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1).

3. (x-y) dy = (x+y+1) dx

4. $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

5. $\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y-1)^2}$ DE0053

(E-3)

[LINEAR]

1.
$$\frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{1}{2x(1+x^2)}$$

2. $(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$ DE0059

- 3. (a) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa. **DE0065**
 - (b) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2sq. units. If the curve passes through (2,1), find the equation of the curve.
- 4. $x(x-1)\frac{dy}{dx} (x-2)y = x^3(2x-1)$
- $5. \quad \sin x \, \frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \cos x$
- **6.** $x(x^2+1) \frac{dy}{dx} = y(1-x^2) + x^3 \cdot \ell nx$

node06 \B080-BA \Kola\UEE(Advanced)\Leader\Walfs\Shee\\Differental Equation & AUC\Eng.p6

7.
$$x \frac{dy}{dx} - y = 2 x^2 \csc 2x$$

DE0063

8.
$$(1 + y^2) dx = (\tan^{-1} y - x) dy$$

DE0064

9. Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t) \tan t dt + \int_0^x \tan(t-x) dt$ where

$$x \in (-\pi/2, \pi/2)$$

DE0067

10.
$$y - x Dy = b(1 + x^2Dy)$$

DE0068

11.
$$2\frac{dy}{dx} - y \sec x = y^3 \tan x$$

DE0070

12.
$$x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$$

DE0071

13.
$$y(2xy + e^x) dx - e^x dy = 0$$

DE0072

$$(E-4)$$

(GENERAL - CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

1.
$$(x-y^2) dx + 2xy dy = 0$$

DE0074

2.
$$(x^3 + y^2 + 2) dx + 2y dy = 0$$

DE0075

3.
$$x \frac{dy}{dx} + y \ln y = xye^x$$

DE0076

4.
$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$$

DE0077

$$5. \qquad \frac{\mathrm{d}\,\mathrm{y}}{\mathrm{d}\,\mathrm{x}} = \frac{\mathrm{e}^{\mathrm{y}}}{\mathrm{x}^2} - \frac{1}{\mathrm{x}}$$

DE0078

6.
$$\frac{dy}{dx} = \frac{y^2 - x}{2y(x+1)}$$

DE0080

7.
$$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

DE0082

$$8. \qquad (x^2 + y)\frac{dy}{dx} = 6x$$

DE0083

EXERCISE (S-2)

(MISCELLANEOUS)

1. $\frac{dy}{dx} - y \ln 2 = 2^{\sin x}$. $(\cos x - 1)\ln 2$, y being bounded when $x \to +\infty$.

DE0084

2. $\frac{ydx - xdy}{(x - y)^2} = \frac{dx}{2\sqrt{1 - x^2}}$, given that y = 2 when x = 1.

DE0086

3. Find the integral curve of the differential equation, $x(1-x \ln y) \cdot \frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

DE0088

- **4.** Let f(x) be a differentiable function and satisfy f(0) = 2, f'(0) = 3 and f''(x) = f(x). Find
 - (a) the range of the function f(x)

DE0095

(b) the value of the function when x = ln2

DE0095

(c) the area enclosed by y = f(x) in the 2nd quadrant

- Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2 + y^2} = c e^{\pm \tan^{-1} \frac{y}{x}}$. **DE0096**
- 6. Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the point of tangency.

 DE0097

(1) $\sec x = (\tan x + c) y$

(3) $y \tan x = \sec x + c$

1.

2.

(2) $y \sec x = \tan x + c$

 $(4) \tan x = (\sec x + c) y$

[AIEEE-2010]

[AIEEE-2011]

DE0137

EXERCISE (JM)

Solution of the differential equation $\cos x \, dy = y(\sin x - y)dx$, $0 < x < \frac{\pi}{2}$ is:

If $\frac{dy}{dx} = y + 3 > 0$ and y(0) = 2, then $y(\ln 2)$ is equal to :-

	3.	(1) 13	(2) –2	(3) 7	(4) 5 DE0098			
	J.	Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where						
		value V(t) depreciate	s at a rate given by di	fferential equation $\frac{dt}{dt}$	$\frac{4}{r} = -k(T - t)$, where			
		k > 0 is a constant and of the equipment is :-	T is the total life in yea	rs of the equipment. The	n the scrap value V(T) [AIEEE-2011]			
		(1) I $-\frac{k(T-t)^2}{2}$	$(2) e^{-kT}$	$(3) T^2 - \frac{I}{k}$	(4) $I - \frac{kT^2}{2}$ DE0099			
	egment of any tangent to							
		it lying between the coordinate axes is bisected by the point of contact, is given by: [AIEEE-2011]						
		()2 ()2			[AILEL-2011]			
		$(1)\left(\frac{x}{2}\right) + \left(\frac{y}{3}\right) = 2$	$(2) \ 2y - 3x = 0$	$(3) y = \frac{6}{x}$	$(4) x^2 + y^2 = 13$			
	5. Consider the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by							
			(у)		[AIEEE-2011]			
		(1) $1^{-1} \cdot \frac{1}{e^y}$	(2) $4^{-2} \frac{1}{e^{y}}$	(3) $3 - \frac{1}{y} + \frac{\frac{1}{e^y}}{e}$	(4) 1. $\frac{1}{e^y}$ DECAY			
	6.	y C	y	y	y C			
	0.	The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5 \text{ p(t)} - 450. \text{ If p(0)} = 850, \text{ then the time at which the population becomes zero is}$						
		$\frac{dp(t)}{dt} = 0.5 p(t) - 450$. If $p(0) = 850$, then the	time at which the popula	ation becomes zero is:			
					[AIEEE-2012]			
		(1) ln18	(2) 2 ln18	(3) ln9	(4) $\frac{1}{2} \ln 18$ DE0102			
Eng.p65	7.	At present a firm is manuf	stimated that the rate of char	nge of production P w.r.t.				
nfal Equation & AUC		additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers						
iheet\Differe		the new level of producti	ion of items is:		[JEE (Main)-2013]			
node06\B0B0-BA\Kola\LEE(Advanced)\Leader\Walts\Shee\\Differental		(1) 2500	(2) 3000	(3) 3500	(4) 4500 DE0103			
vanced)\Lea	8.	_	nere of radius r is increasing	guniformly at the rate 8cm ² /	-			
Kota VEE(Ad		of its volume is: (1) proportional to r ²		(2) constant	-Main (On line)-2013]			
\B080-8A\				(4) proportional to \sqrt{r}	DE0104			
node06		(3) proportional to r		(4) proportional to $\sqrt{1}$	DE0104			
Ε					———			

	a	dy _	y^3
9.	Consider the differential equation	${\mathrm{dx}}$ –	$2(xy^2-x^2)$

Statement 1: The substitution $z = y^2$ transforms the above equation into a first order homogenous differential equation.

Statement 2: The solution of this differential equation is $y^2 e^{-\frac{y^2}{x}} = C$.

- (1) Statement 1 is false and statement 2 is true. (2) Both statements are true.
- (3) Statement 1 is true and statement 2 is false.
- (4) Both statements are false.

DE0105

[JEE-Main (On line)-2013]

- If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 \frac{1}{x^2}\right)$ at any point (x, y) on it, then the 10. ordinate of the point on the curve whose abscissa is -2 is: [JEE-Main (On line)-2013]
- $(2)\frac{5}{2}$

The equation of the curve passing through the origin and satisfying the differential equation 11.

$$(1 + x^2) \frac{dy}{dx} + 2 xy = 4x^2 \text{ is :}$$

[JEE-Main (On line)-2013]

- (1) $(1 + x^2) y = x^3$ (2) $3(1 + x^2) y = 4x^3$ (3) $3(1 + x^2) y = 2x^3$ (4) $(1 + x^2) y = 3x^3$

DE0107

- Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) 200$. **12.**
 - If p(0) = 100, then p(t) equals:

- (1) $400 300 e^{t/2}$
- (2) $300 200 e^{-t/2}$ (3) $600 500 e^{t/2}$
- (4) $400 300 e^{-t/2}$

DE0108

- Let y(x) be the solution of the differential equation (x log x) $\frac{dy}{dx}$ + y = 2x log x, (x \ge 1). **13.** [JEE(Main)-2015]
 - Then y(e) is equal to :

(3) e

(4) 0**DE0109**

(1) 2(2) 2eIf a curve y = f(x) passes through the point (1, -1) and satisfies the differential equation, y(1 + xy) dx = x dy, then **14.**

$$f\left(-\frac{1}{2}\right)$$
 is equal to:

[JEE(Main)-2016]

- $(2) -\frac{2}{5}$
- $(3) \frac{4}{5}$
- $(4) \frac{2}{5}$

DE0110

- **15.** If $(2+\sin x)\frac{dy}{dx} + (y+1)\cos x = 0$ and y(0) = 1, then $y(\frac{\pi}{2})$ is equal to :-[JEE(Main)-2017]

 $(2) \frac{1}{2}$

- $(3) -\frac{2}{3} \qquad (4) -\frac{1}{2}$

- Let y = y(x) be the solution of the differential equation $\sin x \frac{dy}{dx} + y \cos x = 4x$, $x \in (0, \pi)$. If $y(\frac{\pi}{2}) = 0$
 - 0, then $y\left(\frac{\pi}{6}\right)$ is equal to :

[JEE(Main)-2018]

- (1) $\frac{-8}{9\sqrt{3}}\pi^2$
- $(2) \frac{8}{9} \pi^2$
- $(3) \frac{4}{9}\pi^2$
- (4) $\frac{4}{0\sqrt{3}}\pi^2$ **DE0112**

DE0138

DE0114

[JEE(Main)-2019]

[JEE(Main)-2019] (3) $-\frac{4}{3}$ (4) $\frac{1}{2} + e^3$

 $(4) \frac{1}{4}$

- The curve amongst the family of curves, represented by the differential equation, **19.** $(x^2-y^2)dx + 2xy dy = 0$ which passes through (1,1) is: [JEE(Main)-2019]
 - (1) A circle with centre on the y-axis
 - (2) A circle with centre on the x-axis
 - (3) An ellipse with major axis along the y-axis
 - (4) A hyperbola with transverse axis along the x-axis

DE0115

- If y(x) is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$, x > 0, where $y(1) = \frac{1}{2}e^{-2}$, 20. [JEE(Main)-2019] then:
 - (2) y(x) is decreasing in $\left(\frac{1}{2}, 1\right)$ (1) y(x) is decreasing in (0, 1)
 - (3) $y(\log_e 2) = \frac{\log_e 2}{4}$

 $(4) \ y(\log_{2} 2) = \log_{2} 4$

- Let y = y(x) be the solution of the differential equation, $(x^2 + 1)^2 \frac{dy}{dx} + 2x(x^2 + 1)y = 1$ such that y(0) = 0. 21. If $\sqrt{a}y(1) = \frac{\pi}{32}$, then the value of 'a' is: [JEE(Main)-2019]
 - $(1)\frac{1}{2}$

- $(2)\frac{1}{16}$
- $(3) \frac{1}{4}$

- (4)1**DE0140**
- Consider the differential equation, $y^2 dx + \left(x \frac{1}{y}\right) dy = 0$. If value of y is 1 when x = 1, the the value of 22. x for which y = 2, is: [JEE(Main)-2019]
 - $(1) \frac{1}{2} + \frac{1}{\sqrt{e}}$ $(2) \frac{3}{2} \sqrt{e}$
- $(3) \frac{5}{2} + \frac{1}{\sqrt{2}}$
- $(4) \frac{3}{2} \frac{1}{\sqrt{2}}$ **DE0141**

EXERCISE (JA)

Let f be a real valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the 1. y-intercept of the tangent at any point P(x,y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(-3) is equal to [JEE 2010,3]

DE0142

2. (a) Let $f:[1,\infty) \to [2,\infty)$ be a differentiable function such

that
$$f(1) = 2$$
. If $6 \int_{1}^{x} f(t) dt = 3x f(x) - x^3$ for all $x \ge 1$, then the value of $f(2)$ is **[JEE 2011, 4]**

DE0120

(b) Let y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, $x \in R$, where f'(x) denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2)[JEE 2011, 4]

DE0121

If y(x) satisfies the differential equation $y' - y \tan x = 2x \sec x$ and y(0) = 0, then [JEE 2012, 4M] **3.**

$$(A) y \left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$$

(B)
$$y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$$

(C)
$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$$

(A)
$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$$
 (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$ (C) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Let $f: \left| \frac{1}{2}, 1 \right| \to \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that f'(x) < 2f(x) and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x) dx$ lies in the interval

$$(A) (2e - 1, 2e)$$

(B)
$$(e-1, 2e-1)$$

(A)
$$(2e-1, 2e)$$
 (B) $(e-1, 2e-1)$ (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

(D)
$$\left(0, \frac{e-1}{2}\right)$$
 DE0123

A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be

$$\frac{y}{x} + \sec\left(\frac{y}{x}\right)$$
, $x > 0$. Then the equation of the curve is

[JEE(Advanced) 2013, 2M]

DE0124

(A)
$$\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$$
 (B) $\csc\left(\frac{y}{x}\right) = \log x + 2$ (C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

6. The function y = f(x) is the solution of the differential equation

$$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}} \text{ in } (-1, 1) \text{ satisfying } f(0) = 0. \text{ Then } \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x) dx \text{ is } [\textbf{JEE}(\textbf{Advanced}) - \textbf{2014}, \textbf{3}(-1)]$$

(A)
$$\frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

(B)
$$\frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

(B)
$$\frac{\pi}{3} - \frac{\sqrt{3}}{4}$$
 (C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

(D)
$$\frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is 7. represented by the differential equation Py'' + Qy' + 1 = 0, where P,Q are functions of x,y and y' (here

$$y' = \frac{dy}{dx}$$
, $y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true? [**JEE 2015, 4M, -2M**]

$$(A) P = y + x$$

(B)
$$P = y - x$$

(A)
$$P = y + x$$

(C) $P + Q = 1 - x + y + y' + (y')^2$

(B)
$$P = y - x$$

(D) $P - Q = x + y - y' - (y')^2$

DE0130

If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx$, x > 09. and $y(0) = \sqrt{7}$, then y(256) =[JEE(Advanced)-2017, 3(-1)]

(A) 80 (B) 3

(C) 16

(D)9**DE0131**

If $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f'(x) > 2f(x) for all $x \in \mathbb{R}$, and f(0) = 1, then 10. [JEE(Advanced)-2017, 3(-2)]

(A) $f(x) > e^{2x}$ in $(0, \infty)$

(B) f(x) is decreasing in $(0,\infty)$

(C) f(x) is increasing in $(0,\infty)$

(D) $f'(x) < e^{2x}$ in $(0, \infty)$

DE0132

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for 11. all $x \in \mathbb{R}$, and f(1) = g(2) = 1, then which of the following statement(s) is (are) TRUE? **DE0133**

[JEE(Advanced)-2018, 4(-2)]

(A) $f(2) < 1 - \log_2 2$ (B) $f(2) > 1 - \log_2 2$

(C) $g(1) > 1 - \log_2 2$

(D) $g(1) < 1 - \log 2$

Let $f:(0,\pi)\to\mathbb{R}$ be a twice differentiable function such that $\lim_{t\to x}\frac{f(x)\sin t-f(t)\sin x}{t-x}=\sin^2 x$ for **12.**

all $x \in (0, \pi)$. If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE?

[JEE(Advanced)-2018, 4(-2)]

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

DE0134

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with f(0) = 0. If y = f(x) satisfies the differential equation $\frac{dy}{dx} = (2+5y)(5y-2)$, then the value of $\lim_{x \to -\infty} f(x)$ is _____. [JEE(Advanced)-2018, 3(0)]

DE0135

14. Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_p . If PY_p has length 1 for each point P on Γ , then which of the following options is/are correct? [JEE(Advanced)-2019, 4(-1)]

(1) $y = \log_e \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2}$

(2) $xy' - \sqrt{1 - x^2} = 0$

(3) $y = -\log_e \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) + \sqrt{1 - x^2}$

(4) $xy' + \sqrt{1-x^2} = 0$

ANSWER KEY

DIFFERENTIAL EQUATION

EXERCISE (O-1)

EXERCISE (O-2)

EXERCISE (S-1)

$$(E-1)$$

[FORMATION & VARIABLES SEPARABLE]

2. (a)
$$xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx}$$
; (b) $[1 + (y')^2]$. $y''' - 3y'(y'')^2 = 0$; (c) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

3.
$$ln^{2}(\sec x + \tan x) - ln^{2}(\sec y + \tan y) = c$$

4.
$$lnx (1-y)^2 = c - \frac{1}{2}y^2 - 2y + \frac{1}{2}x^2$$

5.
$$\sqrt{x^2-1} - \sec^{-1} x + \sqrt{y^2-1} = c$$

6.
$$y = c (1 - ay) (x + a)$$

7.
$$ln \left[1 + \tan \frac{x+y}{2}\right] = x + c$$

$$8. \quad y \sin y = x^2 lnx + c$$

9. (a)
$$l \ln \left| \tan \frac{y}{4} \right| = c - 2 \sin \frac{x}{2}$$
, (b) $y = e^{\tan(x/2)}$

10. (a)
$$P = 1000 + 1500e^{-kt}$$
 where $k = \frac{1}{10}ln(\frac{5}{3})$; (b) $T = 10 \log_{5/3}(3)$; (c) $P = 1000$ as $t \to \infty$

11.
$$x^2 + y^2 = k^2$$

(E-2)

[HOMOGENEOUS]

1.
$$c(x-y)^{2/3} (x^2 + xy + y^2)^{1/6} = \exp \left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x+2y}{x\sqrt{3}} \right]$$
, where $\exp x = e^x$

$$2. x^2 + y^2 - 2x = 0$$

3. arc tan
$$\frac{2y+1}{2x+1} = l \operatorname{nc} \sqrt{x^2 + y^2 + x + y + \frac{1}{2}}$$

4.
$$(x+y-2)=c(y-x)^3$$

5.
$$e^{-2\tan^{-1}\frac{y+2}{x-3}} = c.(y+2)$$

(E-3)

[LINEAR]

1.
$$y\sqrt{1+x^2} = c + \frac{1}{2}ln\left[\tan\frac{1}{2}\arctan x\right]$$
. Another form is $y\sqrt{1+x^2} = c + \frac{1}{2}ln\frac{\sqrt{1+x^2}-1}{x}$

2.
$$y = c(1-x^2) + \sqrt{1-x^2}$$

3. (a)
$$y = cx^2 \pm x$$
; (b) $xy = 2$

4.
$$y(x-1) = x^2(x^2-x+c)$$

5.
$$\left(\frac{1}{3} + y\right) \tan^3 \frac{x}{2} = c + 2 \tan \frac{x}{2} - x$$

6.
$$4(x^2+1) y+x^3 (1-2 \ln x) = cx$$

7.
$$y = cx + xln tanx$$

8.
$$x = ce^{-arc tany} + arc tan y - 1$$

9.
$$\cos x - 1$$

10.
$$y(1 + bx) = b + cx$$

11.
$$\frac{1}{y^2} = -1 + (c + x)\cot\left(\frac{x}{2} + \frac{\pi}{4}\right)$$
 12. $x^3 y^{-3} = 3\sin x + c$ 13. $y^{-1} e^x = c - x^2$

12.
$$x^3 y^{-3} = 3\sin x + \cos x$$

13.
$$y^{-1} e^x = c - x^2$$

$$(E-4)$$

1.
$$y^2 + x \ln ax = 0$$

$$y^2 + x \ln ax = 0$$
 2. $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$ 3. $x \ln y = e^x(x - 1) + c$
 $\sin y = (e^x + c)(1 + x)$ 5. $cx^2 + 2xe^{-y} = 1$

3.
$$x \ln y = e^{x}(x-1) + c$$

4.
$$\sin y = (e^x + c)(1 + x)$$

$$5. cx^2 + 2xe^{-y} = 1$$

6.
$$y^2 = -1 + (x+1)ln\frac{c}{x+1}$$
 or $x + (x+1)ln\frac{c}{x+1}$ 7. $e^y = c. \exp(-e^x) + e^x - 1$

7.
$$e^y = c. \exp(-e^x) + e^x - 1$$

8.
$$y = 3 ln(x^2 + y + 3) + C$$

EXERCISE (S-2)

1.
$$y = 2^{\sin x}$$
 2. $\frac{\sin^{-1} x}{2} + \frac{y}{x - y} = \frac{\pi}{4} - 2$

3.
$$x(ey + ln y + 1) = 1$$

4. (a)
$$(-\infty,\infty)$$
; (b) $\frac{19}{4}$; (c) $3-\sqrt{5}$ **6.** $y=cx-x^2$

6.
$$y = cx - x^2$$

EXERCISE (JM)

- **1.** 1
- **2.** 3
- **3.** 4
- **4.** 3
- **5.** 4
- **6.** 2
- **7.** 3

- 8.
- **9.** 2 **16.** 2
- **10.** 3 **17.** 3
- **11.** 2 **18.** 1
- **12.** 1

19. 2

- **13.** 1 **20.** 2
- **14.** 1 **21.** 2

15. 2 **22.** 4

EXERCISE (JA)

- **2.** (a) Bonus; (b) 0
- **3.** A,D
- **4.** D
- **5.** A
- В

- **7.** B,C
- 8. Α
- **9.** B
- **10.** A,C
- **11.** B,C
- **12.** B,C,D
- **13.** 0.4

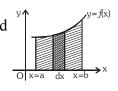
14. 1,4

ALLEN

AREA UNDER THE CURVE

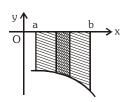
1. AREA UNDER THE CURVES:

(a) Area bounded by the curve y = f(x), the x-axis and the ordinates at x = a and x = b is given by $A = \int_{a}^{b} y \, dx$, where y = f(x) lies above the x-axis

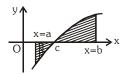


and b > a. Here vertical strip of thickness dx is considered at distance x.

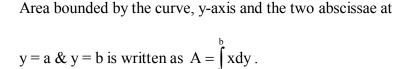
(b) If y = f(x) lies completely below the x-axis then A is negative and we consider the magnitude only, i.e. $A = \left| \int_a^b y \, dx \right|$

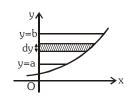


(c) If curve crosses the x-axis at x = c, then $A = \left| \int_{a}^{c} y \, dx \right| + \int_{c}^{b} y dx$



(d) Sometimes integration w.r.t. y is very useful (horizontal strip):





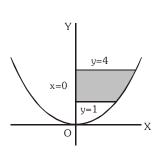
Note: If the curve is symmetric and suppose it has 'n' symmetric portions, then total area = n (Area of one symmetric portion).

Illustration 1: Find the area bounded by $y = \sec^2 x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ & x-axis

Solution: Area bounded = $\int_{\pi/6}^{\pi/3} y dx = \int_{\pi/6}^{\pi/3} sec^2 x dx = [\tan x]_{\pi/6}^{\pi/3} = \tan \frac{\pi}{3} - \tan \frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq.units.

Illustration 2: Find the area in the first quadrant bounded by $y = 4x^2$, x = 0, y = 1 and y = 4.

Solution: Required area $= \int_{1}^{4} x \, dy = \int_{1}^{4} \frac{\sqrt{y}}{2} \, dy = \frac{1}{2} \left[\frac{2}{3} y^{3/2} \right]_{1}^{4}$ $= \frac{1}{3} \left[4^{3/2} - 1 \right] = \frac{1}{3} \left[8 - 1 \right]$ $= \frac{7}{3} = 2 \frac{1}{3} \text{ sq.units.}$

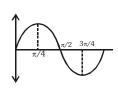


ALLEN

Illustration 3: Find the area bounded by the curve $y = \sin 2x$, x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Solution: Required area $=\int_{\pi/4}^{\pi/2} \sin 2x dx + \left| \int_{\pi/2}^{3\pi/4} \sin 2x dx \right| = \left(-\frac{\cos 2x}{2} \right) \Big|_{\pi/4}^{\pi/2} + \left| \left(-\frac{\cos 2x}{2} \right) \Big|_{\pi/2}^{3\pi/4}$

$$= -\frac{1}{2}[-1-0] + \left| \frac{1}{2}(0+(-1)) \right| = 1 \text{ sq. unit}$$

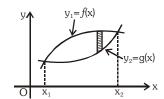


Do yourself - 1:

- (i) Find the area bounded by $y = x^2 + 2$ above x-axis between x = 2 & x = 3.
- (ii) Using integration, find the area of the curve $y = \sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.
- (iii) Find the area bounded by the curve $y = 2\cos x$ and the x-axis from x = 0 to $x = 2\pi$.
- (iv) Find the area bounded by the curve y = x|x|, x-axis and the ordinates $x = -\frac{1}{2}$ and x=1.

2. AREA ENCLOSED BETWEEN TWO CURVES:

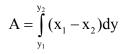
(a) Area bounded by two curves y = f(x) & y = g(x)such that f(x) > g(x) is



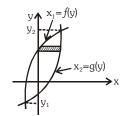
$$A = \int_{x_1}^{x_2} (y_1 - y_2) dy$$

$$A = \int_{x_1}^{x_2} [f(x) - g(x)] dx$$

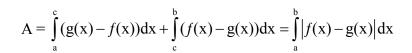
(b) In case horizontal strip is taken we have

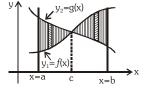


$$A = \int_{y_1}^{y_2} [f(y) - g(y)]dy$$



(c) If the curves $y_1 = f(x)$ and $y_2 = g(x)$ intersect at x = c, then required area





Note: Required area must have all the boundaries indicated in the problem.

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Illustration 4: Find the area bounded by the curve y = (x - 1)(x - 2)(x - 3) lying between the ordinates x = 0 and x = 3 and x-axis

Solution: To determine the sign, we follow the usual rule of change of sign.

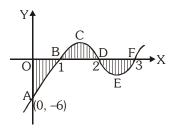
$$y = +ve$$
 for $x > 3$

$$y = -ve$$
 for $2 < x < 3$

$$y = +ve$$
 for $1 < x < 2$

$$y = -ve$$
 for $x < 1$.

$$\int_0^3 |y| dx = \int_0^1 |y| dx + \int_1^2 |y| dx + \int_2^3 |y| dx$$
$$= \int_0^1 -y dx + \int_1^2 y dx + \int_2^3 -y dx$$



Now let $F(x) = \int (x-1)(x-2)(x-3) dx = \int (x^3 - 6x^2 + 11x - 6) dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$.

$$\therefore F(0) = 0, F(1) = -\frac{9}{4}, F(2) = -2, F(3) = -\frac{9}{4}.$$

Hence required Area = $-[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq.units.

Illustration 5: Compute the area of the figure bounded by the straight lines x = 0, x = 2 and the curves $y = 2^x$, $y = 2x - x^2$.

Solution: The required area = $\int_0^2 (y_1 - y_2) dx$

where
$$y_1 = 2^x$$
 and $y_2 = 2x - x^2 = \int_0^2 (2^x - 2x + x^2) dx$

$$= \left[\frac{2^{x}}{\ln 2} - x^{2} + \frac{1}{3}x^{3} \right]^{2} = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3} \right) = \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3} \text{ sq.units.} \frac{Q}{Q}$$

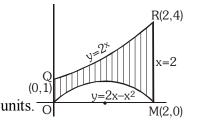


Illustration 6: Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

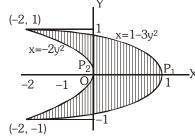
Solving the equations $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$.

The points are (-2, -1) and (-2, 1).

The required area

$$2\int_0^1 (x_1 - x_2) dy = 2\int_0^1 [(1 - 3y^2) - (-2y^2)] dy$$

$$=2\int_0^1 (1-y^2) dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3} \text{ sq.units.}$$



Do yourself - 2:

- (i) Find the area bounded by $y = \sqrt{x}$ and y = x.
- (ii) Find the area bounded by the curves $x = y^2$ and $x = 3 2y^2$.
- (iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, x = 2, $y = \log x$ and $y = 2^x$.

3. CURVE TRACING:

The following procedure is to be applied in sketching the graph of a function y = f(x) which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

- (a) Symmetry: The symmetry of the curve is judged as follows:
 - (i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.
 - (ii) If all the powers of x are even, the curve is symmetrical about the axis of y.
 - (iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.
 - (iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about y = x.
 - (v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.
- **(b)** Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.
- (c) Find the points where the curve crosses the x-axis & also the y-axis.
- (d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

Illustration 7: Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2 (a^2 - x^2)$.

Solution: The curve is symmetrical about both the axes. It cuts x-axis at (0, 0), (-a, 0), (a, 0)

Area of a loop =
$$2 \int_0^a y \, dx = 2 \int_0^a \frac{x}{a} \sqrt{a^2 - x^2} \, dx$$

$$= -\frac{1}{a} \int_0^a \sqrt{a^2 - x^2} (-2x) dx = \frac{1}{a} \left[\frac{2}{3} (a^2 - x^2)^{3/2} \right]_0^a = \frac{2}{3} a^2 \qquad \text{Y'}_{\overline{(-a,0)}}$$

Total area = $2 \times \frac{2}{3} a^2 = \frac{4}{3} a^2$ sq.units.

Illustration 8: Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

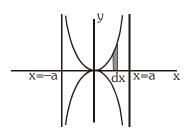
(i) The curve is symmetric about both the axes (even powers of x & y)
 (ii) Asymptotes are x = ± a

$$A = 4\int\limits_0^a y dx$$

$$=4\int_{0}^{a}\frac{ax}{\sqrt{a^{2}-x^{2}}}\,dx$$

$$=4a\left|-\sqrt{a^2-x^2}\right|_0^a$$

$$=4a^2$$



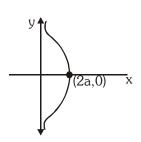
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Solution:

Find the area bounded by the curve $xy^2 = 4a^2(2a-x)$ and its asymptote. Illustration 9:

Solution:

- The curve is symmetrical about the x-axis as it contains even powers of y.
- (ii) It passes through (2a,0).
- (iii) Its asymptote is x = 0, i.e., y-axis.



$$A = 2 \int_{0}^{2a} y dx = 2 \int_{0}^{2a} 2a \sqrt{\frac{2a - x}{x}} dx$$

Put
$$x = 2a \sin^2\theta$$

$$A = 16a^2 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$=4\pi a^{2}$$

4. **IMPORTANT POINTS:**

Since area remains invariant even if the co-ordinate axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the area.

Illustration 10: Find the area enclosed by |x - 1| + |y + 1| = 1.

Solution:

Shift the origin to (1, -1).

$$X = x - 1$$

$$Y = y + 1$$

$$|X| + |Y| = 1$$

Area =
$$\sqrt{2} \times \sqrt{2}$$
 = 2 sq. units

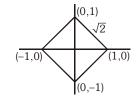


Illustration 11: Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

(-2√3.2

Solution:

Circle is
$$x^2 + y^2 + 4x + 6y - 3 = 0$$

$$\Rightarrow$$
 $(x + 2)^2 + (y + 3)^2 = 16$

Shifting origin to
$$(-2,-3)$$
.

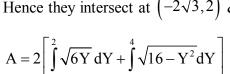
$$X^2 + Y^2 = 16$$

equation of parabola
$$\rightarrow$$
 $(x + 2)^2 = 6(y + 3)$

$$\Rightarrow$$
 X² = 6Y

Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

Hence they intersect at $\left(-2\sqrt{3},2\right)$ & $\left(2\sqrt{3},2\right)$



$$=2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_{0}^{2}+\left[\frac{1}{2}Y\sqrt{16-Y^{2}}+\frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_{2}^{4}\right]=\left(\frac{4\sqrt{3}}{3}+\frac{16\pi}{3}\right)\text{sq. units}$$

Do yourself: 3

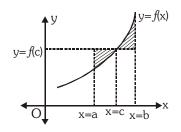
Find the area inside the circle $x^2-2x + y^2 - 4y + 1 = 0$ and outside the ellipse (i) $x^2-2x+4y^2-16y+13=0$

If y = f(x) is a monotonic function in (a, b), then the area bounded by the ordinates at x = a, **(b)** x = b, y = f(x) and y = f(c) [where $c \in (a, b)$] is minimum when $c = \frac{a + b}{2}$.

Proof: Let the function y = f(x) be monotonically increasing.

Required area A =
$$\int_{a}^{c} [f(c) - f(x)] dx + \int_{c}^{b} [f(x) - f(c)] dx$$

For minimum area, $\frac{dA}{da} = 0$



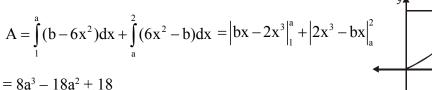
$$\Rightarrow [f'(c).c + f(c) - f'(c)a - f(c)] + [-f(c) - f'(c).b + f'(c).c + f(c)] = 0$$

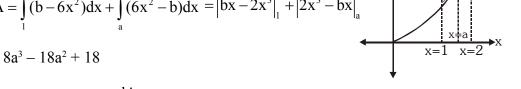
$$\Rightarrow f'(c)\left\{c-\frac{a+b}{2}\right\}=0$$

$$\Rightarrow c = \frac{a+b}{2} \quad (\because f'(c) \neq 0)$$

Illustration 12: Find the value of 'a' for which area bounded by x = 1, x = 2, $y = 6x^2$ and y = f(a) is minimum.

Let b = f(a). **Solution:**





For minimum area $\frac{dA}{da} = 0$

$$\Rightarrow$$
 24a² - 36a = 0 \Rightarrow a = 1.5

Alternatively,
$$y = 6x^2 \implies \frac{dy}{dx} = 12x$$

Hence y = f(x) is monotonically increasing. Hence bounded area is minimum when

$$a = \left(\frac{1+2}{2}\right) = 1.5$$

Do yourself - 4:

- Find the value of 'a' $(0 < a < \frac{\pi}{2})$ for which the area bounded by the curve $f(x) = \sin^3 x + \sin x$, (i) y = f(a) between x = 0 & $x = \pi$ is minimum.
- The area bounded by a curve & an axis is equal to the area bounded by the inverse of that (d) curve & the other axis, i.e., the area bounded by y = f(x) and x-axis (say) is equal to the area bounded by $y = f^{-1}(x)$ and y-axis.

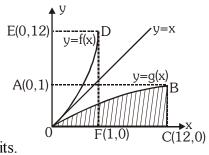
Illustration 13: If y = g(x) is the inverse of a bijective mapping $f : R \to R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by g(x), the x-axis and the ordinate at x = 12.

Solution: f(x) = 12

$$\Rightarrow 6x^5 + 4x^3 + 2x = 12 \Rightarrow x = 1$$

$$\int_{0}^{12} g(x)dx = \text{area of rectangle OEDF} - \int_{0}^{1} f(x)dx$$

=
$$1 \times 12 - \int_{0}^{1} (6x^5 + 4x^3 + 2x) dx = 12 - 3 = 9$$
 sq. units.



Do yourself - 5:

(i) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates x = 0 & x = 44.

5. USEFUL RESULTS:

- (a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is π ab sq.units.
- **(b)** Area enclosed between the parabolas $y^2 = 4$ ax & $x^2 = 4$ by is 16ab/3 sq.units.
- (c) Area included between the parabola $y^2 = 4$ ax & the line y = mx is $8 a^2/3 m^3$ sq.units.
- (d) The area of the region bounded by one arch of sin ax (or cos ax) and x-axis is 2/a sq.units.
- (e) Average value of a function y = f(x) over an interval $a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a} \int_a^b f(x) dx$.

Miscellaneous Illustration:

Illustration 14: Find the smaller of the areas bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.

Solution:

$$C_1$$
 is $4(y^2 - 2y) = 3x - 7$

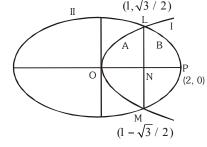
or
$$4(y-1)^2 = 3x - 3 = 3(x-1)$$
(i)

Above is parabola with vertex at (1, 1)

$$C_2$$
 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or
$$(x-1)^2 + 4(y-1)^2 = -1 + 1 + 4$$

or
$$\frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$
(ii)



Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

$$4Y^2 = 3X$$
 and $\frac{X^2}{2^2} + \frac{Y^2}{1} = 1$ They meet at $\left(1, \pm \frac{\sqrt{3}}{2}\right)$

Required area = $2(A + B) = 2 \left[\int Y_1 dX + \int Y_2 dX \right]$

$$=2\left\lceil\frac{\sqrt{3}}{2}\int_0^1\sqrt{X}dX+\int_1^2\frac{\sqrt{4-X^2}}{2}dX\right\rceil=\left\lceil\frac{\sqrt{3}}{6}+\frac{2\pi}{3}\right\rceil\text{ sq.units.}$$

Illustration 15: Find the area bounded by the regions $y \ge \sqrt{x}$, $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$.

Solution: Common region is given by the diagram If area of region $OAB = \lambda$ then area of $OCD = \lambda$

Because
$$y = \sqrt{x} \& x = -\sqrt{y}$$

will bound same area with x & y axes respectively.

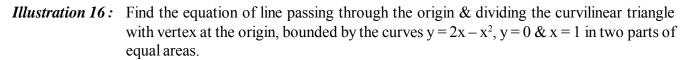
$$y = \sqrt{x} \implies y^2 = x$$

 $x = -\sqrt{y} \implies x^2 = y$ and hence both the curves are symmetric with respect to the line y = x

Area of first quadrant OBC =
$$\frac{\pi r^2}{4} = \frac{\pi}{2}$$
 (: $r = \sqrt{2}$)

Area of region OCA =
$$\frac{\pi}{2} - \lambda$$

Area of shaded region = $(\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq.units.



Solution: Area of region OBA = $\int_0^1 (2x - x^2) dx$

$$=\left[x^{2}-\frac{x^{3}}{3}\right]_{0}^{1}=\frac{2}{3}$$

$$\frac{2}{3} = A_1 + A_1 \Rightarrow A_1 = \frac{1}{3}$$

Let pt. C has coordinates (1, y)

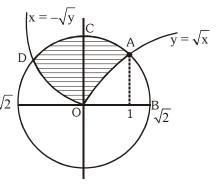
Area of
$$\triangle OCB = \frac{1}{2} \times 1 \times y = \frac{1}{3}$$

$$y = \frac{2}{3}$$

C has coordinates $\left(1, \frac{2}{3}\right)$

Line OC has slope
$$m = \frac{\frac{2}{3} - 0}{1 - 0} = \frac{2}{3}$$

Equation of line OC is $y = mx \Rightarrow y = \frac{2}{3}x$.



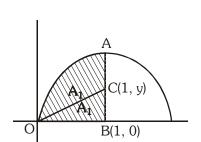


Illustration 17: The area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin is :-

- (A) 8/3 sq. units
- (B) 1/3 sq. units
- (C) 2/3 sq. units
- (D) none of these

Solution:

The parabola is even function & let the equation of tangent is y= mx

Now we calculate the point of intersection of parabola & tangent

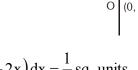
$$mx = x^2 + 1$$

$$x^2 - mx + 1 = 0 \implies D = 0$$

$$\Rightarrow$$
 m²-4=0 \Rightarrow m=±2

Two tangents are possible y = 2x & y = -2x

Intersection of $y = x^2 + 1 \& y = 2x$ is x = 1 & y = 2



Area of shaded region OAB = $\int_{0}^{1} (y_2 - y_1) dx = \int_{0}^{1} ((x^2 + 1) - 2x) dx = \frac{1}{3}$ sq. units

Area of total shaded region = $2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units

Illustration 18: Find the area bounded by x-axis and the curve given by x = asint, $y = acost for <math>0 \le t \le \pi$.

Area = $\int_{0}^{\pi} y \frac{dx}{dt} dt = \int_{0}^{\pi} a \cos t(a \cos t) dt = \frac{a^{2}}{2} \int_{0}^{\pi} (1 + \cos 2t) dt = \frac{a^{2}}{2} \left| t + \frac{\sin 2t}{2} \right|^{\pi} = \frac{a^{2}}{2} \left| \pi \right| = \frac{\pi a^{2}}{2}$ Solution:

Alternatively,

Area
$$=\int_{0}^{\pi} x \frac{dy}{dt} . dt = \left| \int_{0}^{\pi} a \sin t(-a \sin t) dt \right| = \frac{a^{2}}{2} \left| \int_{0}^{\pi} (\cos 2t - 1) dt \right| = \frac{a^{2}}{2} \left| -\frac{\sin 2t}{2} - t \right|_{0}^{\pi} = \frac{\pi a^{2}}{2}$$

ANSWERS FOR DO YOURSELF

- $\frac{25}{3}$ sq. units
- (ii) $\frac{\pi}{4}$ sq. units.
- (iii) 8 sq. units. (iv) $\frac{3}{8}$ sq. units

- (ii) 4 sq. units
- (iii) $\frac{4-\sqrt{2}}{\log 2} \frac{5}{2}\log 2 + \frac{3}{2}$ sq. units

- 2π sq. units
- 60 sq. units. **(i)**

curve is given by:

(A) $\frac{10}{3}$

1.

2.

(C) $\frac{4}{3}$ (D) $\frac{9}{2}$

AU0001

EXERCISE (0-1)

[SINGLE CHOICE OBJECTIVE TYPE]

(B) $\frac{7}{3}$

The area bounded in the first quadrant by the normal at (1,2) on the curve $y^2 = 4x$, x-axis & the

Let 'a' be a positive constant number. Consider two curves C_1 : $y = e^x$, C_2 : $y = e^{a-x}$. Let S be the area of

	the part surrounding by C_1 , C_2 and the y-axis, then $\lim_{a\to 0} \frac{S}{a^2}$ equals						
	(A) 4	(B) 1/2	(C) 0	(D) 1/4	AU0003		
3.	The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{ x }$ is						
	(A) 1/3	(B) 2/3	(C) 1/6	(D) 1	AU0004		
4.	Area enclosed by the graph of the function $y = ln^2x - 1$ lying in the 4 th quadrant is						
	$(A) \frac{2}{e}$	(B) $\frac{4}{e}$	(C) $2\left(e + \frac{1}{e}\right)$	(D) $4\left(e-\frac{1}{e}\right)$	AU0005		
5.	The area bounded by the curve $y = f(x)$ (where $f(x) \ge 0$), the co-ordinate axes & the line $x = x_1$ is give						
	by x_1 . e^{x_1} . Therefore $f(x)$ equals:						
	$(A) e^{x}$	(B) $x e^x$	(C) $xe^x - e^x$	(D) $x e^x + e^x$	AU0006		
6.	The slope of the tangent to a curve $y=f(x)$ at $(x,f(x))$ is $2x+1$. If the curve passes through the point $(1,2)$ then the area of the region bounded by the curve, the x-axis and the line $x=1$ is						
	(A) $\frac{5}{6}$	(B) $\frac{6}{5}$	(C) $\frac{1}{6}$	(D) 1	AU0007		
7.	The area bounded by the curves $y = x(x-3)^2$ and $y = x$ is (in sq. units):						
	(A) 28	(B) 32	(C) 4	(D) 8	AU0008		
8.	Area of the region enclosed between the curves $x = y^2 - 1$ and $x = y \sqrt{1 - y^2}$ is						
	(A) 1	(B) 4/3	(C) 2/3	(D) 2	AU0009		
9.	The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is						
	(A) $\frac{1}{24}$	(B) $\frac{1}{12}$	(C) $\frac{1}{8}$	(D) $\frac{1}{6}$			

ALLEN

EXERCISE (O-2)

[SINGLE CHOICE OBJECTIVE TYPE]

- 1. The area bounded by the curve $y = x e^{-x}$; xy = 0 and x = c where c is the x-coordinate of the curve's inflection point, is
 - (A) $1 3e^{-2}$
- (B) $1 2e^{-2}$
- (C) $1 e^{-2}$
- (D) 1

AU0011

- 2. A function y = f(x) satisfies the differential equation $\frac{dy}{dx} y = \cos x \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by y = f(x), $y = \cos x$ and the y-axis in the 1st quadrant
 - (A) $\sqrt{2} 1$
- (B) $\sqrt{2}$
- (C) 1

(D) $\frac{1}{\sqrt{2}}$

AU0012

- 3. If the area bounded between x-axis and the graph of $y = 6x 3x^2$ between the ordinates x = 1 and x = a is 19 square units then 'a' can take the value
 - (A) 4 or -2

- (B) two values are in (2, 3) and one in (-1, 0)
- (C) two values one in (3,4) and one in (-2,-1)
- (D) none of these

AU0013

[MULTIPLE OBJECTIVE TYPE]

- 4. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between y = cx and $y = x^2$ where c > 0 then
 - (A) Area (R) = $\frac{c^3}{6}$

(B) Area of R = $\frac{c^3}{3}$

(C) $\lim_{c\to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = 3$

(D) $\lim_{c\to 0^+} \frac{\operatorname{Area}(T)}{\operatorname{Area}(R)} = \frac{3}{2}$

AU0014

- Suppose f is defined from R \rightarrow [-1, 1] as $f(x) = \frac{x^2 1}{x^2 + 1}$ where R is the set of real number. Then the statement which does not hold is
 - (A) f is many one onto
 - (B) f increases for x > 0 and decrease for x < 0
 - (C) minimum value is not attained even though f is bounded
 - (D) the area included by the curve y = f(x) and the line y = 1 is π sq. units.

AU0015

- **6.** Which of the following statement(s) is/are True for the function
 - $f(x) = (x-1)^2(x-2) + 1$ defined on [0, 2]?
 - (A) Range of f is $\left[\frac{23}{27}, 1\right]$.
 - (B) The coordinates of the turning point of the graph of y = f(x) occur at (1, 1) and $\left(\frac{5}{3}, \frac{23}{27}\right)$.
 - (C) The value of p for which the equation f(x) = p has 3 distinct solutions lies in interval $\left(\frac{23}{27}, 1\right)$.
 - (D) The area enclosed by y = f(x), the lines x = 0 and y = 1 as x varies from 0 to 1 is $\frac{7}{12}$.

- Let $f(x) = 2 |x 1| & g(x) = (x 1)^2$, then -7.
 - (A) area bounded by f(x) & g(x) is $\frac{7}{6}$
- (B) area bounded by f(x) & g(x) is $\frac{7}{3}$ AU0019
- (C) area bounded by f(x), g(x) & x-axis is $\frac{5}{3}$
- (D) area bounded by f(x), g(x) & x-axis is $\frac{5}{6}$
- If A_1 denotes area of the region bounded by the curves C_1 : $y = (x 1)e^x$, tangent to C_1 at (1,0) & y-axis 8. and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then -

AU0021

$$(A) A_1 > A_2$$

(B)
$$A_1 < A_2$$

(C)
$$2A_1 + A_2 = 2$$

(C)
$$2A_1 + A_2 = 2$$
 (D) $A_1 + 2A_2 = 4$

Area bounded by the curve $y = \cot x$, $x = \frac{\pi}{4}$ and y = 0 is-9.

AU0023

(A)
$$\int_{0}^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx$$
 (B) $\frac{\pi}{4} - \int_{0}^{1} \tan^{-1} x dx$ (C) $1 - \int_{0}^{1} \tan^{-1} x dx$ (D) $\int_{0}^{\pi/4} \tan^{-1} x dx$

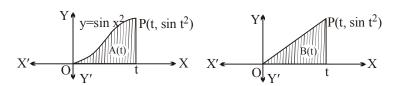
(B)
$$\frac{\pi}{4} - \int_{0}^{1} \tan^{-1} x dx$$

(C)
$$1 - \int_{0}^{1} \tan^{-1} x dx$$

(D)
$$\int_{0}^{\pi.4} \tan^{-1} x dx$$

EXERCISE (S-1)

- Find the area bounded on the right by the line x + y = 2, on the left by the parabola $y = x^2$ and below by the 1. x-axis. AU0026
- Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$. 2. AU0027
- Find the area of the region bounded by curves $f(x) = (x-4)^2$, $g(x) = 16 x^2$ and the x-axis. **3. AU0028**
- Find the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 x$. Also find the ratio in which the y-axis 4. divided this area. AU0030
- If the area enclosed by the parabolas $y = a x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'. **AU0031** 5.
- Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = ln x$ on the xy plane. Let D_1 denotes the region surrounded 6. by C_1 , C_2 and the line x = 1 and D_2 denotes the region surrounded by C_1 , C_2 and the line x = a. If $D_1 =$ D₂. Find the value of 'a'. AU0033
- Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x-axis. 7. AU0034
- Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices 8. $(0, 0), (a, 0), (0, a^2 + 1)$ and $(a, a^2 + 1)$. AU0036
- Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 x^5$, the straight lines 9. x = 1 & x = c & the abscissa axis is equal to 16/3.**AU0038**
- The figure shows two regions in the first quadrant. 10.



A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim_{t\to 0} \frac{A(t)}{B(t)}$ AU0041

EXERCISE (S-2)

1. Compute the area of the region bounded by the curves y = e. x. $ln x & y = ln x/(e \cdot x)$ where ln e=1.

AU0042

- Find the area bounded by the curve $y = x e^{-x}$; xy = 0 and x = c where c is the x-coordinate of the curve's 2. inflection point.
- Consider the curve $y = x^n$ where n > 1 in the 1st quadrant. If the area bounded by the curve, the x-axis and the 3. tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n. AU0046
- Show that the area bounded by the curve $y = \frac{\ln x c}{x}$, the x-axis and the vertical line through the maximum 4. point of the curve is independent of the constant c AU0048
- Let 'c' be the constant number such that c > 1. If the least area of the figure given by the line passing through the **5.** point (1, c) with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$. AU0051
- For what values of $a \in [0, 1]$ does the area of the figure bounded by the graph of the function y = f(x) and the 6. straight lines x = 0, x = 1 & y = f(a) is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1 - x^2}$. Find also the maximum & the minimum areas. AU0054

EXERCISE (JM)

1.	The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$	and $x = \frac{3\pi}{2}$	is :-
		[AIEEE-20)10]

$$(1) 4\sqrt{2} - 2$$

$$(2)$$
 4 $\sqrt{2}$ + 2

$$(3) 4\sqrt{2} - 1$$

(3)
$$4\sqrt{2} - 1$$
 (4) $4\sqrt{2} + 1$ AU0085

- The area of the region enclosed by the curves y = x, x = e, $y = \frac{1}{x}$ and the positive x-axis is:- **AU0056** 2. [AIEEE-2011]
 - (1) $\frac{3}{2}$ square units (2) $\frac{5}{2}$ square units (3) $\frac{1}{2}$ square units

- The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is :-**3.**

[AIEEE-2011]

(1)0

(2) $\frac{32}{3}$ (3) $\frac{16}{3}$

 $(4) \frac{8}{2}$ AU0057

The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line y = 2 is: 4.

[AIEEE-2012]

(1) $10\sqrt{2}$

(2) $20\sqrt{2}$

(3) $\frac{10\sqrt{2}}{2}$

 $(4) \frac{20\sqrt{2}}{2}$ AU0058

The area (in square units) bounded by the curves $y = \sqrt{x}$, 2y - x + 3 = 0, x-axis and lying in the first quadrant **5.** [JEE (Main)-2013] is:

(1)9

(2)36

(3) 18

AU0059

The area bounded by the curve y = ln(x) and the lines y = 0, y = ln(3) and x = 0 is equal to : 6.

[JEE-Main (On line)-2013]

(1) 3ln(3) - 2

(2)3

(3)2

(4) 3ln(3) + 2

AU0060

(1)7/9

 $(1)\frac{15}{4}$

x = 0, y = 1 and y = 4, is :-

(2) 14/3

The area under the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$, and above x-axis is :

7.

8.

The area of the region (in sq. units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines

(3) 14/9

AU0061

AU0062

[JEE-Main (On line)-2013]

(4) 7/3

				[JEE-Main (On line)-2013]		
	$(1) 2\sqrt{2}$	$(2) 2\sqrt{2} + 2$	(3) 0	$(4) \ 2\sqrt{2} - 2$		
9.	Let $f: [-2, 3] \rightarrow [0, \circ]$	e) be a continuous funct	ion such that f (1-	$f(x) = f(x) \text{ for all } x \in [-2, 3].$		
	If R_1 is the numerical value of the area of the region bounded by $y = f(x)$, $x = -2$, $x = 3$ and					
	the axis of x and $R_2 =$	$\int_{-2}^{3} x f(x) dx$, then:		[JEE-Main (On line)-2013]		
			_	(4) $R_1 = 2R_2$ AU0063		
10.	The area of the region of	described by $A = \{(x, y) :$	$x^2 + y^2 \le 1 \text{ and } y^2 :$			
	4	4	2	[JEE(Main)-2014]		
	$(1) \frac{\pi}{2} + \frac{4}{3}$	(2) $\frac{\pi}{2} - \frac{4}{3}$	(3) $\frac{\pi}{2} - \frac{2}{3}$	(4) $\frac{\pi}{2} + \frac{2}{3}$ AU0064		
11.	The area (in sq.units) or	f the region $\{(x, y) : y^2 \ge 2\}$	$2x \text{ and } x^2 + y^2 \le 4x, $	$x \ge 0, y \ge 0$ } is :-		
	[JEE(Main)-2016]					
	$(1) \frac{\pi}{2} - \frac{2\sqrt{2}}{2}$	(2) $\pi - \frac{4}{2}$	(3) $\pi - \frac{8}{3}$	(4) $\pi - \frac{4\sqrt{2}}{3}$ AU0065		
12.	2 3	3	3	_ 3		
	The area (in sq. units) of the region $\{(x, y\} : x \ge 0, x + y \le 3, x^2 \le 4y \text{ and } y \le 1 + \sqrt{x} \}$ is : [JEE(Main)-2017]					
	$(1) \frac{5}{2}$	$(2)\frac{59}{12}$	$(3) \frac{3}{2}$	(4) $\frac{7}{3}$ AU0066		
13.	Let $g(x) = \cos x^2$, $f($	$(x) = \sqrt{x}$ and α , β (α	< β) be the root	s of the quadratic equation		
	$18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines					
				[JEE(Main)-2018]		
	$(1) \frac{1}{2} (\sqrt{3} + 1)$	(2) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$	$(3) \frac{1}{2}(\sqrt{2}-1)$	(4) $\frac{1}{2}(\sqrt{3}-1)$ AU0067		
14.	=	_	-	t at the point $(2, 3)$ to it and the		
	y-axis is : [JEE(Main)-2019]					
	$(1)\frac{14}{2}$	$(2) \frac{56}{3}$	$(3) \frac{8}{3}$	(4) $\frac{32}{3}$ AU0068		
15	5	3	3	<i>5</i>		
15.	The area of the region $A = [(x,y): 0 \le y \le x x + 1 \text{ and } -1 \le x \le 1]$ in sq. units, is:					
				[JEE(Main)-2019]		
	$(1)\frac{2}{3}$	(2) $\frac{1}{3}$	(3) 2	(4) $\frac{4}{3}$ AU0069		
16.	The area (in sq. units	s) of the region bounder	ed by the parabola	a, $y = x^2 + 2$ and the lines,		
	y = x + 1, $x = 0$ and $x = 1$	3, is:		[JEE(Main)-2019]		

 $(3) \frac{21}{2}$

 $(2) \frac{15}{2}$

 $(4) \frac{17}{4}$

AU0086

- The area (in sq. units) of the region $A = \{(x, y) : x^2 \le y \le x + 2\}$ is 17.
- [JEE(Main)-2019]

 $(1) \frac{10}{3}$

(2) $\frac{9}{2}$

 $(3) \frac{31}{6}$

 $(4) \frac{13}{6}$

AU0087

- The area (in sq. units) of the region bounded by the curves $y = 2^x$ and y = |x + 1|, in the first quadrant **18.** [JEE(Main)-2019]
 - $(1) \frac{3}{2} \frac{1}{\log_2 2}$
- $(2) \frac{1}{2}$

- (3) $\log_e 2 + \frac{3}{2}$
- $(4) \frac{3}{2}$

AU0071

- If the area (in sq. units) of the region $\{(x, y): y^2 \le 4x, x + y \le 1, x \ge 0, y \ge 0\}$ is $a\sqrt{2} + b$, then a b**19.** [JEE(Main)-2019] is equal to:
 - $(1) \frac{8}{3}$

- $(2) \frac{10}{3}$
- (3)6

 $(4) -\frac{2}{3}$

AU0072

EXERCISE (JA)

- (a) Let the straight line x = b divide the area enclosed by $y = (1 x)^2$, y = 0 and x = 0 into 1. two parts $R_1(0 \le x \le b) d$ and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals
 - (A) $\frac{3}{4}$

- (B) $\frac{1}{2}$
- (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

- (b) Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that f(x) = f(1-x) for all $x \in [-1,2]$. Let $R_1 = \int_0^x x f(x) dx$, and R_2 be the area of the region bounded by y = f(x), x=-1, x=2, and the x-axis. Then -[JEE 2011, 3+3]
 - (A) $R_1 = 2R_2$

- (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$ **AU0078**
- The area enclosed by the curve $y = \sin x + \cos x$ and $y = |\cos x \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is 2.
 - (A) $4(\sqrt{2}-1)$

- (B) $2\sqrt{2}(\sqrt{2}-1)$
- [JEE(Advanced) 2013, 2M]

(C) $2(\sqrt{2}+1)$

(D) $2\sqrt{2}(\sqrt{2}+1)$

- AU0079
- Let $F(x) = \int_{0}^{x^2 + \frac{\pi}{6}} 2\cos^2 t dt$ for all $x \in \mathbb{R}$ and $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, **3.** if F'(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is **AU0080** [JEE 2015, 4M, -0M]
- If the line $x=\alpha$ divides the area of region $R=\{(x,\,y)\in\mathbb{R}^2:x^3\leq y\leq x,\,0\leq x\leq 1\}$ into two equal 4. [JEE(Advanced)-2017, 3(-2)] parts, then

- (A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 1 = 0$ (C) $0 < \alpha \le \frac{1}{2}$ (D) $2\alpha^4 4\alpha^2 + 1 = 0$

AU0081

- Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function such that $f(x)=1-2x+\hat{\int} e^{x-t}f(t)dt$ 5. for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE?
 - (A) The curve y = f(x) passes through the point (1, 2)[JEE(Advanced)-2018, 4(-2)]
 - (B) The curve y = f(x) passes through the point (2, -1)
 - (C) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1 x^2} \}$ is $\frac{\pi 2}{4}$
 - (D) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1 x^2} \}$ is $\frac{\pi 1}{4}$ AU0082
- **6.** A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of Δ PQR, then the value of n is _____. [JEE(Advanced)-2018, 3(0)]

AU0083

- The area of the region $\{(x, y) : xy \le 8, 1 \le y \le x^2\}$ is [JEE(Advanced)-2019, 3(-1)] 7.
 - (1) $8\log_e 2 \frac{14}{3}$ (2) $16\log_e 2 \frac{14}{3}$ (3) $16\log_e 2 6$ (4) $8\log_e 2 \frac{7}{3}$
- AU0084

Ε

ANSWER KEY

AREA UNDER THE CURVE

EXERCISE (O-1)

- **1.** A
- **2.** D
- **3.** B
- **4.** B **5.** D **6.** A **7.** D
- **8.** D

9. A

EXERCISE (O-2)

- **1.** A
- **2.** A
- **3.** C

- **4.** A,C **5.** A,C,D **6.** B,C,D **7.** B,C **8.** B,C

9. A,B

EXERCISE (S-1)

- **1.** 5/6 sq. units **2.** 23/6 sq. units **3.** 64 **4.** $\frac{\pi}{2}$; $\frac{\pi-1}{\pi+1}$ **5.** a = 9 **6.** e

- **7.** 2 sq. units **8.** $\sqrt{3}$ **9.** C = -1 or $\left(8 \sqrt{17}\right)^{1/3}$ **10.** 2/3

EXERCISE (S-2)

- **1.** $(e^2 5)/4$ e sq. units **2.** $1 3e^{-2}$ **3.** $\sqrt{2} + 1$ **4.** 1/2 **5.** 104

- **6.** a = 1/2 gives minima, $A(\frac{1}{2}) = \frac{3\sqrt{3} \pi}{12}$; a = 0 gives local maxima $A(0) = 1 \frac{\pi}{4}$;

a = 1 gives maximum value, $A(1) = \pi/4$

EXERCISE (JM)

- **1.** 1
- **2.** 1
- **3.** 3
- **4.** 4
- **5.** 1 **6.** 3
- **7.** 3
- **8.** 4

- **9.** 4
- **10.** 1
- **11.** 3
- **12.** 1

- **13.** 4 **14.** 3 **15.** 3
- **16.** 2

- **17.** 2
- **18.** 1
- **19.** 3

EXERCISE (JA)

- **1.** (a) B; (b) C
- **2.** B
- **3.** 3 **4.** A,D **5.** B,C **6.** 4