PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the **focus**.

(b) The fixed straight line is called the **directrix**.

(c) The constant ratio is called the **eccentricity** denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the **axis**.

(e) A point of intersection of a conic with its axis is called a **vertex**.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus **(p, q)** & directrix $\mathbf{lx + my + n = 0}$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $\mathbf{D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$ & the general equation of a conic represents a pair of straight lines and if :

$\mathbf{e > 1}$ the lines will be real & distinct intersecting at S.

$\mathbf{e = 1}$ the lines will coincident.

$\mathbf{e < 1}$ the lines will be imaginary.

Case (ii) When the focus does not lie on the directrix :

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1$; $D \neq 0$ $h^2 = ab$	$0 < e < 1$; $D \neq 0$ $h^2 < ab$	$D \neq 0$; $e > 1$; $h^2 > ab$	$e > 1$; $D \neq 0$ $h^2 > ab$; $a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $\mathbf{y^2 = 4ax}$. For this parabola :

(i) Vertex is **(0, 0)** (ii) Focus is **(a, 0)** (iii) Axis is $\mathbf{y = 0}$ (iv) Directrix is $\mathbf{x + a = 0}$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the **focal distance of the point.**

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a **focal chord**.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a **double ordinate**.

(d) Latus rectum :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **latus rectum**. For $y^2 = 4ax$.

- Length of the latus rectum = **4a.**
- Length of the semi latus rectum = **2a**.
- Ends of the latus rectum are **L(a, 2a) & L'(a, – 2a)**

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

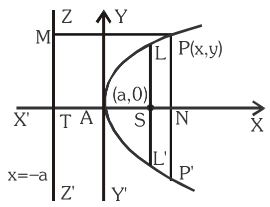
(iii) Two parabolas are said to be equal if they have the same latus rectum.

5. PARAMETRIC REPRESENTATION :

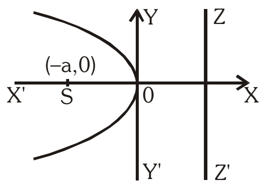
The simplest & the best form of representing the co-ordinates of a point on the parabola is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. TYPE OF PARABOLA :

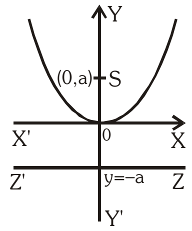
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



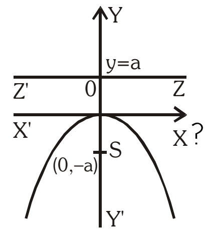
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2,2at)$	x + a
$y^2 = -4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2,2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a– h =0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x–h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y–q+b

Illustration 1 : Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Solution : The given equation can be rewritten as $\left(y-\frac{2}{3}\right)^2 = \frac{16}{9}\left(x+\frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$. Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$

The axis is $y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$

The directrix is $X + A = 0 \Rightarrow x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow x = -\frac{613}{144}$

The focus is X = A and Y = 0 $\Rightarrow x + \frac{61}{16} = \frac{4}{9}$ and $y - \frac{2}{3} = 0$

$\Rightarrow$ focus $= \left(-\frac{485}{144}, \frac{2}{3}\right)$

Length of the latus rectum $= 4A = \frac{16}{9}$

The tangent at the vertex is $X = 0 \Rightarrow x = -\frac{61}{16}$. **Ans.**

Illustration 2 : The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

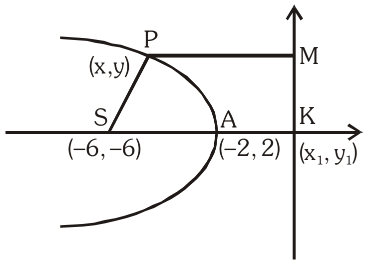
(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$

Solution : The length of latus rectum = 2 × perp. from focus to the directrix

$$= 2 \times \left|\frac{2 - 4(3) + 3}{\sqrt{(1)^2 + (4)^2}}\right| = \frac{14}{\sqrt{17}}$$ **Ans. (D)**

Illustration 3 : Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

Solution : Let S(–6, –6) be the focus and A(–2, 2) is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that SA = AK. Draw KM perpendicular on SK. Then KM is the directrix of the parabola. Since A bisects SK, $\left(\frac{-6 + x_1}{2}, \frac{-6 + y_1}{2}\right) = (-2, 2)$



$\Rightarrow -6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$

Hence the equation of the directrix KM is

$y - 10 = m(x - 2)$ .... (i)

Also gradient of SK $= \frac{10 - (-6)}{2 - (-6)} = \frac{16}{8} = 2; \Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x - 2)$ (from (i))

$\Rightarrow x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola. From SP = PM, the equation of the parabola is $\sqrt{\{(x+6)^2 + (y+6)^2\}} = \frac{|x + 2y - 22|}{\sqrt{(1^2 + 2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$ or $(2x - y)^2 + 104x + 148y - 124 = 0$.

Ans.

Illustration 4 : The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Solution : Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y - 4 = \frac{0}{5-3}(x-7) \Rightarrow y = 4$$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightwards and the other concave leftwards.

The vertex of the first parabola is (6.5, 4) and its equation is $(y-4)^2 = 2(x-6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y-4)^2 = -2(x-7.5)$. It meets the x-axis at (–0.5, 0). **Ans.**

Do yourself - 1 :

(i) Name the conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$, where a, b $\in$ R, a, b, > 0.

(ii) Find the vertex, axis, focus, directrix, latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

(iii) Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

(iv) Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y = 4$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.

7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point $(\mathbf{x_1}, \mathbf{y_1})$ lies **outside**, **on** or **inside** the parabola $\mathbf{y^2 = 4ax}$ according as the expression $\mathbf{y_1^2 - 4ax_1}$ is **positive**, **zero** or **negative**.

Illustration 5 : Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Solution : $\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore$ $y_1^2 - 4ax_1 < 0$

$\Rightarrow$ $\alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow$ $\alpha^2 - 4\alpha + 4 < 0$

$(\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$ **Ans.**

8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is

$\mathbf{y(t_1 + t_2) = 2x + 2at_1t_2}$

Note :

(i) If PQ is focal chord then $\mathbf{t_1t_2 = -1}$.

(ii) Extremities of focal chord can be taken as $\mathbf{(at^2, 2at)}$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Illustration 6 : Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Solution : The given parabola is $y^2 = 4x$ ...... (i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$

Slope of OP $= \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of OQ $= \frac{2}{t_2}$

Since OP $\perp$ OQ, $\frac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ ...... (ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow y\left(t_1 - \frac{4}{t_1}\right) = 2(x - 4)$ [from (ii)]

$\Rightarrow 2(x - 4) - y\left(t_1 - \frac{4}{t_1}\right) = 0 \Rightarrow L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$

i.e. (4, 0) **Ans.**

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm \Rightarrow$ condition of tangency is, $\mathbf{c = \frac{a}{m}}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : Length of the focal chord making an angle α with the x - axis is $4a\,\text{cosec}^2\,\alpha$.

Illustration 7 : If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

(A) $(3, \infty)$ (B) $(-\infty, 1/3)$ (C) $(1/3, 3)$ (D) none of these

Solution : Putting value of y from the line in the parabola -

$(3x + \lambda)^2 = 4x$

$\Rightarrow \quad 9x^2 + (6\lambda - 4)x + \lambda^2 = 0$

$\because$ line cuts the parabola at two distinct points

$\therefore \quad D > 0$

$\Rightarrow \quad 4(3\lambda - 2)^2 - 4.9\lambda^2 > 0$

$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0$

$\Rightarrow \quad \lambda < 1/3$

Hence, $\lambda \in (-\infty, 1/3)$ **Ans.(B)**

Do yourself - 2 :

(i) Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

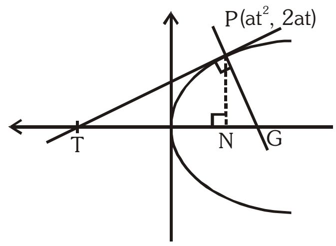
(ii) The focal distance of a point on the parabola $(x - 1)^2 = 16(y - 4)$ is 8. Find the co-ordinates.

(iii) Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with x-axis is of length $4a\ \text{cosec}^2\alpha$.

(iv) Find the condition that the straight line $ax + by + c = 0$ touches the parabola $y^2 = 4kx$.

(v) Find the length of the chord of the parabola $y^2 = 8x$, whose equation is $x + y = 1$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$$\mathbf{yy_1 = 2a(x + x_1)}$$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form :

Equation of tangent to the given parabola at its point P(t), is $\mathbf{ty = x + at^2}$

Note : Point of intersection of the tangents at the point $\mathbf{t_1}$ & $\mathbf{t_2}$ is $\mathbf{[at_1t_2, a(t_1 + t_2)]}$.

Illustration 8 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution : Let the slope of the tangent be m

$$\therefore \quad \tan 45° = \left|\frac{3-m}{1+3m}\right| \Rightarrow \quad 1+3m = \pm(3-m)$$

$$\therefore \quad m = -2 \quad \text{or} \quad \frac{1}{2}$$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

for $m = -2$, equation of tangent is $y = -2x - 1$ and point of contact is $\left(\frac{1}{2}, -2\right)$

for $m = \frac{1}{2}$, equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is (8, 8)

Ans.

Illustration 9 : Find the equation of the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \quad \Rightarrow \quad 16m^2 - 40\,m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$ **Ans.**

Illustration 10: Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that -

(i) $m_1^2 + m_2^2 = \lambda$ (constant) (ii) $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$

Let it passes through P(h, k)

$$\therefore \quad m^2h - mk + a = 0$$

(i) $m_1^2 + m_2^2 = \lambda$

$$(m_1 + m_2)^2 - 2m_1m_2 = \lambda$$

$$\frac{k^2}{h^2} - 2.\frac{a}{h} = \lambda$$

$\therefore$ locus of P(h, k) is $y^2 - 2ax = \lambda x^2$

(ii) $\theta_1 - \theta_2 = \theta_0$

$\tan(\theta_1 - \theta_2) = \tan\theta_0$

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \tan\theta_0$$

$$(m_1 + m_2)^2 - 4m_1m_2 = \tan^2\theta_0(1 + m_1m_2)^2$$

$$\frac{k^2}{h^2} - \frac{4a}{h} = \tan^2\theta_0\left(1 + \frac{a}{h}\right)^2$$

$$k^2 - 4ah = (h + a)^2\tan^2\theta_0$$

$\therefore$ locus of P(h, k) is $y^2 - 4ax = (x + a)^2\tan^2\theta_0$ **Ans.**

Do yourself - 3 :

(i) Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

(ii) Find the equation of the tangents to the parabola $y^2 = 16x$, which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$. Find also the co-ordinates of their points of contact.

(iii) Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle θ is $(x + a)^2\tan^2\theta = y^2 - 4ax$.

12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$\mathbf{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$\mathbf{y = mx - 2am - am^3}$

foot of the normal is $\mathbf{(am^2, -2am)}$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$\mathbf{y + tx = 2at + at^3}$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $\mathbf{(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))}$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $\mathbf{t_2 = -\left(t_1 + \frac{2}{t_1}\right)}$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $\mathbf{t_1t_2 = 2\ ;\ t_3 = -(t_1 + t_2)}$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $\mathbf{am^3 + m(2a - h) + k = 0}$.

This gives $\mathbf{m_1 + m_2 + m_3 = 0}$; $\mathbf{m_1m_2 + m_2m_3 + m_3m_1} = \frac{2a-h}{a}$; $\mathbf{m_1m_2m_3} = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

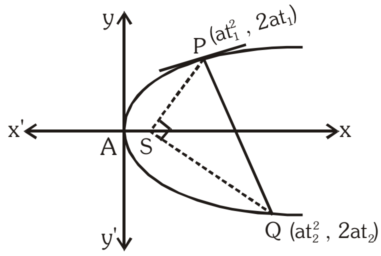
- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).

Illustration 11 : Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Solution : Let the normal at $P(at_1^2, 2at_1)$ meet the curve at $Q(at_2^2, 2at_2)$

$\therefore$ PQ is a normal chord.

and $t_2 = -t_1 - \frac{2}{t_1}$ ....(i)



By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a)

but focus S(a, 0)

$\therefore$ Slope of SP $= \frac{4a-0}{4a-a} = \frac{4a}{3a} = \frac{4}{3}$

and Slope of SQ $= \frac{-6a-0}{9a-a} = \frac{-6a}{8a} = -\frac{3}{4}$

$\therefore$ Slope of SP $\times$ Slope of SQ $= \frac{4}{3} \times -\frac{3}{4} = -1$

$\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Illustration 12 : If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha \cdot \tan\beta = 2$, then find the locus of this point.

Solution : Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

passes through (h, k)

$k = mh - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation, then $m_1 \cdot m_2 \cdot m_3 = -\frac{k}{a}$

but $m_1m_2 = 2$, $m_3 = -\frac{k}{2a}$

m_3 is root of (i) $\therefore$ $a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a-h) + k = 0 \Rightarrow k^2 = 4ah$

Thus locus is $y^2 = 4ax$. **Ans.**

Illustration 13 : Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the coordinates of the feet of the normals.

Solution : The given parabola is $y^2 - 16x - 8y = 0$ .... (i)

Let the co-ordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$. Now the equation of the tangent at $P(\alpha, \beta)$ to parabola (i) is

$y\beta - 8(x + \alpha) - 4(y + \beta) = 0$

or $(\beta - 4)y = 8x + 8a + 4\beta$ .... (ii)

Its slope $= \dfrac{8}{\beta - 4}$

Equation of the normal to parabola (i) at (α, β) is $y - \beta = \dfrac{4-\beta}{8}(x - \alpha)$

It passes through (14, 7)

$\Rightarrow \quad 7 - \beta = \dfrac{4-\beta}{8}(14 - \alpha) \qquad \Rightarrow \quad \alpha = \dfrac{6\beta}{\beta - 4}$ .... (iii)

Also (α, β) lies on parabola (i) i.e. $\beta^2 - 16\alpha - 8\beta = 0$ .... (iv)

Putting the value of α from (iii) in (iv), we get $\beta^2 - \dfrac{96\beta}{\beta - 4} - 8\beta = 0$

$\Rightarrow \quad \beta^2(\beta - 4) - 96\beta - 8\beta(\beta - 4) = 0 \qquad \Rightarrow \quad \beta(\beta^2 - 4\beta - 96 - 8\beta + 32) = 0$

$\Rightarrow \quad \beta(\beta^2 - 12\beta - 64) = 0 \qquad \Rightarrow \quad \beta(\beta - 16)(\beta + 4) = 0$

$\Rightarrow \quad \beta = 0, 16, -4$

from (iii), $\alpha = 0$ when $\beta = 0$; $\alpha = 8$, when $\beta = 16$; $\alpha = 3$ when $\beta = -4$

Hence the feet of the normals are (0, 0), (8, 16) and (3, –4) **Ans.**

Do yourself - 4 :

(i) If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

(ii) Find the number of distinct normal that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

(iii) If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

(iv) Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the co-ordinates of the feet of the normals.

13. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $\mathbf{SS_1 = T^2}$ where :

$\mathbf{S \equiv y^2 - 4ax}$; $\mathbf{S_1 \equiv y_1^2 - 4ax_1}$; $\mathbf{T \equiv yy_1 - 2a(x + x_1)}$.

14. DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **director circle.** It's equation is $\mathbf{x + a = 0}$ which is parabola's own directrix.

Illustration 15 : The angle between the tangents drawn from a point (–a, 2a) to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

Solution : The given point (–a, 2a) lies on the directrix x = –a of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle. **Ans.(B)**

Illustration 16 : The circle drawn with variable chord x + ay – 5 = 0 (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

(A) x + 5 = 0 (B) y + 5 = 0 (C) x + y + 5 = 0 (D) x – y + 5 = 0

Solution : Clearly x + ay – 5 = 0 will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e. the line x + 5 = 0. **Ans.(A)**

Do yourself - 5 :

(i) Find the angle between the tangents drawn from the origin to the parabola, $y^2 = 4a(x - a)$.

15. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $\mathbf{yy_1 = 2a(x + x_1)}$

Note : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\dfrac{(y_1^2 - 4ax_1)^{3/2}}{2a}$ i.e. $\dfrac{(S_1)^{3/2}}{2a}$, **also note that the chord of contact exists only if the point P is not inside.**

Illustration 17 : If the line x – y – 1 = 0 intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$$yk = 4(x + h)$$

$$4x - yk + 4h = 0 \quad \text{....... (i)}$$

But given line is

$$x - y - 1 = 0 \quad \text{....... (ii)}$$

Comparing (i) and (ii)

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \quad \Rightarrow \quad h = -1,\ k = 4$$

$\therefore$ point ≡ (–1, 4) **Ans.**

Illustration 18 : Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$ .... (i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x+h)$

$$y = \frac{2b}{k}x + \frac{2bh}{k} \qquad \text{......... (ii)}$$

From (i) & (ii)

$$m = \frac{2b}{k}, \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$$

locus of P is $y^2 = \frac{4b^2}{a}x$. **Ans.**

---

16. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $\mathbf{y - y_1 = \frac{2a}{y_1}(x - x_1)}$.

This reduced to $\mathbf{T = S_1}$, where $\mathbf{T \equiv yy_1 - 2a(x + x_1)}$ & $\mathbf{S_1 \equiv y_1^2 - 4ax_1}$.

Illustration 19 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Solution : Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$

$\therefore$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Illustration 20 : Find the locus of the middle point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Solution : The equation of the chord of the parabola whose middle point is (α, β) is

$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha$

$\Rightarrow$ $y\beta - 2ax = \beta^2 - 2a\alpha$

or $$\frac{y\beta - 2ax}{\beta^2 - 2a\alpha} = 1 \qquad \text{......... (i)}$$

Now, the equation of the pair of the lines OP and OQ joining the origin O i.e. the vertex to the points of intersection P and Q of the chord with the parabola $y^2 = 4ax$ is obtained by making the equation homogeneous by means of (i). Thus the equation of lines OP and

OQ is $$y^2 = \frac{4ax(y\beta - 2ax)}{\beta^2 - 2a\alpha}$$

$\Rightarrow$ $y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

If the lines OP and OQ are at right angles, then the coefficient of x^2 + the coefficient of $y^2 = 0$

Therefore, $\beta^2 - 2a\alpha + 8a^2 = 0 \Rightarrow \beta^2 = 2a(\alpha - 4a)$

Hence the locus of (α, β) is $y^2 = 2a(x - 4a)$

Do yourself - 6 :

(i) Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

(ii) Find the co-ordinates of the middle point of the chord of the parabola $y^2 = 16x$, the equation of which is $2x - 3y + 8 = 0$

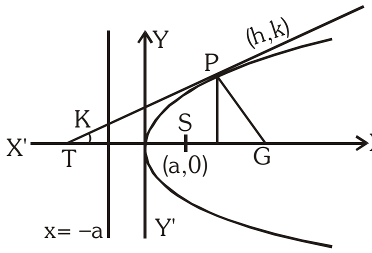
(iii) Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangent at the extremities of the chords are perpendicular.

17. DIAMETER :

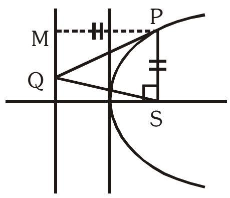
The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is **$y = 2a/m$**, where **m = slope of parallel chords**.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.



(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

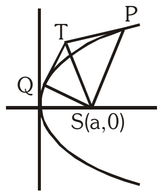
(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; **$2a = \frac{2bc}{b+c}$** **i.e.** $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

(f) If the tangents at P and Q meet in T, then :

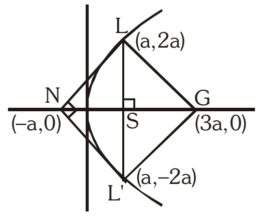
(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP\ .\ SQ$ &

(iii) The triangles SPT and STQ are similar.



(g) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.



Note :

(i) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(ii) Figure LNL'G is square of side $2\sqrt{2}a$

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

Do yourself - 7 :

(i) The parabola $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the line $x = 4$, $y = 4$ and the co-ordinates axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then find $S_1 : S_2 : S_3$.

(ii) Let P be the point (1, 0) and Q a point on the parabola $y^2 = 8x$, then find the locus of the mid point of PQ.

Miscellaneous Illustrations :

Illustration 21 : The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is -

(A) $y = x + a$ (B) $x + y + a = 0$ (C) $x + y + 2a = 0$ (D) $y = x + 2a$

Solution : Any tangent to parabola is $y = mx + \frac{2a}{m}$

Solving with the circle $x^2 + (mx + \frac{2a}{m})^2 = 2a^2 \Rightarrow x^2(1 + m^2) + 4ax + \frac{4a^2}{m^2} - 2a^2 = 0$

$B^2 - 4AC = 0$ gives $m = \pm 1$

Tangent $y = \pm x \pm 2a$ **Ans. (C,D)**

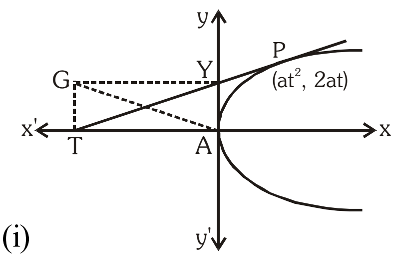
Illustration 22 : If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Solution : Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then tangent at $P(at^2, 2at)$ is $ty = x + at^2$

Since tangent meet the axis of parabola in T and tangent at the vertex in Y.

$\therefore$ Co-ordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.



Let co-ordinates of G be (x_1, y_1).

Since TAYG is rectangle.

$\therefore$ Mid-points of diagonals TY and GA is same

$\Rightarrow \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2} \Rightarrow x_1 = -at^2$ .... (i)

and $\frac{y_1 + 0}{2} = \frac{0 + at}{2} \Rightarrow y_1 = at$ .... (ii)

Eliminating t from (i) and (ii) then we get $x_1 = -a\left(\frac{y_1}{a}\right)^2$

or $\quad y_1^2 = -ax_1 \quad$ or $\quad y_1^2 + ax_1 = 0$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$

Illustration 23 : If P(–3, 2) is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is -

(A) –1/2 (B) 2 (C) 1/2 (D) –2

Solution : The equation of the tangent at (–3, 2) to the parabola $y^2 + 4x + 4y = 0$ is

$2y + 2(x - 3) + 2(y + 2) = 0$

or $2x + 4y - 2 = 0 \Rightarrow x + 2y - 1 = 0$

Since the tangent at one end of the focal chord is parallel to the normal at the other end, the slope of the normal at the other end of the focal chord is $-\frac{1}{2}$. **Ans.(A)**

Illustration 24 : Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.

Solution : Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are $y = mx - 2am - am^3$ and $y = m(x - b) - 2cm - cm^3$ respectively then normals must be identical, compare the co-efficients

$$1 = \frac{2am + am^3}{mb + 2cm + cm^3}$$

$\Rightarrow \quad m[(c - a)m^2 + (b + 2c - 2a)] = 0, m \neq 0 \qquad (\because$ other than axis)

and $\quad m^2 = \frac{2a - 2c - b}{c - a}, \; m = \pm\sqrt{\frac{2(a-c)-b}{c-a}}$

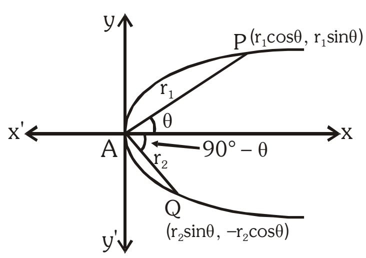
or $\quad m = \pm\sqrt{\left(-2 - \frac{b}{c-a}\right)}$

$\therefore \quad -2 - \frac{b}{c-a} > 0$

or $\quad -2 + \frac{b}{a-c} > 0 \Rightarrow \frac{b}{a-c} > 2$

Illustration 25 : If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1r_2)^{4/3} = 16a^2\left(r_1^{2/3} + r_2^{2/3}\right)$.

Solution : Since chord are perpendicular, therefore if one makes an angle θ then the other will make an angle (90° – θ) with x-axis



Let $\quad AP = r_1$ and $AQ = r_2$

If $\quad \angle PAX = \theta$

then $\quad \angle QAX = 90° - \theta$

$\therefore \quad$ Co-ordinates of P and Q are $(r_1\cos\theta, r_1\sin\theta)$ and $(r_2\sin\theta, -r_2\cos\theta)$ respectively.

Since P and Q lies on $y^2 = 4ax$

$\therefore \quad r_1^2 \sin^2\theta = 4ar_1\cos\theta$ and $r_2^2\cos^2\theta = 4ar_2\sin\theta$

$\Rightarrow \quad r_1 = \dfrac{4a\cos\theta}{\sin^2\theta}$ and $r_2 = \dfrac{4a\sin\theta}{\cos^2\theta}$

$$\therefore \quad (r_1r_2)^{4/3} = \left(\frac{4a\cos\theta}{\sin^2\theta}\cdot\frac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3} \qquad \text{....... (i)}$$

and $$16a^2.\left(r_1^{2/3}+r_2^{2/3}\right) = 16a^2\left\{\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^{2/3}+\left(\frac{4a\sin\theta}{\cos^2\theta}\right)^{2/3}\right\}$$

$$=16a^2.(4a)^{2/3}\left\{\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{4/3}}+\frac{(\sin\theta)^{2/3}}{(\cos\theta)^{4/3}}\right\} = 16a^2.(4a)^{2/3}\left\{\frac{\cos^2\theta+\sin^2\theta}{(\sin\theta)^{4/3}(\cos\theta)^{4/3}}\right\}$$

$$=\frac{16a^2.(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}} = \left(\frac{16a^2}{\cos\theta\cos\theta}\right)^{4/3} = (r_1r_2)^{4/3} \qquad \text{\{from (i)\}}$$

Illustration 26 : The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Solution : Let the three points on the parabola be

$(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$

The area of the triangle formed by these points

$$\Delta_1 = \frac{1}{2}[\,at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_2^2(2a_1 - 2at_2)]$$

$= -a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2).$

The points of intersection of the tangents at these points are

$(at_2t_3, a(t_2 + t_3)), (at_3t_1, a(t_3 + t_1))$ and $(at_1t_2, a(t_1 + t_2))$

The area of the triangle formed by these three points

$$\Delta_2 = \frac{1}{2}\{at_2t_3(at_3 - at_2) + at_3t_1(at_1 - at_3) + at_1t_2(at_2 - at_1)\}$$

$$=\frac{1}{2}a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)$$

Hence $\Delta_1 = 2\Delta_2$

Illustration 27: Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

Solution : Let the equations of the three tangents be

$t_1y = x + at_1^2$ ....(i)

$t_2y = x + at_2^2$ ....(ii)

and $t_3y = x + at_3^2$ ....(iii)

The point of intersection of (ii) and (iii) is found, by solving them, to be $(at_2t_3, a(t_2 + t_3))$

The equation of the straight line through this point & perpendicular to (i) is

$$y - a(t_2 + t_3) = -t_1(x - at_2t_3)$$

i.e. $y + t_1x = a(t_2 + t_3 + t_1t_2t_3)$ ....(iv)

Similarly, the equation of the straight line through the point of intersection of (iii) and (i) & perpendicular to (ii) is

$y + t_2x = a(t_3 + t_1 + t_1t_2t_3)$ ....(v)

and the equation of the straight line through the point of intersection of (i) and (ii) & perpendicular to (iii) is

$y + t_1x = a(t_1 + t_2 + t_1t_2t_3)$ ....(vi)

The point which is common to the straight lines (iv), (v) and (vi)

i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$$x = -a,\ y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$$

and this point lies on the directrix.

ANSWERS FOR DO YOURSELF

1 : **(i)** Parabola **(ii)** Vertex : $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Axis : $y = \frac{5}{2}$, Focus : $\left(-\frac{17}{4}, \frac{5}{2}\right)$, Directrix : $x = -\frac{11}{4}$; LR = 3

(iii) $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$; Axis : $2x - y = 3$; LR $= 4\sqrt{5}$ unit

(iv) $(3x + 4y - 4)^2 = 20(4x - 3y + 7)$

2 : **(i)** $\left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$ **(ii)** (–7, 8), (9, 8) **(iv)** $kb^2 = ac$ **(v)** $8\sqrt{3}$

3 : **(i)** $x - y + 3 = 0$, (3, 6); $3x - 2y + 4 = 0$, $\left(\frac{4}{3}, 4\right)$

(ii) $2x - y + 2 = 0$, (1, 4) ; $x + 2y + 16 = 0$, (16, –16)

4 : **(i)** C **(ii)** 1 **(iii)** 48 **(iv)** (0, 0), (–4, 3) and (16, 8)

5 : **(i)** $\pi/2$

6 : **(i)** $2x = y + 1$ **(ii)** (14, 12) **(iii)** $y^2 = 2a(x - a)$

7 : **(i)** 1 : 1 : 1 **(ii)** $y^2 - 4x + 2 = 0$

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. The equation of the directrix of the parabola, $y^2 + 4y + 4x + 2 = 0$ is -

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$ **PR0001**

2. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is-

(A) 4 (B) 2 (C) 1/5 (D) 2/5 **PR0002**

3. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$ then one of the values of 'k' is -

(A) 1/8 (B) 8 (C) 4 (D) 1/4 **PR0003**

4. The length of the intercept on y– axis cut off by the parabola, $y^2 - 5y = 3x - 6$ is

(A) 1 (B) 2 (C) 3 (D) 5 **PR0004**

5. Maximum number of common chords of a parabola and a circle can be equal to

(A) 2 (B) 4 (C) 6 (D) 8 **PR0005**

6. A variable circle is drawn to touch the line $3x - 4y = 10$ and also the circle $x^2 + y^2 = 1$ externally then the locus of its centre is -

(A) straight line (B) circle

(C) pair of real, distinct straight lines (D) parabola **PR0006**

7. The locus of the point of trisection of all the double ordinates of the parabola $y^2 = \ell x$ is a parabola whose latus rectum is -

(A) $\frac{\ell}{9}$ (B) $\frac{2\ell}{9}$ (C) $\frac{4\ell}{9}$ (D) $\frac{\ell}{36}$ **PR0007**

8. The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)$ (D) $m \in R - \{0\}$ **PR0008**

9. The vertex A of the parabola $y^2 = 4ax$ is joined to any point P on it and PQ is drawn at right angles to AP to meet the axis in Q. Projection of PQ on the axis is equal to

(A) twice the length of latus rectum (B) the latus length of rectum

(C) half the length of latus rectum (D) one fourth of the length of latus rectum **PR0009**

10. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

(A) $x^2 + y^2 - 4y = 0$ (B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$ (D) $x^2 + y^2 - 2x - 4y + 1 = 0$ **PR0011**

11. If a focal chord of $y^2 = 4x$ makes an angle $\alpha, \alpha \in \left(0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then minimum length of this focal chord is -

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$ (C) 8 (D) 16 **PR0014**

12. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha,0)$ $(\beta,0)$ both to the right of the origin. A circle also passess through these two points. The length of a tangent from the origin to the circle is :

(A) $\sqrt{\frac{bc}{a}}$ (B) ac^2 (C) $\frac{b}{a}$ (D) $\sqrt{\frac{c}{a}}$ **PR0015**

13. If (2,–8) is one end of a focal chord of the parabola $y^2 = 32x$, then the other end of the focal chord, is-

(A) (32,32) (B) (32,–32) (C) (–2,8) (D) (2,8) **PR0016**

14. The length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then
(A) $2a^2 = bc$ (B) $a^3 = b^2c$ (C) $ac = b^2$ (D) $b^2c = 4a^3$ **PR0018**

15. From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :
(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$
PR0021

16. y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$ is
(A) – 18 (B) – 12 (C) – 9 (D) – 6 **PR0022**

17. The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are
(A) (9, 6) and (1, 2) (B) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $(\frac{1}{4}, 1)$
PR0023

18. If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then
(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$ **PR0024**

19. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is - **PR0026**
(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x + 3)$ (C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x + 1)$

20. If $x + y = k$ is normal to $y^2 = 12x$, then 'k' is-
(A) 3 (B) 9 (C) –9 (D) –3 **PR0028**

21. Equation of the other normal to the parabola $y^2 = 4x$ which passes through the intersection of those at (4,–4) and (9a, –6a) is - **PR0029**
(A) $5x - y + 115 = 0$ (B) $5x + y - 135 = 0$ (C) $5x - y - 115 = 0$ (D) $5x + y + 115 = 0$

22. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is:
(A) 8 (B) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$ **PR0030**

23. The normal chord of a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus is :
(A) $\frac{\pi}{4}$ (B) $\tan^{-1}\sqrt{2}$ (C) $\tan^{-1}2$ (D) $\frac{\pi}{2}$ **PR0032**

24. Which one of the following lines cannot be the normals to $x^2 = 4y$?
(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$
(C) $x - 2y + 12 = 0$ (D) $x + 2y + 12 = 0$ **PR0033**

25. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :
(A) (3,2) (B) (2,4) (C) (3,4) (D) (4,1) **PR0034**

26. Tangents are drawn from the point (– 1, 2) on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line x = 2 is :
(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these **PR0038**

27. If the locus of the middle points of the chords of the parabola $y^2 = 2x$ which touches the circle $x^2 + y^2 - 2x - 4 = 0$ is given by $(y^2 + 1 - x)^2 = \lambda(1 + y^2)$, then the value of λ is equal to-
(A) 3 (B) 4 (C) 5 (D) 6 **PR0039**

[MULTIPLE OBJECTIVE TYPE]

28. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 8x$ is a parabola whose-
(A) Latus rectum is half the latus rectum of the original parabola
(B) Vertex is (1,0)
(C) Directrix is y-axis
(D) Focus has the co-ordinates (2,0) **PR0041**

29. Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is

(A) $\left(\frac{p}{2}, p\right)$ (B) $\left(\frac{p}{2}, -p\right)$ (C) $\left(-\frac{p}{2}, p\right)$ (D) $\left(-\frac{p}{2}, -p\right)$ **PR0042**

30. Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then :
(A) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D) $a < 0, b < 0$ **PR0043**

31. The focus of the parabola is (1,1) and the tangent at the vertex has the equation $x + y = 1$. Then :
(A) equation of the parabola is $(x - y)^2 = 2(x + y - 1)$
(B) equation of the parabola is $(x - y)^2 = 4(x + y - 1)$
(C) the co-ordinates of the vertex are $\left(\frac{1}{2}, \frac{1}{2}\right)$
(D) length of the latus rectum is $2\sqrt{2}$ **PR0044**

32. The straight line $y + x = 1$ touches the parabola
(A) $x^2 + 4y = 0$ (B) $x^2 - x + y = 0$
(C) $4x^2 - 3x + y = 0$ (D) $x^2 - 2x + 2y = 0$ **PR0045**

[COMPREHENSION TYPE]

Paragraph for question nos. 33 & 34

Consider the parabola $y^2 = 8x$

33. Area of the figure formed by the tangents and normals drawn at the extremities of its latus rectum is
(A) 8 (B) 16 (C) 32 (D) 64 **PR0048**

34. Distance between the tangent to the parabola and a parallel normal inclined at 30° with the x-axis, is

(A) $\frac{16}{3}$ (B) $\frac{16\sqrt{3}}{9}$ (C) $\frac{2}{3}$ (D) $\frac{16}{\sqrt{3}}$ **PR0048**

[MATRIX MATCH TYPE]

35. Identify the conic whose equations are given in column-I.

Column-I (Equation of a conic)		Column-II (Nature of conic)	
(A) $xy + a^2 = a(x + y)$	(P) Ellipse	PR0049
(B) $2x^2 - 72xy + 23y^2 - 4x - 28y - 48 = 0$	(Q) Hyperbola	PR0050
(C) $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$	(R) Parabola	PR0051
(D) $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$	(S) line pair	PR0052
(E) $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$		PR0053

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. PN is an ordinate of the parabola $y^2 = 4ax$(P on $y^2 = 4ax$ and N on x-axis). A straight line is drawn parallel to the axis to bisect NP and meets the curve in Q. NQ meets the tangent at the vertex in a point T such that AT = kNP, then the value of k is (where A is the vertex)

(A) 3/2 (B) 2/3 (C) 1 (D) none **PR0055**

2. Locus of the feet of the perpendiculars drawn from vertex of the parabola $y^2 = 4ax$ upon all such chords of the parabola which subtend a right angle at the vertex is

(A) $x^2 + y^2 - 4ax = 0$ (B) $x^2 + y^2 - 2ax = 0$

(C) $x^2 + y^2 + 2ax = 0$ (D) $x^2 + y^2 + 4ax = 0$ **PR0057**

3. Through the focus of the parabola $y^2 = 2px (p > 0)$ a line is drawn which intersects the curve at $A(x_1,y_1)$ and $B(x_2,y_2)$. The ratio $\frac{y_1y_2}{x_1x_2}$ equals-

(A) 2 (B) –1

(C) –4 (D) some function of p **PR0059**

4. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is

(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$

(C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$ **PR0061**

5. If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet pass through a fixed point whose co-ordinates are :

(A) (–2a,0) (B) (a,0) (C) (2a,0) (D) none **PR0062**

6. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β respectively with the x-axis then $\tan\alpha (\tan\alpha + \tan\beta)$ has the value equal to

(A) 0 (B) – 2 (C) $-\frac{1}{2}$ (D) – 1 **PR0063**

7. Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line y = x are

(A) (– 4, 10) (B) (– 3, 8) (C) (4, – 10) (D) (– 3, 10) **PR0066**

8. Normals are drawn at points A, B, and C on the parabola $y^2 = 4x$ which intersect at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2 , is

(A) x + y = 1 (B) x – y = 3 (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$ **PR0068**

9. TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point (– a, b) then the locus of T is :

(A) ay = 2b (x – b) (B) bx = 2a (y – a) (C) by = 2a (x – a) (D) ax = 2b (y – b) **PR0069**

[MULTIPLE OBJECTIVE TYPE]

10. P is a point on the parabola $y^2 = 4ax (a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x-axis at a point whose co-ordinates are :

(A) $(-3a, 0)$ (B) $(-a, 0)$ (C) $(-2a, 0)$ (D) $(a, 0)$ **PR0072**

11. If from the vertex of a parabola $y^2 = 4x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further end of the rectangle is -

(A) an equal parabola

(B) a parabola with focus at (9, 0) **PR0073**

(C) a parabola with directrix as $x - 7 = 0$

(D) a parabola having tangent at its vertex $x = 8$

12. A circle 'S' is described on the focal chord of the parabola $y^2 = 4x$ as diameter. If the focal chord is inclined at an angle of 45° with axis of x, then which of the following is/are true ?

(A) Radius of the circle is 4.

(B) Centre of the circle is (3,2)

(C) The line $x + 1 = 0$ touches the circle

(D) The circle $x^2 + y^2 + 2x - 6y + 3 = 0$ is orthogonal to 'S'. **PR0074**

13. PQ is a double ordinate of the parabola $y^2 = 4ax$. If the normal at P intersect the line passing through Q and parallel to axis of x at G, then locus of G is a parabola with -

(A) length of latus rectum equal to 4a

(B) vertex at (4a,0)

(C) directrix as the line $x - 3a = 0$

(D) focus at (5a,0) **PR0075**

[COMPREHENSION TYPE]

Paragraph for question nos. 14 to 16

Tangents are drawn to the parabola $y^2 = 4x$ from the point $P(6, 5)$ to touch the parabola at Q and R. C_1 is a circle which touches the parabola at Q and C_2 is a circle which touches the parabola at R. Both the circles C_1 and C_2 pass through the focus of the parabola.

14. Area of the ΔPQR equals

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{4}$ **PR0080**

15. Radius of the circle C_2 is

(A) $5\sqrt{5}$ (B) $5\sqrt{10}$ (C) $10\sqrt{2}$ (D) $\sqrt{210}$ **PR0080**

16. The common chord of the circles C_1 and C_2 passes through the

(A) incentre of the ΔPQR

(B) circumcenter of the ΔPQR

(C) centroid of the ΔPQR

(D) orthocenter of the ΔPQR **PR0080**

EXERCISE (S-1)

1. 'O' is the vertex of the parabola $y^2 = 4ax$ & L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H, prove that the length of the double ordinate through H is $4a\sqrt{5}$. **PR0085**

2. A point P on a parabola $y^2 = 4x$, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle, find the area of the equilateral triangle. **PR0086**

3. Through the vertex O of a parabola $y^2 = 4x$, chords OP & OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. **PR0087**

4. Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact. **PR0088**

5. Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2,5). Also find the point of contact. **PR0089**

6. In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T & the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5. **PR0091**

7. Show that the normals at the points (4a, 4a) & at the upper end of the latus rectum of the parabola $y^2 = 4ax$ intersect on the same parabola. **PR0092**

8. Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that if one of the normals is given by $y = x - 3$ & find the equations of the others. **PR0093**

9. Prove that the locus of the middle point of portion of a normal to $y^2 = 4ax$ intercepted between the curve & the axis is another parabola. Find the vertex & the latus rectum of the second parabola. **PR0095**

10. If the normal at P(18, 12) to the parabola $y^2= 8x$ cuts it again at Q, show that $9PQ = 80\sqrt{10}$ **PR0096**

11. Prove that, the normal to $y^2 = 12x$ at (3, 6) meets the parabola again in (27, –18) & circle on this normal chord as diameter is $x^2 + y^2 - 30x + 12y - 27 = 0$. **PR0097**

12. Show that the normals at two suitable distinct real points on the parabola $y^2 = 4ax$ $(a > 0)$ intersect at a point on the parabola whose abscissa > 8a. **PR00101**

13. The normal to the parabola $y^2 = 4x$ at the point P,Q & R are concurrent at the point(15,12). Find

(a) the equation of the circle circumscribing the triangle PQR **PR00102**

(b) the co-ordinates of the centroid of the triangle PQR. **PR00102**

14. From the point (–1, 2) tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact & the tangents. **PR0103**

15. Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point (6, 9). **PR0104**

16. Find the equations of the chords of the parabola $y^2 = 4ax$ which pass through the point (–6a, 0) and which subtends an angle of 45° at the vertex. **PR0105**

EXERCISE (S-2)

1. PC is the normal at P to the parabola $y^2 = 4ax$, C being on the axis. CP is produced outwards to Q so that PQ = CP; show that the locus of Q is a parabola. **PR0106**

2. A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola. **PR0107**

3. Let P(a,b) and Q(c,d) are the two points on the parabola $y^2 = 8x$ such that the normals at them meet in (18,12). Find the product (abcd). **PR0110**

4. A variable circle passes through the point A(2,1) and touches the x-axis. Locus of the other end of the diameter through A is a parabola.

(a) Find the length of the latus rectum of the parabola. **PR0111**

(b) Find the co-ordinates of the foot of the directrix of the parabola. **PR0111**

(c) The two tangents and two normals at the extremities of the latus rectum of the parabola constitutes a quadrilateral. Find area of quadrilateral. **PR0111**

5. Two tangents to the parabola $y^2= 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2=8(x+2)$. **PR0113**

6. A variable chords of the parabola $y^2 = 8x$ touches the parabola $y^2 = 2x$. The locus of the point of intersection of the tangent at the end of the chord is a parabola. Find its latus rectum. **PR0115**

7. PQ, a variable chord of the parabola $y^2 = 4x$ subtends a right angle at the vertex. The tangents at P and Q meet at T and the normals at those points meet at N. If the locus of the mid point of TN is a parabola, then find its latus rectum. **PR0118**

EXERCISE (JM)

1. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is :-

(1) $x = 1$ (2) $2x + 1 = 0$ **[AIEEE-2010]**

(3) $x = -1$ (4) $2x - 1 = 0$ **PR0119**

2. Given : A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement–I : An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement–II : If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$. **[JEE (Main)-2013]**

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true. **PR0120**

3. Statement 1 : The slope of the tangent at any point P on a parabola, whose axis is the axis of *x* and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement 2 : The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

(1) Statement 1 is True Statement 2 is True, Statement 2 is a correct explanation for Statement 1.

(2) Statement 1 is True, Statement 2 is False.

(3) Statement 1 is True, Statement 2 is True statement 2 is not a correct explanation for statement 1.

(4) Statement 1 is False, Statement 2 is True **[JEE-Main (On line)-2013]**

PR0121

4. **Statement 1 :** The line $x - 2y = 2$ meets the parabola, $y^2 + 2x = 0$ only at the point $(-2, -2)$

Statement 2 : The line $y = mx - \frac{1}{2m}$ $(m \neq 0)$ is tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$.

(1) Statement 1 is false; Statement 2 is true.

(2) Statement l is true; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

(3) Statement 1 is true; Statement 2 is false.

(4) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1. **PR0122**

[JEE-Main (On line)-2013]

5. The point of intersection of the normals tothe parabola $y^2 = 4x$ at the ends of its latus rectum is :

[JEE-Main (On line)-2013]

(1) (0, 3) (2) (2, 0) (3) (3, 0) (4) (0, 2) **PR0123**

6. The slope of the line touching both, the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[JEE(Main)-2014]**

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{1}{8}$ (4) $\frac{2}{3}$ **PR0124**

7. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is :- **[JEE(Main)-2015]**

(1) $y^2 = 2x$ (2) $x^2 = 2y$ (3) $x^2 = y$ (4) $y^2 = x$ **PR0125**

8. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is :

[JEE(Main)-2016]

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$ (2) $x^2 + y^2 - 4x + 8y + 12 = 0$

(3) $x^2 + y^2 - x + 4y - 12 = 0$ (4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$ **PR0126**

9. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is :- **[JEE-Main 2017]**

(1) $4(\sqrt{2}+1)$ (2) $2(\sqrt{2}+1)$ (3) $2(\sqrt{2}-1)$ (4) $4(\sqrt{2}-1)$ **PR0127**

10. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is - **[JEE-Main 2018]**

(1) 2 (2) 3 (3) $\frac{4}{3}$ (4) $\frac{1}{2}$ **PR0128**

11. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is : **[JEE-Main 2018]**

(1) 185 (2) 85 (3) 95 (4) 195 **PR0129**

12. Let A(4,–4) and B(9,6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is: **[JEE (Main)-Jan 19]**

(1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$ **PR0130**

13. If the parabolas $y^2=4b(x-c)$ and $y^2=8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a,b,c) **[JEE (Main)-Jan 19]**

(1) (1, 1, 0) (2) $\left(\frac{1}{2},2,3\right)$ (3) $\left(\frac{1}{2},2,0\right)$ (4) (1, 1, 3) **PR0131**

14. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to : **[JEE (Main)-Apr 19]**

(1) $2\sqrt{2}+1$ (2) $\sqrt{2}-1$ (3) $\sqrt{2}+1$ (4) $2\sqrt{2}-1$ **PR0132**

15. The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is :- **[JEE (Main)-Apr 19]**

(1) $4\pi(2-\sqrt{2})$ (2) $8\pi(3-2\sqrt{2})$ (3) $4\pi(3+\sqrt{2})$ (4) $8\pi(2-\sqrt{2})$ **PR0133**

16. The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point : **[JEE (Main)-Apr 19]**

(1) $\left(-\frac{5}{2},-1\right)$ (2) $\left(-\frac{5}{2},1\right)$ (3) $\left(\frac{5}{2},-1\right)$ (4) $\left(\frac{5}{2},1\right)$ **PR0134**

17. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is : **[JEE (Main)-Apr 19]**

(1) $x + y + 4 = 0$ (2) $x - 2y + 16 = 0$ (3) $2x - y + 2 = 0$ (4) $x - y + 4 = 0$

PR0135

EXERCISE (JA)

1. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$ **[JEE 2009, 4]**

(C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0) **PR0140**

2. Let A and B be two distinct point on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be - **[JEE 2010,3]**

(A) $\frac{-1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $\frac{-2}{r}$ **PR0141**

3. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2},2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[JEE 2011,4]**

PR0142

4. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x,y) in the ratio 1 : 3. Then the locus of P is- **[JEE 2011,3]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$ **PR0143**

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by - **[JEE 2011,4]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

PR0144

6. Let S be the focus of the parabola $y^2 = 8x$ & let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[JEE 2012, 4M]**

PR0145

Paragraph for Question 7 and 8

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

7. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

[JEE(Advanced) 2013, 3, (–1)]

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$ **PR0146**

8. Length of chord PQ is **[JEE(Advanced) 2013, 3, (–1)]**

(A) 7a (B) 5a (C) 2a (D) 3a **PR0146**

9. A line L : $y = mx + 3$ meets y – axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0,y_0)$. The tangent to the parabola at $F(x_0,y_0)$ intersects the y-axis at $G(0,y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		List-II
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

PR0147

[JEE(Advanced) 2013, 3, (–1)]

10. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the point P, Q and the parabola at the points R,S. Then the area of the quadrilateral PQRS is -

(A) 3 (B) 6 (C) 9 (D) 15 **PR0148**

[JEE(Advanced)-2014, 3(–1)]

Paragraph For Questions 11 and 12

Let a,r,s,t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a,0).

11. The value of r is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$ **PR0149**

12. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$ **PR0149**

13. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is **[JEE 2015, 4M, –0M]**

PR0150

14. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is **[JEE 2015, 4M, –0M]**

PR0151

15. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE 2015, 4M, –2M]**

(A) $(4, 2\sqrt{2})$ (B) $(9, 3\sqrt{2})$ (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1, \sqrt{2})$ **PR0152**

16. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then - **[JEE(Advanced)-2016, 4(–2)]**

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$ **PR0153**

17. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then-

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = (\sqrt{5}+1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6 **PR0154**

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$ **[JEE(Advanced)-2016, 4(–2)]**

18. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE(Advanced)-2017, 4(–2)]**

(A) p = 5, h = 4, k = –3 (B) p = –1, h = 1, k = –3

(C) p = –2, h = 2, k = –4 (D) p = 2, h = 3, k = –4 **PR0155**

ELLIPSE

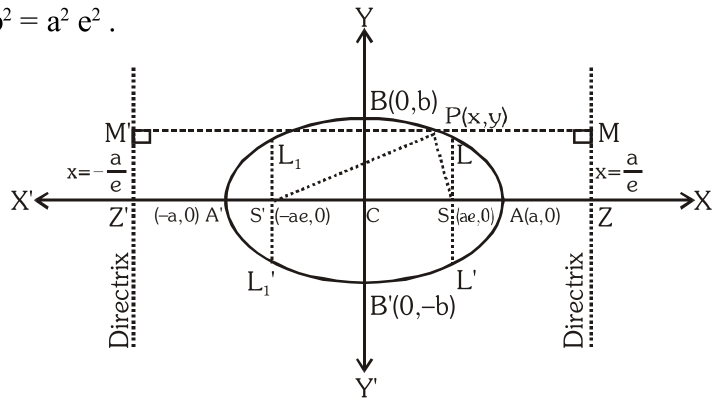
1. STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

where **a > b & b^2 = a^2 (1 – e^2)** $\Rightarrow a^2 - b^2 = a^2 e^2$.

where e = eccentricity **(0 < e < 1)**.

FOCI : **S ≡ (ae, 0) & S' ≡ (– ae, 0)**.



(a) Equation of directrices :

$$x = \frac{a}{e} \quad \& \quad x = -\frac{a}{e}.$$

(b) Vertices :

A' ≡ (–a, 0) & A ≡ (a, 0).

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z)** $\left(\pm\frac{a}{e}, 0\right)$.

(d) Minor Axis : The y-axis intersects the ellipse in the points B’ ≡ (0,- b) & B ≡ (0, b). The line segment B’B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axes : The major & minor axis together are called **Principal Axes** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. C ≡ (0,0) the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate.**

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum (LL') $= \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : **x = ± ae**.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$ **and** $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) Focal radii : SP = a –ex & S'P = a + ex ⇒ SP + S 'P = 2a = Major axis.

(l) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

Note :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e BS = CA.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & nothing is mentioned, then the rule is to assume that a > b.

Illustration 1 : If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$

Solution : As given $\frac{2b^2}{a} = b$ ⇒ $2b = a$ ⇒ $4b^2 = a^2$

⇒ $4a^2(1 - e^2) = a^2$ ⇒ $1 - e^2 = 1/4$

∴ $e = \sqrt{3}/2$ **Ans. (C)**

Illustration 2 : Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Solution : Here S is (2, 3) & S' is (–2, 3) and $b = \sqrt{5}$ ⇒ SS' = 4 = 2ae ⇒ ae = 2

but $b^2 = a^2(1 - e^2)$ ⇒ $5 = a^2 - 4$ ⇒ $a = 3$.

Hence the equation to major axis is y = 3

Centre of ellipse is midpoint of SS' i.e. (0, 3)

∴ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$ or $\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$ **Ans.**

Illustration 3 : Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Solution : With centre at (1, 2), the equation of the ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$. It passes through the point (4, 6)

⇒ $\frac{9}{a^2} + \frac{16}{b^2} = 1$ .... (i)

Distance between the focus and the centre = (6 – 1) = 5 = ae

⇒ $b^2 = a^2 - a^2e^2 = a^2 - 25$ .... (ii)

Solving for a^2 and b^2 from the equations (i) and (ii), we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$ **Ans.**

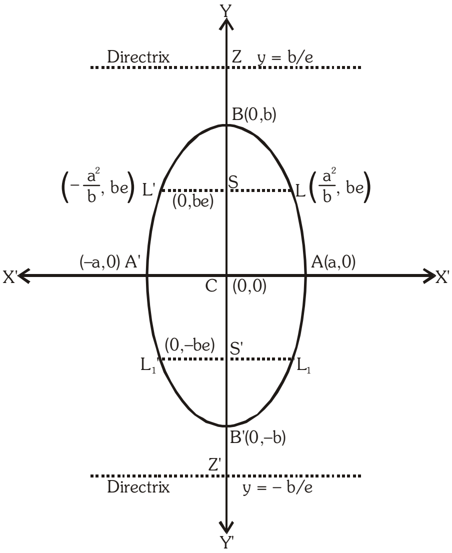
Do yourself - 1 :

(i) If LR of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a < b) is half of its major axis, then find its eccentricity.

(ii) Find the equation of the ellipse whose foci are (4, 6) & (16, 6) and whose semi-minor axis is 4.

(iii) Find the eccentricity, foci and the length of the latus-rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

2. ANOTHER FORM OF ELLIPSE : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a <b)



(a) **AA' = Minor axis = 2a**

(b) **BB' = Major axis = 2b**

(c) **$a^2 = b^2 (1 - e^2)$**

(d) **Latus rectum LL' = L_1L_1' = $\frac{2a^2}{b}$, equation y = ± be**

(e) **Ends of the latus rectum are :**

$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) **Equation of directrix y = ± b/e**

(g) **Eccentricity :** $e=\sqrt{1-\frac{a^2}{b^2}}$

Illustration 4 : The equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10, will be-

(A) $2x^2 + y^2 = 100$ (B) $x^2 + 2y^2 = 100$ (C) $2x^2 + 3y^2 = 80$ (D) none of these

Solution : When a > b

As given $2b = 2ae \Rightarrow b = ae$ ...... (i)

Also $\frac{2b^2}{a}=10 \Rightarrow b^2 = 5a$ ...... (ii)

Now since $b^2 = a^2 - a^2e^2 \Rightarrow b^2 = a^2 - b^2$ [From (i)]

$\Rightarrow 2b^2 = a^2$ ...... (iii)

(ii), (iii) $\Rightarrow a^2 = 100, b^2 = 50$

Hence equation of the ellipse will be $\frac{x^2}{100}+\frac{y^2}{50}=1 \Rightarrow x^2 + 2y^2 = 100$

Similarly when a < b then required ellipse is $2x^2 + y^2 = 100$ **Ans. (A, B)**

Do yourself - 2 :

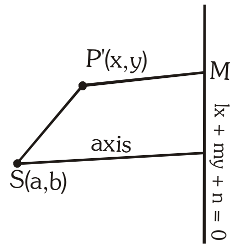
(i) The foci of an ellipse are (0, ±2) and its eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation

(ii) Find the centre, the length of the axes, eccentricity and the foci of ellipse
$12x^2 + 4y^2 + 24x - 16y + 25 = 0$

(iii) The equation $\frac{x^2}{8-t}+\frac{y^2}{t-4}=1$, will represent an ellipse if

(A) $t \in (1, 5)$ (B) $t \in (2, 8)$ (C) $t \in (4, 8) - \{6\}$ (D) $t \in (4, 10) - \{6\}$

3. GENERAL EQUATION OF AN ELLIPSE



Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let P(x, y) be any point on the ellipse. Then by definition.

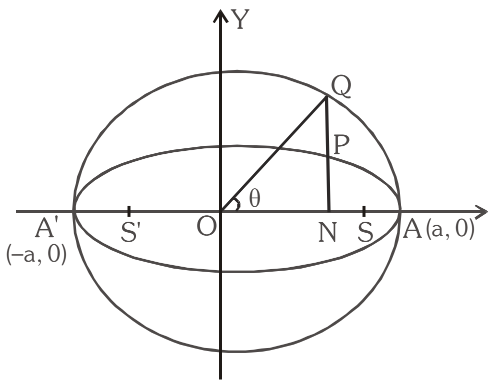
$\Rightarrow$ **SP = e PM** (e is the eccentricity) $\Rightarrow (x-a)^2 + (y-b)^2 = e^2\dfrac{(lx+my+n)^2}{(l^2+m^2)}$

$\Rightarrow$ $(l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = e^2\{lx + my + n\}^2$

4. POSITION OF A POINT W.R.T. AN ELLIPSE :

The point $\mathbf{P(x_1, y_1)}$ lies **outside**, **inside** or **on** the ellipse according as ; $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$ **> < or = 0**.

5. AUXILLIARY CIRCLE/ECCENTRIC ANGLE :



A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.

Note that $\dfrac{l(PN)}{l(QN)} = \dfrac{b}{a} = \dfrac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6. PARAMETRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ; $\mathbf{Q(\theta) \equiv (a\cos\theta, a\sin\theta)}$ is on the auxiliary circle.

7. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as $\mathbf{c^2}$ **is < = or >** $\mathbf{a^2m^2 + b^2}$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\mathbf{c^2 = a^2m^2 + b^2}$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by

$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

Illustration 5 : For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Solution : $\because$ Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$ $\therefore$ $m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow$ $\lambda^2 = 16 \times 1^2 + 9$ $\Rightarrow$ $\lambda^2 = 25$ $\therefore \lambda = \pm 5$ **Ans.**

Illustration 6 : If α, β are eccentric angles of end points of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\tan \alpha/2.\ \tan \beta/2$ is equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{e+1}{e-1}$ (D) $\frac{e-1}{e+1}$

Solution : Equation of line joining points 'α' and 'β' is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

If it is a focal chord, then it passes through focus (ae, 0), so $e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

$$\Rightarrow \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}=\frac{e}{1} \quad \Rightarrow \quad \frac{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}}=\frac{e-1}{e+1}$$

$$\Rightarrow \frac{2\sin\alpha/2\ \sin\beta/2}{2\cos\alpha/2\ \cos\beta/2}=\frac{e-1}{e+1} \quad \Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

using (–ae, 0) , we get $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e+1}{e-1}$ **Ans. (A,C)**

Do yourself - 3 :

(i) Find the position of the point (4, 3) relative to the ellipse $2x^2 + 9y^2 = 113$.

(ii) A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$ having slope –1 intersects the axis of x & y in point A & B respectively. If O is the origin then find the area of triangle OAB.

(iii) Find the condition for the line $x\cos\theta + y\sin\theta = P$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

8. TANGENT TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Note : For general ellipse replace x^2 by (xx_1), y^2 by (yy_1), $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$, $2xy$ by $(xy_1 + yx_1)$ & c by (c).

(b) **Slope form :** Equation of tangent to the given ellipse whose slope is 'm', is $\mathbf{y = mx \pm \sqrt{a^2m^2+b^2}}$

Point of contact are $\left(\frac{\mp a^2 m}{\sqrt{a^2m^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(c) **Parametric form :** Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Note :

(i) The eccentric angles of point of contact of two parallel tangents differ by π.

(ii) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Illustration 7 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution : Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$\therefore \quad mx - 2 = -1 \Rightarrow \quad m = \frac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \quad a^2 = 4$ and $b^2 = 3$

So the equation of the tangent are $y=\frac{1}{2}x\pm\sqrt{4\times\frac{1}{4}+3}$

$\Rightarrow \quad y=\frac{1}{2}x\pm 2$ or $\quad x - 2y \pm 4 = 0.$ **Ans.**

Illustration 8 : The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Solution : Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P(acosθ, bsinθ) be a point on the ellipse.

The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. It meets the major axis at T ≡ (a secθ, 0).

The coordinates of N are (a cosθ, 0). The equation of the circle with NT as its diameter is $(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$.

$\Rightarrow \quad x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$

It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally if

$2g \cdot 0 + 2f \cdot 0 = a^2 - a^2 = 0$, which is true. **Ans.**

Do yourself - 4 :

(i) Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ which are parallel to the line $x + 3y + k = 0$.

(ii) Find the equation of the tangent to the ellipse $7x^2 + 8y^2 = 100$ at the point (2, –3).

9. NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of the normal to the given ellipse at (x_1, y_1) is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=\mathbf{a^2-b^2=a^2e^2}.$$

(b) **Slope form :** Equation of a normal to the given ellipse whose slope is 'm' is

$$\mathbf{y = mx} \mp \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}.$$

(c) **Parametric form :** Equation of the normal to the given ellipse at the point **(acosθ, bsinθ)** is **axsecθ – bycosecθ = (a² – b²)**.

Illustration 9 : Find the condition that the line $\ell x + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Equation of normal to the given ellipse at (a cos θ, b sin θ) is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(i)

If the line $\ell x + my = n$ is also normal to the ellipse then there must be a value of θ for which line (i) and line $\ell x + my = n$ are identical. For that value of θ we have

$$\frac{\ell}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)} \quad \text{or} \quad \cos\theta=\frac{an}{\ell(a^2-b^2)} \quad \text{...... (iii)}$$

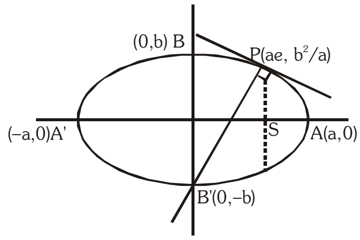
and $\sin\theta=\frac{-bn}{m(a^2-b^2)}$ ...... (iv)

Squaring and adding (iii) and (iv), we get $1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{\ell^2}+\frac{b^2}{m^2}\right)$ which is the required condition.

Illustration 10 : If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Solution : The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$.

The equation of normal at $P(ae, b^2/a)$ is



$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2$ or $\frac{ax}{e}-ay=a^2-b^2$

It passes through one extremity of the minor axis whose co-ordinates are $(0, -b)$

$\therefore\quad 0+ab=a^2-b^2 \quad\Rightarrow\quad (a^2b^2)=(a^2-b^2)^2$

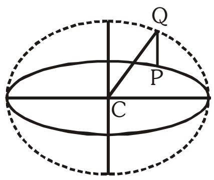
$\Rightarrow\quad a^2.a^2(1-e^2)=(a^2e^2)^2 \quad\Rightarrow\quad 1-e^2=e^4$

$\Rightarrow\quad e^4+e^2-1=0 \quad\Rightarrow\quad (e^2)^2+e^2-1=0$

$\therefore\quad e^2=\frac{-1\pm\sqrt{1+4}}{2} \quad\Rightarrow\quad e=\sqrt{\frac{\sqrt{5}-1}{2}}$ (taking positive sign) **Ans.**

Illustration 11 : P and Q are corresponding points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Solution : Let $P\equiv(a\cos\theta, b\sin\theta)$



$\therefore\quad Q\equiv(a\cos\theta, a\sin\theta)$

Equation of normal at P is

$(a\sec\theta)x-(b\,\text{cosec}\theta)y=a^2-b^2$ .... (i)

equation of CQ is $y=\tan\theta\,.\,x$ .... (ii)

Solving equation (i) & (ii), we get $(a-b)x=(a^2-b^2)\cos\theta$

$x=(a+b)\cos\theta$, & $y=(a+b)\sin\theta$

$\therefore\quad R\equiv((a+b)\cos\theta, (a+b)\sin\theta)$

$\therefore\quad CR=a+b$ **Ans.**

Do yourself - 5 :

(i) Find the equation of the normal to the ellipse $9x^2+16y^2=288$ at the point (4, 3)

(ii) Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then find maximum value of A.

(iii) If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$ intersects it again at the point $Q(2\theta)$, then find $\cos\theta$.

(iv) Show that for all real values of 't' the line $2tx+y\sqrt{1-t^2}=1$ touches a fixed ellipse. Find the eccentricity of the ellipse.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ **or T = 0 (at x_1,y_1)**.

Illustration 12 : If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B, the find the locus of point of intersection of tangents at A and B.

Solution : Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

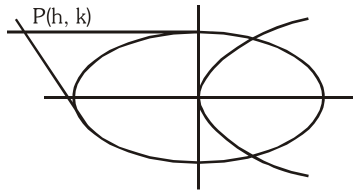
$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$ .... (i)

which touches the parabola.

Equation of tangent to parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$\Rightarrow$ $mx - y = -\frac{a}{m}$ .... (ii)

equation (i) & (ii) as must be same



$\therefore$ $\frac{m}{\left(\frac{h}{a^2}\right)}=\frac{-1}{\left(\frac{k}{b^2}\right)}=\frac{-\frac{a}{m}}{1} \Rightarrow m=-\frac{h}{k}\frac{b^2}{a^2}$ & $m=\frac{ak}{b^2}$

$\therefore$ $-\frac{hb^2}{ka^2}=\frac{ak}{b^2} \Rightarrow$ locus of P is $y^2=-\frac{b^4}{a^3}.x$ **Ans.**

Do yourself - 6 :

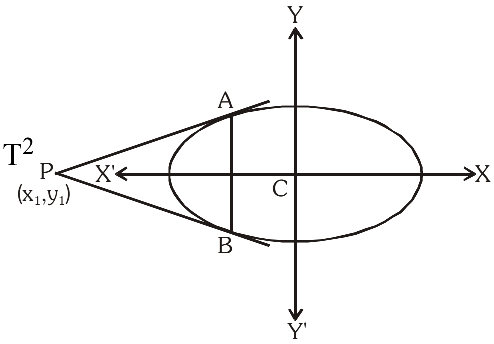
(i) Find the equation of chord of contact to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at the point (1, 3).

(ii) If the chord of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find $\frac{x_1x_2}{y_1y_2}$.

(iii) If a line $3x - y = 2$ intersects ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$ at points A & B, then find co-ordinates of point of intersection of tangents at points A & B.

11. PAIR OF TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,
and a pair of tangents PA, PB can be drawn to it from P.
Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$



where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

12. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $\mathbf{x^2 + y^2 = a^2 + b^2}$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

Illustration 13 : A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Solution : Given ellipse are $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ .... (i)

and, $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ .... (ii)

any tangent to (i) is $\dfrac{x\cos\theta}{2} + \dfrac{y\sin\theta}{1} = 1$ .... (iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\dfrac{hx}{6} + \dfrac{ky}{3} = 1$ .... (iv)

comparing (iii) and (iv), we get $\dfrac{\cos\theta}{h/3} = \dfrac{\sin\theta}{k/3} = 1$

$\Rightarrow \cos\theta = \dfrac{h}{3}$ and $\sin\theta = \dfrac{k}{3}$ $\Rightarrow h^2 + k^2 = 9$

locus of the point (h, k) is $x^2 + y^2 = 9$ $\Rightarrow$ $x^2 + y^2 = 6 + 3 = a^2 + b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

13. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, whose mid-point be (x_1,y_1) is $\mathbf{T = S_1}$

where $T = \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$, $S_1 = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$, i.e. $\left(\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1\right) = \left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1\right)$

Illustration 14 : Find the locus of the mid-point of focal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

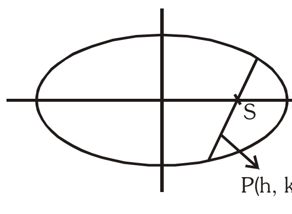
Solution : Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given $\dfrac{xh}{a^2} + \dfrac{yk}{b^2} - 1 = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ It passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)



$\therefore$ locus is $\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

If it passes through (–ae, 0)

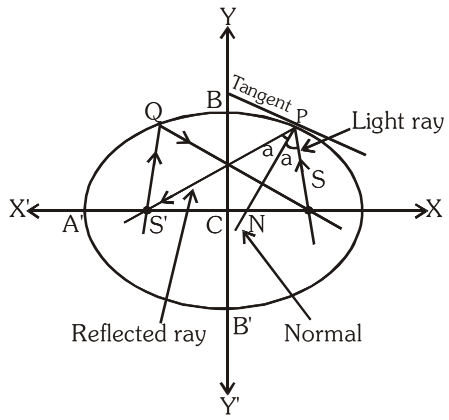
$\therefore$ locus is $-\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ **Ans.**

Do yourself - 7 :

(i) Find the equation of chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ whose mid point be (–1, 1).

14. IMPORTANT POINTS :

Referring to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



(a) If P be any point on the ellipse with S & S' as its foci then $\ell(SP)+\ell(S'P)=2a$.

(b) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice versa .

(c) **The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2** and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.

(d) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a **right angle** at the corresponding focus.

(e) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) **PF . PG = b^2** **(ii)** **PF . Pg = a^2**

(iii) **PG . Pg = SP . S' P** **(iv)** **CG . CT = CS^2**

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S' are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(f) Atmost four normals & two tangents can be drawn from any point to an ellipse.

(g) The circle on any focal distance as diameter touches the auxiliary circle.

(h) Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(i) If the tangent at the point P of a standard ellipse meets the axes in T and t and CY is the perpendicular on it from the centre then,

(i) **Tt . PY = $a^2 - b^2$ and** **(ii)** **least value of Tt is a + b.**

Do yourself - 8 :

(i) A man running round a racecourse note that the sum of the distance of two flag-posts from him is always 20 meters and distance between the flag-posts is 16 meters. Find the area of the path be encloses in square meters

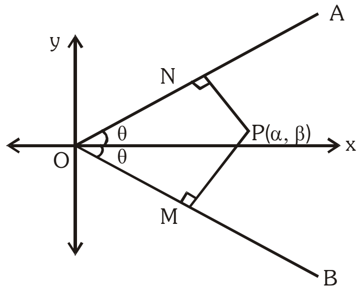
(ii) If chord of contact of the tangent drawn from the point (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2 + y^2 = k^2$, then find the locus of the point (α, β).

Miscellaneous Illustration :

Illustration 15 : A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.

Solution : Let two intersecting lines OA and OB, intersect at origin O and let both lines OA and OB makes equal angles with x axis.

i.e., $\angle XOA = \angle XOB = \theta$.



$\therefore$ Equations of straight lines OA and OB are

$y = x \tan\theta$ and $y = -x \tan\theta$

or $x \sin\theta - y \cos\theta = 0$ .... (i)

and $x \sin\theta + y \cos\theta = 0$ .... (ii)

Let $P(\alpha, \beta)$ is the point whose locus is to be determine.

According to the example $(PM)^2 + (PN)^2 = 2\lambda^2$ (say)

$\therefore$ $(\alpha \sin\theta + \beta \cos\theta)^2 + (\alpha \sin\theta - \beta \cos\theta)^2 = 2\lambda^2 \Rightarrow 2\alpha^2 \sin^2\theta + 2\beta^2\cos^2\theta = 2\lambda^2$

or $\alpha^2 \sin^2\theta + \beta^2\cos^2\theta = \lambda^2 \Rightarrow \frac{\alpha^2}{\lambda^2\text{cosec}^2\theta}+\frac{\beta^2}{\lambda^2\sec^2\theta}=1 \Rightarrow \frac{\alpha^2}{(\lambda\ \text{cosec}\ \theta)^2}+\frac{\beta^2}{(\lambda\sec\theta)^2}=1$

Hence required locus is $\frac{x^2}{(\lambda\ \text{cosec}\ \theta)^2}+\frac{y^2}{(\lambda\sec\theta)^2}=1$ **Ans.**

Illustration 16 : Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1$ passing through (a, –b) are bisected by the line x + y = b.

Solution : Let (t, b – t) be a point on the line x + y = b.

Then equation of chord whose mid point (t, b – t) is

$$\frac{tx}{2a^2}+\frac{y(b-t)}{2b^2}-1=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}-1 \quad \text{.......... (i)}$$

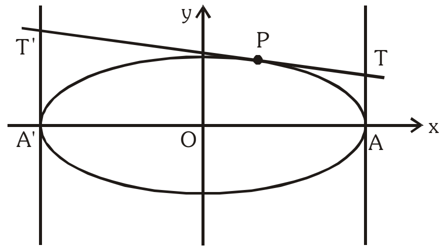
(a, –b) lies on (i) then $\frac{ta}{2a^2}-\frac{b(b-t)}{2b^2}=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2} \Rightarrow t^2(a^2+b^2)-ab(3a+b)t+2a^2b^2=0$

Since t is real $B^2 - 4AC \geq 0 \Rightarrow a^2b^2(3a + b)^2 - 4(a^2 + b^2)2a^2b^2 \geq 0$

$\Rightarrow a^2 + 6ab - 7b^2 \geq 0 \Rightarrow a^2 + 6ab \geq 7b^2$, which is the required condition.

Illustration 17 : Any tangent to an ellipse is cut by the tangents at the ends of the major axis in T and T '. Prove that circle on TT' as diameter passes through foci.

Solution : Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



and let P(acosϕ, bsinϕ) be any point on this ellipse

∴ Equation of tangent at P(acosϕ, bsinϕ) is

$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$ .....(i)

The two tangents drawn at the ends of the major axis are x = a and x = –a

Solving (i) and x = a we get $T=\left\{a, \frac{b(1-\cos\phi)}{\sin\phi}\right\}\equiv\left\{a, b\tan\left(\frac{\phi}{2}\right)\right\}$

and solving (i) and x = – a we get $T'=\left\{-a, \frac{b(1+\cos\phi)}{\sin\phi}\right\}\equiv\left\{-a, b\cot\left(\frac{\phi}{2}\right)\right\}$

Equation of circle on TT' as diameter is $(x-a)(x+a)+(y-b\tan(\phi/2))(y-b\cot(\phi/2))=0$

or $x^2+y^2-by(\tan(\phi/2)+\cot(\phi/2))-a^2+b^2=0$ .... (ii)

Now put x = ± ae and y = 0 in LHS of (ii), we get

$a^2e^2+0-0-a^2+b^2=a^2-b^2-a^2+b^2=0=$ RHS

Hence foci lie on this circle

ANSWERS FOR DO YOURSELF

1 : **(i)** $e=\frac{1}{\sqrt{2}}$ **(ii)** $\frac{(x-10)^2}{52}+\frac{(y-6)^2}{16}=1$ **(iii)** $e=\frac{\sqrt{3}}{2}$; foci $=(1\pm\sqrt{3}, -1)$; LR $=1$

2 : **(i)** $\frac{x^2}{4}+\frac{y^2}{8}=1$

(ii) C ≡ (–1, 2), length of major axis = 2b = $\sqrt{3}$, length of minor axis = 2a = 1; $e=\sqrt{\frac{2}{3}}$;

$f\left(-1,\ 2\pm\frac{1}{\sqrt{2}}\right)$

(iii) C

3 : **(i)** On the ellipse **(ii)** $\frac{1}{2}(a^2+b^2)$ **(iii)** $P^2=a^2\cos^2\theta+b^2\sin^2\theta$

4 : **(i)** $3y+x\pm\sqrt{97}=0$ **(ii)** $7x-12y=50$

5 : **(i)** $4x-3y=7$ **(ii)** abe **(iii)** –1 **(iv)** $\frac{\sqrt{3}}{2}$

6 : **(i)** $\frac{x}{16}+\frac{y}{3}=1$ **(ii)** $-\frac{a^4}{b^4}$ **(iii)** (12, –2)

7 : **(i)** $-9x+16y=25$ **8 :** **(i)** 60π **(ii)** $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{k^2}$

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. Let 'E' be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ & 'C' be the circle $x^2+y^2=9$. Let P & Q be the points (1 , 2) and (2, 1) respectively. Then :

(A) Q lies inside C but outside E (B) Q lies outside both C & E

(C) P lies inside both C & E (D) P lies inside C but outside E. **EL0001**

2. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{\sqrt{3}}$ **EL0002**

3. The equation, $2x^2+3y^2-8x-18y+35=K$ represents

(A) no locus if $K>0$ (B) an ellipse if $K<0$

(C) a point if $K=0$ (D) a hyperbola if $K>0$ **EL0003**

4. If the ellipse $\frac{(x-h)^2}{M}+\frac{(y-k)^2}{N}=1$ has major axis on the line $y=2$, minor axis on the line $x=-1$, major axis has length 10 and minor axis has length 4. The number h,k,M,N (in this order only) are -

(A) –1,2,5,2 (B) –1,2,10,4 (C) 1,–2,25,4 (D) –1,2,25,4 **EL0004**

5. The y-axis is the directrix of the ellipse with eccentricity $e=1/2$ and the corresponding focus is at (3, 0), equation to its auxiliary circle is

(A) $x^2+y^2-8x+12=0$ (B) $x^2+y^2-8x-12=0$

(C) $x^2+y^2-8x+9=0$ (D) $x^2+y^2=4$ **EL0005**

6. The latus rectum of a conic section is the width of the function through the focus. The positive difference between the length of the latus rectum of $3y=x^2+4x-9$ and $x^2+4y^2-6x+16y=24$ is-

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) $\frac{5}{2}$ **EL0007**

7. Let S(5,12) and S'(–12,5) are the foci of an ellipse passing through the origin. The eccentricity of ellipse equals -

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$ **EL0008**

8. A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

(A) 11 (B) 12 (C) 13 (D) none **EL0009**

9. (a) Which of the following is an equation of the ellipse with centre (–2,1), major axis running from (–2,6) to (–2,–4) and focus at (–2,5) ?

(A) $\frac{(x-2)^2}{25}+\frac{(y+1)^2}{16}=1$ (B) $\frac{(x+2)^2}{25}+\frac{(y-1)^2}{9}=1$

(C) $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{25}=1$ (D) $\frac{(x+2)^2}{9}+\frac{(y-1)^2}{25}=1$ **EL0011**

(b) Which of the following statement(s) is/are correct for the ellipse of 9(a) ?

(A) auxiliary circle is $(x+2)^2+(y-1)^2=25$

(B) director circle is $(x+2)^2+(y-1)^2=34$

(C) Latus rectum $=\frac{18}{5}$

(D) eccentricity $=\frac{4}{5}$ **EL0011**

10. $x-2y+4=0$ is a common tangent to $y^2=4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then the value of b and the other common tangent are given by :

(A) $b=\sqrt{3}$; $x+2y+4=0$ (B) $b=3$; $x+2y+4=0$

(C) $b=\sqrt{3}$; $x+2y-4=0$ (D) $b=\sqrt{3}$; $x-2y-4=0$ **EL0012**

11. Consider the particle travelling clockwise on the elliptical path $\frac{x^2}{100}+\frac{y^2}{25}=1$. The particle leaves the orbit at the point (–8, 3) and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

(A) $\left(0,\frac{25}{3}\right)$ (B) $\left(0,\frac{23}{3}\right)$ (C) (0, 9) (D) $\left(0,\frac{26}{3}\right)$ **EL0013**

[MULTIPLE OBJECTIVE TYPE]

12. Consider the ellipse $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$ where $\alpha\in(0,\pi/2)$.

Which of the following quantities would vary as α varies ?

(A) degree of flatness (B) ordinate of the vertex

(C) coordinates of the foci (D) length of the latus rectum **EL0014**

13. The equation of the common tangents of the parabola $y^2=4x$ and an ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ are -

(A) $x-2y+4=0$ (B) $x+2y+4=0$ (C) $2x-y+1=0$ (D) $2x+y+1=0$ **EL0015**

14. If length of perpendicular drawn from origin to any normal of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is ℓ, then ℓ cannot be -

(A) 4 (B) $\frac{5}{2}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ **EL0017**

[COMPREHENSION TYPE]

Paragraph for question nos. 15 to 17

Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the parabola $y^2 = 2x$. They intersect at P and Q in the first and fourth quadrants respectively. Tangents to the ellipse at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

15. The ratio of the areas of the triangles PQS and PQR, is

(A) 1 : 3 (B) 1 : 2 (C) 2 : 3 (D) 3 : 4 **EL0020**

16. The area of quadrilateral PRQS, is

(A) $\frac{3\sqrt{15}}{2}$ (B) $\frac{15\sqrt{3}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $\frac{5\sqrt{15}}{2}$ **EL0020**

17. The equation of circle touching the parabola at upper end of its latus rectum and passing through its vertex, is

(A) $2x^2 + 2y^2 - x - 2y = 0$ (B) $2x^2 + 2y^2 + 4x - \frac{9}{2}y = 0$

(C) $2x^2 + 2y^2 + x - 3y = 0$ (D) $2x^2 + 2y^2 - 7x + y = 0$ **EL0020**

[MATRIX MATCH TYPE]

18. **Column-I** **Column-II**

(A) The eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the X-axis and the straight line $4x + 5y = 20$ on the Y-axis and whose principal axes lie along the coordinate axes, is **EL0021** (P) $\frac{1}{2}$

(B) A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one end describes a conic whose eccentricity is **EL0022** (Q) $\frac{1}{\sqrt{2}}$

(C) If one extremity of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci form an equilateral triangle, then its eccentricity, is **EL0023** (R) $\frac{\sqrt{5}}{3}$

(D) There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Eccentricity of this ellipse is equal to **EL0024** (S) $\frac{\sqrt{7}}{4}$

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Equation of the common tangent to the ellipses, $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$ is -

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$ (B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$ (D) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$ **EL0027**

2. The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is π/4, is :

(A) $\frac{(a^2 - b^2)\,ab}{a^2 + b^2}$ (B) $\frac{(a^2 - b^2)}{(a^2 + b^2)ab}$ (C) $\frac{(a^2 - b^2)}{ab\,(a^2 + b^2)}$ (D) $\frac{a^2 + b^2}{(a^2 - b^2)\,ab}$ **EL0029**

3. The locus of the middle point of chords of an ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0, \frac{3}{5}\right)$ and $\left(0, \frac{-3}{5}\right)$ (B) (0, – 4) and (0, 1)

(C) (0, 4) and (0, 1) (D) $\left(0, \frac{11}{2}\right)$ and $\left(0, \frac{-1}{2}\right)$ **EL0030**

[MULTIPLE OBJECTIVE TYPE]

4. Extremities of the latus rectum of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a > b)$ having a given major axis 2a lies on-

(A) $x^2 = a(a - y)$ (B) $x^2 = a(a + y)$ (C) $y^2 = a(a + x)$ (D) $y^2 = a(a - x)$ **EL0031**

5. If a number of ellipse (whose axes are x & y axes) be described having the same major axis 2a but a variable minor axis then the tangents at the ends of their latus rectum pass through fixed points which can be -

(A) (0,a) (B) (0,0) (C) (0,–a) (D) (a,a) **EL0032**

6. Tangents are drawn from any point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ to the circle $x^2 + y^2 = 1$ and respective chord of contact always touches a conic C , then -

(A) minimum distance between C & ellipse is $\frac{3}{2}$ (B) maximum distance between C & ellipse is $\frac{10}{3}$

(C) eccentricity of C is $\frac{\sqrt{5}}{3}$ (D) product of eccentricity of C & ellipse is 1 **EL0033**

7. Two lines are drawn from point P(α, β) which touches $y^2 = 8x$ at A, B and touches $\frac{x^2}{4}+\frac{y^2}{6}=1$ at C, D, then -

(A) $\alpha + \beta = -4$ (B) $\alpha\beta = 4$

(C) Area of triangle PAB is $128\sqrt{2}$ (D) Area of triangle PAB is $32\sqrt{2}$ **EL0034**

EXERCISE (S-1)

1. **(a)** Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and passing through the point (4, 6). **EL0038**

(b) An ellipse passes through the points (– 3, 1) & (2, –2) & its principal axis are along the coordinate axes in order. Find its equation. **EL0039**

2. Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$ then find the maximum value of $(4x - 9y)$. **EL0041**

3. Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of t ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then find the product (AB) (CD). **EL0042**

4. 'O' is the origin & also the centre of two concentric circles having radii of the inner & the outer circle as 'a' & 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the focii of this ellipse lie on the inner circle, find the ratio of inner : outer radii & find also the eccentricity of the ellipse. **EL0043**

5. Find the condition so that the line $px + qy = r$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\pi/4$. **EL0045**

6. The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point A (a, 0) in T and T is joined to B, the other end of the diameter through A, prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $1/\sqrt{2}$. **EL0047**

7. Find the equations of the lines with equal intercepts on the axes & which touch the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

EL0048

8. A tangent having slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$, intersects the axis of x & y in points A & B respectively. If O is the origin, find the area of triangle OAB. **EL0049**

9. Tangents drawn from the point $P(2,3)$ to the circle $x^2 + y^2 - 8x + 6y + 1 = 0$ touch the circle at the points A and B. The circumcircle of the ΔPAB cuts the director circle of ellipse $\frac{(x+5)^2}{9} + \frac{(y-3)^2}{b^2} = 1$ orthogonally. Find the value of b^2. **EL0050**

10. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant & meets the coordinate axes in A & B respectively. If P divides AB in the ratio 3 : 1 reckoning from the x-axis find the equation of the tangent. **EL0052**

11. If the normal at the point P(θ) to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, intersects it again at the point Q(2θ), show that $\cos\theta = -(2/3)$. **EL0054**

12. Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$. **EL0055**

13. A ray emanating from the point (– 4, 0) is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection. **EL0057**

14. Prove that, in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **EL0060**

EXERCISE (S-2)

1. Rectangle ABCD has area 200. An ellipse with area 200π passes through A and C and has foci at B and D. Find the perimeter of the rectangle. **EL0061**

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P & Q. Prove that the tangents at P & Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **EL0062**

3. If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity 'e' of the ellipse is given by the absolute value of $\frac{\cos\beta}{\cos\alpha}$. **EL0064**

4. Prove that the equation to the circle, having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (with eccentricity e) at the ends of a latus rectum, is $x^2 + y^2 - 2ae^3x = a^2(1 - e^2 - e^4)$. **EL0072**

5. Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre 'O' where a > b > 0. Tangent at any point P on the ellipse meets the coordinate axes at X and Y and N is the foot of the perpendicular from the origin on the tangent at P. Minimum length of XY is 36 and maximum length of PN is 4.

(a) Find the eccentricity of the ellipse.

(b) Find the maximum area of an isosceles triangle inscribed in the ellipse if one of its vertex coincides with one end of the major axis of the ellipse.

(c) Find the maximum area of the triangle OPN. **EL0073**

EXERCISE (JM)

1. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is :-

[AIEEE-2009]

(1) $4x^2 + 48y^2 = 48$ (2) $4x^2 + 64y^2 = 48$ (3) $x^2 + 16y^2 = 16$ (4) $x^2 + 12y^2 = 16$

EL0075

2. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{2/5}$ is :- **[AIEEE-2011]**

(1) $3x^2 + 5y^2 - 15 = 0$ (2) $5x^2 + 3y^2 - 32 = 0$

(3) $3x^2 + 5y^2 - 32 = 0$ (4) $5x^2 + 3y^2 - 48 = 0$ **EL0076**

3. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : **[AIEEE-2012]**

(1) $x^2 + 4y^2 = 16$ (2) $4x^2 + y^2 = 4$ (3) $x^2 + 4y^2 = 8$ (4) $4x^2 + y^2 = 8$ **EL0077**

4. **Statement–1 :** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement–2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true. **EL0078**

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

5. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is : **[JEE (Main)-2013]**

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$ **EL0079**

6. If a and c are positive real number and the ellipse $\frac{x^2}{4c^2} + \frac{y^2}{c^2} = 1$ has four distinct points in common with the circle $x^2 + y^2 = 9a^2$, then **[JEE-Main (On line)-2013]**

(1) $6ac + 9a^2 - 2c^2 > 0$ (2) $6ac + 9a^2 - 2c^2 < 0$

(3) $9ac - 9a^2 - 2c^2 < 0$ (4) $9ac - 9a^2 - 2c^2 > 0$ **EL0080**

7. Equation of the line passing through the points of intersection of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is - **[JEE-Main (On line)-2013]**

(1) $y + 3 = 0$ (2) $3y + 1 = 0$ (3) $3y - 1 = 0$ (4) $y - 3 = 0$ **EL0081**

8. Let the equations of two ellipses be $E_1 : \frac{x^2}{3} + \frac{y^2}{2} = 1$ and $E_2 : \frac{x^2}{16} + \frac{y^2}{b^2} = 1$. If the product of their eccentricities is $\frac{1}{2}$, then the length of the minor axis of ellipse E_2 is :- **[JEE-Main (On line)-2013]**

(1) 9 (2) 8 (3) 2 (4) 4 **EL0082**

9. If the curves $\frac{x^2}{\alpha}+\frac{y^2}{4}=1$ and $y^3 = 16x$ intersect at right angles, then a value of α is :

[JEE-Main (On line)-2013]

(1) $\frac{4}{3}$ (2) $\frac{3}{4}$ (3) $\frac{1}{2}$ (4) 2 **EL0083**

10. A point on the ellipse, $4x^2 + 9y^2 = 36$, where the normal is parallel to the line, $4x - 2y - 5 = 0$, is :

[JEE-Main (On line)-2013]

(1) $\left(\frac{8}{5},-\frac{9}{5}\right)$ (2) $\left(-\frac{9}{5},\frac{8}{5}\right)$ (3) $\left(\frac{8}{5},\frac{9}{5}\right)$ (4) $\left(\frac{9}{5},\frac{8}{5}\right)$ **EL0084**

11. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is : **[JEE(Main)-2014]**

(1) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(3) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 + y^2)^2 = 6x^2 - 2y^2$ **EL0085**

12. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is : **[JEE (Main)-2015]**

(1) $\frac{27}{2}$ (2) 27 (3) $\frac{27}{4}$ (4) 18 **EL0086**

13. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is :- **[JEE(Main) 2017]**

(1) $x + 2y = 4$ (2) $2y - x = 2$ (3) $4x - 2y = 1$ (4) $4x + 2y = 7$ **EL0087**

14. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices then the mid points of the tangents intercepted betwen the coordinate axes lie on the curve :

[JEE (Main)-Jan 19]

(1) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (2) $\frac{x^2}{4}+\frac{y^2}{2}=1$ (3) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (4) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

EL0088

15. Let S and S' be the foci of the ellipse and B be any one of the extremities of its minor axis. If ΔS'BS is a right angled triangle with right angle at B and area (ΔS'BS) = 8 sq. units, then the length of a latus rectum of the ellipse is : **[JEE (Main)-Jan 19]**

(1) $2\sqrt{2}$ (2) 2 (3) 4 (4) $4\sqrt{2}$ **EL0089**

16. If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $\left(3,\frac{-9}{2}\right)$, then the length of the latus recturm of the ellipse is : **[JEE (Main)-Apr 19]**

(1) 9 (2) $8\sqrt{3}$ (3) $12\sqrt{2}$ (4) 5 **EL0090**

17. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through Q(4, 4) then PQ is equal to : **[JEE (Main)-Apr 19]**

(1) $\frac{\sqrt{221}}{2}$ (2) $\frac{\sqrt{157}}{2}$ (3) $\frac{\sqrt{61}}{2}$ (4) $\frac{5\sqrt{5}}{2}$ **EL0091**

EXERCISE (JA)

PARAGRAPH :

Tangents are drawn from the point P(3,4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

1. The coordinates of A and B are **[JEE 2010, 3+3+3]**

(A) (3,0) and (0,2)

(B) $\left(-\frac{8}{5},\frac{2\sqrt{261}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(C) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0,2)

(D) (3,0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

EL0096

2. The orthocenter of the triangle PAB is

(A) $\left(5,\frac{8}{7}\right)$ (B) $\left(\frac{7}{5},\frac{25}{8}\right)$ (C) $\left(\frac{11}{5},\frac{8}{5}\right)$ (D) $\left(\frac{8}{25},\frac{7}{5}\right)$

EL0096

3. The equation of the locus of the point whose distances from the point P and the line AB are equal, is -

(A) $9x^2+y^2-6xy-54x-62y+241=0$

(B) $x^2+9y^2+6xy-54x+62y-241=0$

(C) $9x^2+9y^2-6xy-54x-62y-241=0$

(D) $x^2+y^2-2xy+27x+31y-120=0$

EL0096

4. The ellipse $E_1:\frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is - **[JEE 2012, 3M, –1M]**

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

EL0097

5. A vertical line passing through the point (h,0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1=\max_{1/2\le h\le 1}\Delta(h)$ and $\Delta_2=\min_{1/2\le h\le 1}\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=$ **[JEE-Advanced 2013, 4, (–1)]**

EL0098

6. **List-I** **List-II**

List-I		List-II	
P.	Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals **EL0099**	**1.**	1
Q.	Let $A_1, A_2, ..., A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\overrightarrow{a_k}$ be the position vector of the point A_k, k = 1, 2, ... n. If $\left\|\sum_{k=1}^{n-1}(\overrightarrow{a_k}\times\overrightarrow{a_{k+1}})\right\| = \left\|\sum_{k=1}^{n-1}(\overrightarrow{a_k}.\overrightarrow{a_{k+1}})\right\|$, then the minimum value of n is **EL0100**	**2.**	2
R.	If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line x + y = 8, then the value of h is **EL0101**	**3.**	8
S.	Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is **EL0102**	**4.**	9

Codes :

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

[JEE(Advanced)-2014, 3(–1)]

7. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **EL0103**

[JEE 2015, 4M, –0M]

8. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line x + y = 3 touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are)

[JEE 2015, 4M, –0M]

(A) $e_1^2+e_2^2=\frac{43}{40}$ (B) $e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2-e_2^2\right|=\frac{5}{8}$ (D) $e_1e_2=\frac{\sqrt{3}}{4}$ **EL0104**

PARAGRAPH :

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

9. The orthocentre of the triangle F_1MN is- **[JEE(Advanced)-2016, 4(–2)]**

(A) $\left(-\frac{9}{10},0\right)$ (B) $\left(\frac{2}{3},0\right)$ (C) $\left(\frac{9}{10},0\right)$ (D) $\left(\frac{2}{3},\sqrt{6}\right)$ **EL0105**

10. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is - **[JEE(Advanced)-2016, 3(0)]**

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3 **EL0105**

11. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi-2)$

(D) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi-2)$

EL0106

12. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) The eccentricities of E_{18} and E_{19} are NOT equal

(2) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

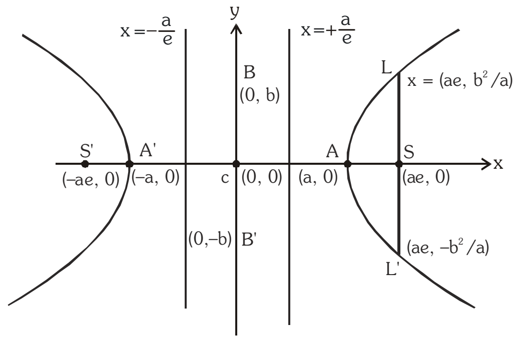
(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N **EL0107**

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. **($e > 1$)**.

1. STANDARD EQUATION & DEFINITION(S) :



Standard equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2 - 1)$$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$$= 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$$x = \frac{a}{e} \;\;\&\;\; x = -\frac{a}{e}.$$

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ **(distance from focus to directrix)**

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right); \left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the **Transverse Axis of the Hyperbola.**

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **Conjugate Axis of the Hyperbola.**

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the **Principal axes of the hyperbola.**

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on Hyperbola from foci **PS = ex – a & PS′ = ex + a.**

Illustration 1 : Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity $\sqrt{3}$.

Solution : Let P(x, y) be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition SP = e PM

$\Rightarrow \quad (SP)^2 = e^2 (PM)^2 \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12\,xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 2 : The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\frac{\sqrt{5}}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{\sqrt{13}}{2}$ (D) $\frac{3}{2}$

Solution : $4x^2 - 9y^2 - 8x = 32 \Rightarrow 4(x-1)^2 - 9y^2 = 36 \Rightarrow \frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$

Here $a^2 = 9, b^2 = 4$

$\therefore \quad$ eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$ **Ans.(B)**

Illustration 3 : If foci of a hyperbola are foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola be 2, then its equation is -

(A) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (B) $\frac{x^2}{12} - \frac{y^2}{4} = 1$ (C) $\frac{x^2}{12} + \frac{y^2}{4} = 1$ (D) none of these

Solution : For ellipse $e = \frac{4}{5}$, so foci = (±4, 0)

For hyperbola e = 2, so $a = \frac{ae}{e} = \frac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **Ans.(A)**

Illustration 4 : Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Solution : Equation can be rewritten as $\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$ so a = 4, b = 3

$b^2 = a^2(e^2 - 1)$ given $e = \frac{5}{4}$

Foci : X = ± ae, Y = 0 gives the foci as (9, 3), (–1, 3)

Centre : X = 0, Y = 0 i.e. (4, 3)

Directrices : $X = \pm \frac{a}{e}$ i.e. $x - 4 = \pm \frac{16}{5}$ $\therefore$ directrices are 5x – 36 = 0; 5x – 4 = 0

Latus-rectum $= \frac{2b^2}{a} = 2.\frac{9}{4} = \frac{9}{2}$

Do yourself - 1 :

(i) Find the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

(ii) Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is 4x – 3y = a.

(iii) In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

(iv) Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **Conjugate Hyperbolas** of each other. eg. $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

& $\mathbf{-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ are conjugate hyperbolas of each other .

Note that :

(i) If $\mathbf{e_1}$ & $\mathbf{e_2}$ are the eccentricities of the hyperbola & its conjugate then $\mathbf{e_1^{-2} + e_2^{-2} = 1}$.

(ii) The foci of a **hyperbola** and its **conjugate** are **concyclic and form the vertices of a square.**

(iii) Two hyperbolas are said to be **similar** if they have the **same eccentricity**.

Illustration 5 : The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is-

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution : Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1$, $b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$ **Ans. (A)**

Do yourself - 2 :

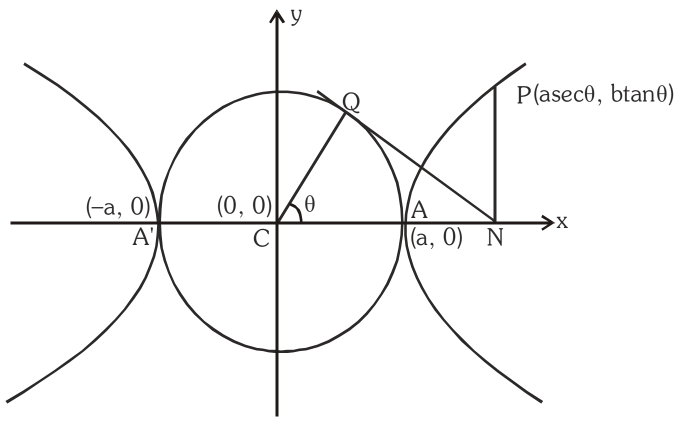
(i) Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and **the length of it's latus rectum is equal to it's transverse or conjugate axis.**

4. AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $\mathbf{x^2 + y^2 = a^2}$.



Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the **eccentric angle** of the point 'P' on the hyperbola. $(\mathbf{0 \le \theta < 2\pi})$.

Parametric Equation :

The equations $\mathbf{x = a \sec\theta}$ & $\mathbf{y = b\tan\theta}$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations ; $\mathbf{x = a\cos h\,\phi, y = b\sin h\,\phi}$ also represents the same hyperbola.

General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-\mathbf{b^2}$ instead of $\mathbf{b^2}$ it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of $\mathbf{b^2}$.

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is **positive , zero** or **negative** according as the point (x_1, y_1) lies within , upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant , a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $\mathbf{P(\alpha)}$ & $\mathbf{Q(\beta)}$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

Illustration 6 : Show that the line $x \cos\alpha + y \sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution : The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow \quad y = -x\cot\alpha + p\,\text{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha,\ c = p\,\text{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \quad \Rightarrow \quad p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Illustration 7 : If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Solution : Equation of chord connecting the points $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right) \qquad \text{........ (i)}$$

If it passes through (ae, 0); we have, $e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$

$$\Rightarrow \quad e=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}} \Rightarrow \quad \tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1-e}{1+e}$$

Similarly if (i) passes through (–ae, 0), $\tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1+e}{1-e}$ **Ans. (B, C)**

Do yourself - 3 :

(i) Find the condition for the line $\ell x+my+n=0$ to touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(ii) If the line y = 5x +1 touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{b^2}=1$ $\{b>4\}$, then -

(A) $b^2=\frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\mathbf{\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1}$.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $\mathbf{y-y_1=m_1(x-x_1)}$ & $\mathbf{y-y_1=m_2(x-x_1)}$, where $\mathbf{m_1}$ & $\mathbf{m_2}$ are roots of the equation $\mathbf{(x_1^2-a^2)\,m^2-2\,x_1y_1\,m+y_1^2+b^2=0}$. If **D < 0**, then **no tangent** can be drawn from $(x_1 y_1)$ to **the hyperbola.**

(b) **Slope form :** The equation of tangents of slope m to the given hyperbola is $\mathbf{y=mx\pm\sqrt{a^2m^2-b^2}}$. Point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}},\frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$

Note that there **are two parallel tangents having the same slope m.**

(c) **Parametric form :** Equation of the tangent to the given hyperbola at the point **(a sec θ , b tan θ)** is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is $\mathbf{x=a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}}$, $\mathbf{y=b\tan\left(\frac{\theta_1+\theta_2}{2}\right)}$

Illustration 8 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution : Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \times 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \quad \Rightarrow \quad x + y \pm 3\sqrt{3} = 0$ **Ans.**

Illustration 9 : The locus of the point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$ (B) $y^2 + b^2 = c^2(x^2 - a^2)$

(C) $y^2 + a^2 = c^2(x^2 - b^2)$ (D) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution : Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \quad \Rightarrow \quad (x_1^2 - a^2)\, m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2 \quad \Rightarrow \quad \frac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$ **Ans.(B)**

Illustration 10 : A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is -

(A) $y = 3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$ (B) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (C) $y = -3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (D) $y = -3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$

Solution : $\frac{x^2}{16} - \frac{y^2}{9} = 1,\ x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and $3\sqrt{1 + m'^2} = \sqrt{16m^2 - 9}$

or $\quad 9 + 9m^2 = 16m^2 - 9$

or $\quad 7m^2 = 18 \quad \Rightarrow \quad m = \pm 3\sqrt{\frac{2}{7}}$

$\therefore$ required equation is $y = \pm 3\sqrt{\frac{2}{7}}\, x \pm 3\sqrt{1 + \frac{18}{7}}$

or $\quad y = \pm 3\sqrt{\frac{2}{7}}\, x \pm \frac{15}{\sqrt{7}}$ **Ans. (A,B,C,D)**

Do yourself - 4 :

(i) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

(ii) Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \frac{16}{3}\right)$.

(iii) Find the common tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is $\mathbf{\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2 e^2}$.

(b) **Slope form :** The equation of normal of slope m to the given hyperbola is $\mathbf{y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2 - m^2b^2)}}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2 - m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2 - m^2b^2)}}\right)$

(c) **Parametric form :** The equation of the normal at the point **P (a secθ , b tanθ)** to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

Illustration 11 : Line $x \cos\alpha + y \sin\alpha = p$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if -

(A) $a^2 \sec^2\alpha - b^2 \operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (C) $a^2 \sec^2\alpha + b^2 \operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

(C) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (D) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

Solution : Equation of a normal to the hyperbola is $ax\cos\theta + by\cot\theta = a^2 + b^2$
comparing it with the given line equation

$$\frac{a\cos\theta}{\cos\alpha} = \frac{b\cot\theta}{\sin\alpha} = \frac{a^2+b^2}{p} \Rightarrow \sec\theta = \frac{ap}{\cos\alpha(a^2+b^2)}, \tan\theta = \frac{bp}{\sin\alpha(a^2+b^2)}$$

Eliminating θ, we get

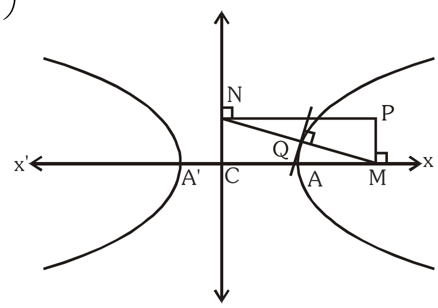
$$\frac{a^2p^2}{\cos^2\alpha(a^2+b^2)^2} - \frac{b^2p^2}{\sin^2\alpha(a^2+b^2)^2} = 1 \Rightarrow a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$$

Ans.(A)

Illustration 12 : The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Solution : Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2 + b^2$

$\therefore$ $M \equiv \left(\dfrac{a^2+b^2}{a}\sec\theta,\ 0\right),\ N \equiv \left(0,\ \dfrac{a^2+b^2}{b}\tan\theta\right)$



$\therefore$ Let $P \equiv (h, k)$

$\Rightarrow$ $h = \dfrac{a^2+b^2}{a}\sec\theta,\quad k = \dfrac{a^2+b^2}{b}\tan\theta$

$\Rightarrow$ $\dfrac{a^2h^2}{(a^2+b^2)} - \dfrac{b^2k^2}{(a^2+b^2)^2} = \sec^2\theta - \tan^2\theta = 1$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)$.

Do yourself - 5 :

(i) Find the equation of normal to the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ at (5, 0).

(ii) Find the equation of normal to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ at the point $\left(6,\ \dfrac{3}{2}\sqrt{5}\right)$.

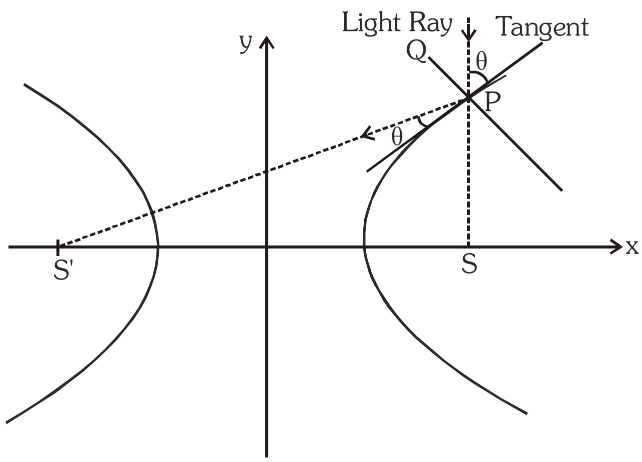
(iii) Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

9. HIGHLIGHTS ON TANGENT AND NORMAL :

(a) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\mathbf{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ upon any tangent is its auxiliary circle i.e. $\mathbf{x^2 + y^2 = a^2}$ & the product of lengths to these perpendiculars is $\mathbf{b^2}$ **(semi Conjugate Axis)²**

(b) The portion of the tangent between the **point of contact** & **the directrix** subtends a **right angle** at the corresponding **focus**.

(c) The tangent & normal at any point of a hyperbola **bisect** the angle between the **focal radii**. This spells the **reflection property of the hyperbola** as **"An incoming light ray "** aimed towards one **focus is reflected from the outer surface of the hyperbola towards the other focus.** It follows that if an ellipse and a hyperbola have the same foci, they cut at **right angles** at any of their **common point.**



Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **& the hyperbola** $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ **(a > k > b > 0) are confocal and therefore orthogonal.**

(d) The **foci of the hyperbola** and the **points P** and **Q** in which any tangent meets the tangents at the vertices are **concyclic** with **PQ** as **diameter of the circle.**

10. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at **right angles** is known as the **Director Circle** of the hyperbola. The equation to the **director circle is** : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, this circle is **real ;** if $b^2 = a^2$ the **radius** of the **circle is zero** & it reduces to a **point circle at the origin.** In this case the **centre is the only point** from which the **tangents at right angles** can be drawn to the **curve**.

If $b^2 > a^2$, the **radius** of the **circle is imaginary**, so that there is **no such circle** & so **no tangents at right angle** can be drawn to the curve.

Note : Equations of chord of contact, chord with a given middle point, pair of tangents from an external point are to be interpreted in the similar way as in ellipse.

11. ASYMPTOTES :

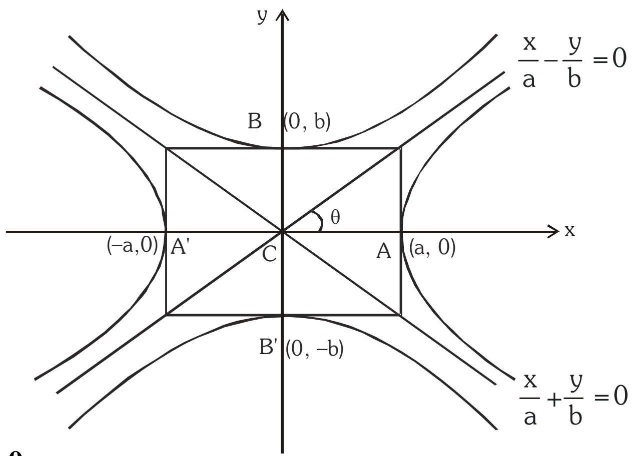
Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the **asymptote** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)\,x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$ ....(1)

In order that $y = mx + c$ **be an asymptote**, both **roots** of equation (1) must approach infinity, the conditions for which are : **coefficient of** $x^2 = 0$ & **coefficient of** $x = 0$.



$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2mc = 0 \Rightarrow c = 0$.

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.

combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Particular Case :

When **b = a** the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are $y = \pm x$ which are at **right angles**.

Note :

(i) **Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.**

(ii) If a **hyperbola is equilateral** then the **conjugate hyperbola is also equilateral**.

(iii) A **hyperbola** and its **conjugate** have the **same asymptote.**

(iv) The equation of the **pair of asymptotes** differ the **hyperbola** & **the conjugate hyperbola** by the **same constant** only.

(v) The asymptotes pass through the **centre of the hyperbola** & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the co–ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $\mathbf{f(x, y) = 0}$ represents a hyperbola.

Find $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ & $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$. Then the point of intersection of $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \mathbf{0}$ & $\frac{\partial \mathbf{f}}{\partial \mathbf{y}} = \mathbf{0}$ gives the **centre** of the **hyperbola**.

Illustration 13 : Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Solution : Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line

so $4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$

$\Rightarrow$ $\lambda = 2$

$\Rightarrow$ $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow$ $(2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

Illustration 14 : Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution : The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is $(3x + y - 7)(2x - y - 3) + k = 0$

It passes through (1, 1)

$\Rightarrow$ $k = -6$.

Hence the equation of the hyperbola is $(2x - y - 3)(3x + y - 7) = 6$.

Do yourself - 6 :

(i) Find the equation to the chords of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3)

(ii) If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ which pass through the point (6, 2), then find the value of $11m_1m_2$ and $11(m_1 + m_2)$.

(iii) Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.

(iv) The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at (1, 2) and it passes through (5, 3). Find its equation.

12. HIGHLIGHTS ON ASYMPTOTES

(a) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

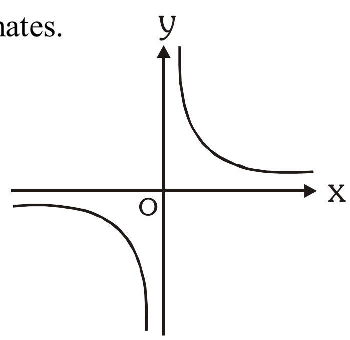
(b) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

(c) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with **centre C**, meets the **asymptotes in Q** and **R** and cuts off a **Δ CQR** of **constant area** equal to **ab** from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact . This implies that locus of the centre of the circle circumscribing the **Δ CQR** in case of a rectangular hyperbola is the **hyperbola** itself.

(d) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the **hyperbola** is $\sec\theta$.

13. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation
$x = ct,\ \ y = c/t,\ \ t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is
$x + t_1t_2y = c(t_1 + t_2)$ with slope, $m = \frac{-1}{t_1t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$
& at P(t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as **(h, k)** is $kx + hy = 2hk$.

Note :

For the hyperbola, $\mathbf{xy = c^2}$

(i) Vertices : **(c, c) & (–c, –c)**.

(ii) Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

(iii) Directrices : $x + y = \pm\sqrt{2}c$

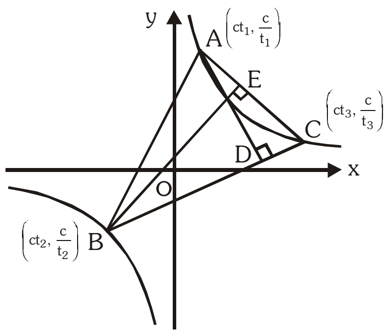
(iv) Latus rectum : $\ell = 2\sqrt{2}c$ = T . A = C . A

Illustration 15 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let t_1, t_2 and t_3 are the vertices of the triangle ABC, described on the rectangular hyperbola xy $= c^2$.

$\therefore$ co-ordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

Now slope of BC is $\dfrac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2 t_3}$



$\therefore$ Slope of AD is $t_2 t_3$

Equation of altitude AD is $y - \frac{c}{t_1} = t_2 t_3(x - ct_1)$

or $t_1 y - c = xt_1t_2t_3 - ct_1^2 t_2 t_3$ .... (i)

Similarly equation of altitude BE is

$t_2 y - c = xt_1t_2t_3 - ct_1 t_2^2 t_3$ .... (ii)

Solving (i) and (ii), we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ which lies on $xy = c^2$.

Do yourself - 7 :

(i) If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola then write required conditions.

(ii) Find the equation of tangent at the point (1, 2) to the rectangular hyperbola xy = 2.

(iii) Prove that the locus of point, tangents from where to hyperbola $x^2 - y^2 = a^2$ inclined at an angle α & β with x-axis such that $\tan\alpha \tan\beta = 2$ is also a hyperbola. Find the eccentricity of this hyperbola.

Miscellaneous Illustrations :

Illustration 16 : Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Prove that locus of their middle point is the curve $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Solution : Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = a^2$, so that its equation by $T = S_1$ is $hx + ky = h^2 + k^2$

or $y = -\frac{h}{k}x + \frac{h^2+k^2}{k}$ i.e. of the form $y = mx + c$

It will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$\therefore \left(\frac{h^2+k^2}{k}\right)^2 = a^2\left(-\frac{h}{k}\right)^2 - b^2$ or $(h^2 + k^2)^2 = a^2h^2 - b^2k^2$

Generalising, the locus of mid-point (h, k) is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

Illustration 17 : C is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at any point P on this hyperbola meets the straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively. Show that $CQ . CR = a^2 + b^2$.

Solution : P is $(a\sec\theta, b\tan\theta)$

Tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

It meets $bx - ay = 0$ i.e. $\frac{x}{a} = \frac{y}{b}$ in Q

$\therefore$ Q is $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$

It meets $bx + ay = 0$ i.e. $\frac{x}{a} = -\frac{y}{b}$ in R.

$\therefore$ R is $\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

$\therefore$ $CQ.CR = \frac{\sqrt{(a^2+b^2)}}{\sec\theta - \tan\theta}.\frac{\sqrt{(a^2+b^2)}}{\sec\theta + \tan\theta} = a^2 + b^2$ $(\because \sec^2\theta - \tan^2\theta = 1)$ **Ans.**

Illustration 18 : A circle of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of triangle PQR.

Solution : Let the circle be $(x - h)^2 + (y - k)^2 = r^2$ where r is variable. Its intersection with $x^2 - y^2 = 9a^2$ is obtained by putting $y^2 = x^2 - 9a^2$.

$x^2 + x^2 - 9a^2 - 2hx + h^2 + k^2 - r^2 = 2k\sqrt{(x^2 - 9a^2)}$

or $[2x^2 - 2hx + (h^2 + k^2 - r^2 - 9a^2)]^2 = 4k^2(x^2 - 9a^2)$

or $4x^4 - 8hx^3 + ... = 0$

$\therefore$ Above gives the abscissas of the four points of intersection.

$\therefore \quad \Sigma x_1 = \dfrac{8h}{4} = 2h$

$x_1 + x_2 + x_3 + x_4 = 2h$

Similarly $y_1 + y_2 + y_3 + y_4 = 2k$.

Now if (α, β) be the centroid of ΔPQR, then $3\alpha = x_1 + x_2 + x_3$, $3\beta = y_1 + y_2 + y_3$

$\therefore \quad x_4 = 2h - 3\alpha,\ y_4 = 2k - 3\beta$

But (x_4, y_4) lies on $x^2 - y^2 = 9a^2$

$\therefore \quad (2h - 3\alpha)^2 + (2k - 3\beta)^2 = 9a^2$

Hence the locus of centroid (α, β) is $(2h - 3x)^2 + (2k - 3y)^2 = 9a^2$

or $\left(x - \dfrac{2h}{3}\right)^2 + \left(y - \dfrac{2k}{3}\right)^2 = a^2$

Illustration 19 : If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then prove that :

(a) $t_1t_2t_3t_4 = 1$

(b) The centre of mean position of the four points bisects the distance between the centres of the two curves.

Solution : (a) Let the equation of the hyperbola referred to rectangular asymptotes as axes be $xy = c^2$ or its parametric equation be

$x = ct,\ y = c/t$ .... (i)

and that of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ .... (ii)

Solving (i) and (ii), we get

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$$

or $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$ .... (iii)

Above equation being of fourth degree in t gives us the four parameters t_1, t_2, t_3, t_4 of the points of intersection.

$$\therefore \quad t_1 + t_2 + t_3 + t_4 = -\frac{2gc}{c^2} = -\frac{2g}{c} \qquad \text{........... (iv)}$$

$$t_1t_2t_3 + t_1t_2t_4 + t_3t_4t_1 + t_3t_4t_2$$

$$= -\frac{2fc}{c^2} = -\frac{2f}{c} \qquad \text{........... (v)}$$

$$t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1. \text{ It proves (a)} \qquad \text{............ (vi)}$$

Dividing (v) by (vi), we get

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4} = -\frac{2f}{c} \qquad \text{............ (vii)}$$

(b) The centre of mean position of the four points of intersection is

$$\left[\frac{c}{4}(t_1+t_2+t_3+t_4),\ \frac{c}{4}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right)\right] = \left[\frac{c}{4}\left(-\frac{2g}{c}\right),\ \frac{c}{4}\left(-\frac{2f}{c}\right)\right],$$ by (iv) and (vii)

$= (-g/2, -f/2)$

Above is clearly the mid-point of (0, 0) and $(-g, -f)$ i.e. the join of the centres of the two curves.

ANSWERS FOR DO YOURSELF

1 : **(i)** $\sqrt{3}$ **(ii)** $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

(iii) 6, 4; $(\pm\sqrt{13}, 0)$; $\sqrt{13}/3$; 8/3 **(iv)** $x^2 - y^2 = 32$

2 : **(i)** $\sqrt{5}$ & 40 sq. units

3 : **(i)** $n^2 = a^2\ell^2 - b^2m^2$ **(ii)** B

4 : **(i)** $24y = 30x \pm \sqrt{161}$ **(ii)** $5x - 3y = 9$ **(iii)** $y = \pm x \pm \sqrt{7}$

5 : **(i)** $y = 0$; **(ii)** $8\sqrt{5}x + 18y = 75\sqrt{5}$ **(iii)** $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

6 : **(i)** $5x + 3y = 16$ **(ii)** 20 & 24 **(iii)** $(x^2 + y^2)^2 = 16x^2 - 9y^2$

(iv) $(2x + 3y - 8)(3x + 2y - 7) = 154$

7 : **(i)** $\Delta \neq 0, h^2 > ab, a + b = 0$ **(ii)** $2x + y = 4$ **(iii)** $e = \sqrt{3}$

EXERCISE (O-1)

1. Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:

(a) centre **(b)** eccentricity **(c)** focii **(d)** equation of directrix

(e) length of the latus rectum **(f)** equation of auxilary circle

(g) equation of director circle HB0001

[STRAIGHT OBJECTIVE TYPE]

2. Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(A) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$ HB0003

3. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Then the equation of the hyperbola with eccentricity 2 is HB0004

(A) $\frac{x^2}{12} - \frac{y^2}{4} = 1$ (B) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (C) $3x^2 - y^2 + 12 = 0$ (D) $9x^2 - 25y^2 - 225 = 0$

4. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$ HB0005

5. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is-

(A) 5 (B) 7 (C) 9 (D) 4 HB0006

6. The graph of the equation $x + y = x^3 + y^3$ is the union of -

(A) line and an ellipse (B) line and a parabola (C) line and hyperbola (D) line and a point

HB0007

7. The focal length of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$, is-

(A) 4 (B) 6 (C) 8 (D) 10 HB0008

8. The equation $\frac{x^2}{29-p} + \frac{y^2}{4-p} = 1$ $(p \neq 4, 29)$ represents -

(A) an ellipse if p is any constant greater than 4

(B) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve is p is less than 29. HB0009

9. If $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$ represents family of hyperbolas where 'α' varies then -

(A) distance between the foci is constant

(B) distance between the two directrices is constant

(C) distance between the vertices is constant

(D) distances between focus and the corresponding directrix is constant HB0010

10. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\ t = 0$ & $\sqrt{3}tx + ty - 4\sqrt{3}\ = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

(A) $\sqrt{3}$ (B) 2 (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{4}{3}$ **HB0011**

11. The magnitude of the gradient of the tangent at an extremity of latera recta of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to (where e is the eccentricity of the hyperbola)

(A) be (B) e (C) ab (D) ae **HB0013**

12. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is :

(A) zero (B) 1 (C) 2 (D) 4 **HB0014**

13. Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2} - \phi$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is :

(A) $x = a$ (B) $y = b$ (C) $x = ab$ (D) $y = ab$ **HB0015**

14. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D) $x^2 + y^2 = 1/16$

HB0016

15. A tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with centre C meets its director circle at P and Q. Then the product of the slopes of CP and CQ, is -

(A) $\frac{9}{4}$ (B) $\frac{-4}{9}$ (C) $\frac{2}{9}$ (D) $-\frac{1}{4}$ **HB0017**

16. In which of the following cases maximum number of normals can be drawn from a point P lying in the same plane

(A) circle (B) parabola (C) ellipse (D) hyperbola **HB0019**

17. With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre , a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

(A) less than 2 (B) 2 (C) $\frac{11}{3}$ (D) none **HB0021**

[MULTIPLE OBJECTIVE TYPE]

18. Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents

(A) two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.

(B) two circles, when $p = q$ and r is of sign opposite to that of p.

(C) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$. **HB0025**

(D) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

19. If θ is eliminated from the equations $a \sec\theta - x\tan\theta = y$ and $b\sec\theta + y\tan\theta = x$ (a and b are constant), then the eliminant denotes the equation of

(A) the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(B) auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(C) Director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(D) Director circle of the circle $x^2 + y^2 = \frac{a^2+b^2}{2}$. **HB0027**

20. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $\left(\sqrt{3},0\right)$ when associated with two asymptotes constitutes -

(A) isosceles triangle which is not equilateral (B) an equilateral triangle

(C) a triangles whose area is $\sqrt{3}$ sq. units (D) a right isosceles triangle. **HB0028**

21. If latus rectum of a hyperbola subtends a right angle at other focus of hyperbola, then eccentricity is equal to-

(A) $1-\sqrt{2}$ (B) $\tan\frac{\pi}{8}$ (C) $\cot\frac{\pi}{8}$ (D) $\left(\frac{1}{\sqrt{2}-1}\right)$ **HB0029**

22. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$, $S(x_4,y_4)$, then -

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1x_2x_3x_4 = c^4$ (D) $y_1y_2y_3y_4 = c^4$ **HB0030**

[COMPREHENSION TYPE]

Paragraph for question nos. 23 to 25

The graph of the conic $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through the origin. the point of tangency being (a, b). Then

23. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(A) $\frac{5\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$ **HB0031**

24. Length of the latus rectum of the conic is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) none **HB0031**

25. Eccentricity of the conic is

(A) $\frac{4}{3}$ (B) $\sqrt{3}$ (C) 2 (D) none **HB0031**

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Let F_1, F_2 are the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and F_3, F_4 are the foci of its conjugate hyperbola. If e_H and e_C are their eccentricities respectively then the statement which holds true is

(A) Their equations of the asymptotes are different.

(B) $e_H > e_C$

(C) Area of the quadrilateral formed by their foci is 50 sq. units.

(D) Their auxiliary circles will have the same equation. **HB0041**

2. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$ **HB0043**

3. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the centre of the hyperbola, then OT.ON is equal to :

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2 **HB0044**

4. Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is **HB0045**

(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

[MULTIPLE OBJECTIVE TYPE]

5. Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$ **HB0051**

6. Let A(–1,0) and B(2,0) be two points on the x - axis . A point M is moving in xy-plane (other than x - axis) in such a way that $\angle MBA = 2\angle MAB$, then the point M moves along a conic whose

(A) coordinate of vertices are (± 3, 0).

(B) length of latus-rectum equals 6.

(C) eccentricity equals 2.

(D) equation of directrices are $x = \pm\frac{1}{2}$. **HB0052**

7. Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{3}=1$ of eccentricity e is confocal with the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$. Let A,B,C & D are points of intersection of hyperbola & ellipse, then-

(A) $e=\frac{5}{2}$

(B) e = 2

(C) A,B,C,D are concyclic points

(D) Number of common tangents of hyperbola & ellipse is 2 **HB0053**

8. If the ellipse $4x^2+9y^2+12x+12y+5=0$ is confocal with a hyperbola having same principal axes, then -

(A) angle between normals at their each point of intersection is 90º.

(B) centre of the ellipse is $\left(-\frac{3}{2},-\frac{2}{3}\right)$

(C) distance between foci of the hyperbola is $\frac{2\sqrt{10}}{3}$

(D) ellipse and hyperbola has same length of latus rectum **HB0054**

9. If the eccentricity of the ellipse $\frac{x^2}{(\log a)^2}+\frac{y^2}{(\log b)^2}=1$ $(a>b>0, a, b\neq 1)$ is $\frac{1}{\sqrt{2}}$ and 'e' be the eccentricity of the hyperbola $\frac{x^2}{(\log_b a)^2}-y^2=1$, then e^2 is greater than (where $\log x=\ell nx$)-

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{4}$ **HB0055**

EXERCISE (S-1)

1. Find the equation to the hyperbola whose directrix is $2x+y=1$, focus (1, 1) & eccentricity $\sqrt{3}$. Find also the length of its latus rectum. **HB0060**

2. The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passes through the point of intersection of the lines, $7x+13y-87=0$ and $5x-8y+7=0$ & the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'. **HB0061**

3. For the hyperbola $\frac{x^2}{100}-\frac{y^2}{25}=1$, prove that

(i) eccentricity $=\sqrt{5}/2$ **HB0062**

(ii) SA. S′A = 25, where S & S′ are the foci & A is the vertex. **HB0062**

4. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes of the hyperbola $16x^2-9y^2+32x+36y-164=0$. **HB0063**

5. Find the equation of the tangent to the hyperbola $x^2-4y^2=36$ which is perpendicular to the line $x-y+4=0$. **HB0065**

6. Tangents are drawn to the hyperbola $3x^2-2y^2=25$ from the point (0, 5/2). Find their equations. **HB0066**

7. A conic C satisfies the differential equation, $(1 + y^2)dx - xy\,dy = 0$ and passes through the point (1,0). An ellipse E which is confocal with C having its eccentricity equal to $\sqrt{2/3}$.

(a) Find the length of the latus rectum of the conic C **HB0067**

(b) Find the equation of the ellipse E. **HB0067**

(c) Find the locus of the point of intersection of the perpendicular tangents to the ellipse E. **HB0067**

8. A hyperbola has one focus at the origin and its eccentricity $= \sqrt{2}$ and one of its directrix is $x + y + 1 = 0$. Find the equation to its asymptotes. **HB0068**

9. If the lines $x + y + 1 = 0$ and $2x - y + 2 = 0$ are the asymptotes of a hyperbola. If the line $x - 2 = 0$ touches the hyperbola then the equation of the hyperbola is $4(x + y + 1)(2x - y + 2) = \lambda$. Find the value of λ. **HB0069**

10. If C is the centre of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, S, S′ its foci and P a point on it.
Prove that SP. S′P $= CP^2 - a^2 + b^2$. **HB0070**

11. Chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle drawn on the line joining the foci as diameter. Find the locus of the point of intersection of tangents at the extremities of the chords. **HB0071**

12. If two points P & Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then prove that $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$. **HB0073**

13. Locus of the feet of the perpendicular from centre of the hyperbola $x^2 - 4y^2 = 4$ upon a variable normal to it has the equation, $(x^2 + y^2)^2 (4y^2 - x^2) = \lambda x^2y^2$, find λ. **HB0074**

14. Let P $(a \sec\theta, b \tan\theta)$ and Q $(a \sec\phi, b \tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P & Q, then find k. **HB0075**

EXERCISE (S-2)

1. Tangent and normal are drawn at the upper end (x_1,y_1) of the latus rectum P with $x_1 > 0$ and $y_1 > 0$, of the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$, intersecting the transverse axis at T and G respectively. Find the area of the triangle PTG. **HB0076**

2. Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact. **HB0077**

3. An ellipse and a hyperbola have their principal axes along the coordinate axes and have a common foci separated by a distance $2\sqrt{13}$, the difference of their focal semi axes is equal to 4. If the ratio of their eccentricities is 3/7. Find the equation of these curves. **HB0079**

4. From the centre C of the hyperbola $x^2 - y^2 = 9$, CM is drawn perpendicular to the tangent at any point of the curve, meeting the tangent at M and the curve at N. Find the value of the product (CM)(CN). **HB0080**

5. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x –axis. If $\tan\theta . \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$. **HB0085**

EXERCISE (JM)

1. The equation of the hyperbola whose foci are (–2,0) and (2, 0) and eccentricity is 2 is given by : **[AIEEE-2011]**

(1) $-3x^2 + y^2 = 3$ (2) $x^2 - 3y^2 = 3$ (3) $3x^2 - y^2 = 3$ (4) $-x^2 + 3y^2 = 3$

HB0086

2. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that OPRQ is a rectangle (where O is the origin). Then R lies on : **[JEE-Main (On line)-2013]**

(1) $\frac{2}{x^2} - \frac{4}{y^2} = 1$ (2) $\frac{4}{x^2} - \frac{2}{y^2} = 1$ (3) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (4) $\frac{2}{x^2} + \frac{4}{y^2} = 1$

HB0087

3. A common tangent to the conics $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is : **[JEE-Main (On line)-2013]**

(1) $x + y = \frac{9}{2}$ (2) $x + y = 1$ (3) $x - y = \frac{3}{2}$ (4) $x - y = 1$

HB0088

4. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE (Main) 2016]**

(1) $\sqrt{3}$ (2) $\frac{4}{3}$ (3) $\frac{4}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$ **HB0089**

5. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point : **[JEE (Main) 2017]**

(1) $(-\sqrt{2},-\sqrt{3})$ (2) $(3\sqrt{2},2\sqrt{3})$ (3) $(2\sqrt{2},3\sqrt{3})$ (4) $(\sqrt{3},\sqrt{2})$ **HB0090**

6. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is - **[JEE (Main) 2018]**

(1) $54\sqrt{3}$ (2) $60\sqrt{3}$ (3) $36\sqrt{5}$ (4) $45\sqrt{5}$ **HB0091**

7. Let $S = \left\{(x,y) \in R^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1\right\}$, where $r \neq \pm 1$. Then S represents : **[JEE (Main)-Jan 19]**

(1) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, where $0 < r < 1$.

(2) An ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, where $r > 1$

(3) A hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$.

(4) An ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$ **HB0092**

8. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbole $xy = 2$ is : **[JEE (Main)-Jan 19]**

(1) $x + 2y + 4 = 0$ (2) $x - 2y + 4 = 0$

(3) $x + y + 1 = 0$ (4) $4x + 2y + 1 = 0$ **HB0093**

9. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is :- **[JEE (Main)-Jan 19]**

(1) 2 (2) $\frac{13}{6}$ (3) $\frac{13}{8}$ (4) $\frac{13}{12}$ **HB0094**

10. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is

[JEE (Main)-Apr 19]

(1) $\frac{\sqrt{5}}{2}$ (2) $\frac{3}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{15}}{2}$ **HB0095**

11. If a directrix of a hyperbola centred at the origin and passing through the point $\left(4, -2\sqrt{3}\right)$ is $5x = 4\sqrt{5}$ and its eccentricity is e, then : **[JEE (Main)-Apr 19]**

(1) $4e^4 - 24e^2 + 35 = 0$ (2) $4e^4 + 8e^2 - 35 = 0$

(3) $4e^4 - 12e^2 - 27 = 0$ (4) $4e^4 - 24e^2 + 27 = 0$ **HB0096**

12. Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and S' denote the foci of the hyperbola where S lies on the positive x-axis then P divides SS' in a ratio : **[JEE (Main)-Apr 19]**

(1) 5:4 (2) 14:13 (3) 2:1 (4) 13:11 **HB0097**

EXERCISE (JA)

1. Consider a branch of the hyperbola, $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[JEE 2008, 3]**

(A) $1-\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}}-1$ (C) $1+\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}}+1$ **HB0100**

2. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is **[JEE 2009, 3]**

(A) a hyperbola (B) a parabola (C) an ellipse (D) a straight line

HB0101

3. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[JEE 2009, 4]**

(A) Equation of ellipse is $x^2 + 2y^2 = 2$ (B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$ (D) The foci of ellipse are $(\pm\sqrt{2}, 0)$ **HB0102**

4. Match the conics in **Column I** with the statements/expressions in **Column II.** **[JEE 2009, 8]**

	Column I		**Column II**	
(A)	Circle	(p)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$	HB0103
(B)	Parabola	(q)	Points z in the complex plane satisfying $\|z+2\| - \|z-2\| = \pm 3$	HB0104
(C)	Ellipse	(r)	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right),\ y = \frac{2t}{1+t^2}$	HB0105
(D)	Hyperbola			
		(s)	The eccentricity of the conic lies in the interval $1 \le x < \infty$	HB0106
		(t)	Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \|z\|^2 + 1$	HB0107

Comprehension : **[JEE 2010, 3+3]**

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

5. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is -

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$
(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$ **HB0108**

6. Equation of the circle with AB as its diameter is -

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$
(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$ **HB0108**

7. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **HB0109**

[JEE 2010, 3]

8. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then - **[JEE 2011, 4]**

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
(B) a focus of the hyperbola is (2,0)
(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$ **HB0110**

9. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9,0), then the eccentricity of the hyperbola is - **[JEE 2011, 3]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$ **HB0111**

10. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contact of the tangents on the hyperbola are **[JEE 2012, 4M]**

(A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3},-2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3},2\sqrt{2}\right)$ **HB0112**

11. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1 > 1$ (B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1 > 1$ (D) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1 > 0$ **HB0113**

12. If 2x – y + 1 = 0 is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced)-2017, 4(–2)]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2 **HB0114**

Column 1, 2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively.

Column 1	**Column 2**	**Column 3**
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

13. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced)-2017, 3(–1)]**

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R) **HB0115**

14. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8,16), then which of the following options is the only **CORRECT** combination ?

[JEE(Advanced)-2017, 3(–1)]

(A) (III) (i) (P) (B) (III) (ii) (Q) (C) (II) (iv) (R) (D) (I) (ii) (Q) **HB0115**

15. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only **CORRECT** combination for obtaining its equation ?

[JEE(Advanced)-2017, 3(–1)]

(A) (II) (ii) (Q) (B) (III) (i) (P) (C) (I) (i) (P) (D) (I) (ii) (Q) **HB0115**

16. Let $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I	**LIST-II**
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is : **HB0116**

(A) **P → 4; Q → 2, R → 1; S → 3**
(B) **P → 4; Q → 3; R → 1; S → 2**
(C) **P → 4; Q → 1, R → 3; S → 2**
(D) **P → 3; Q → 4; R → 2; S → 1**

[JEE(Advanced)-2018, 3(–1)]

ANSWER KEY

PARABOLA

EXERCISE (O-1)

1. D **2.** D **3.** C **4.** A **5.** C **6.** D **7.** A **8.** D
9. B **10.** D **11.** C **12.** D **13.** A **14.** D **15.** C **16.** B
17. B **18.** C **19.** C **20.** B **21.** B **22.** B **23.** D **24.** D
25. C **26.** B **27.** C **28.** A,B,C,D **29.** A,B **30.** A,D
31. B,C,D **32.** A,B,C **33.** C **34.** A **35.** (A) S, (B) Q, (C) S, (D) P, (E) S

EXERCISE (O-2)

1. B **2.** A **3.** C **4.** B **5.** B **6.** B **7.** A **8.** B
9. C **10.** A,D **11.** A,B,C,D **12.** A,B,C **13.** A,B,C,D **14.** A
15. B **16.** C

EXERCISE (S-1)

2. $4\sqrt{3}$ **3.** $(4, 0)$; $y^2 = 2a(x - 4a)$ **4.** $2x - y + 2 = 0, (1, 4)$; $x + 2y + 16 = 0, (16, -16)$
5. $3x - 2y + 4 = 0$; $x - y + 3 = 0$ **8.** $y = -4x + 72, y = 3x - 33$
9. $(a, 0)$; a **13.** (a) $x^2 + y^2 - 17x - 6y = 0$; (b) $(26/3, 0)$ **14.** $x - y = 1$; $8\sqrt{2}$ sq. units
15. $x^2 + y^2 + 18x - 28y + 27 = 0$ **16.** $7y \pm 2(x + 6a) = 0$

EXERCISE (S-2)

3. 512 **4.** (a) 4, (b) (2,–1), (c) 8 sq. units **6.** 32 **7.** 25/2

EXERCISE (JM)

1. 3 **2.** 2 **3.** 3 **4.** 4 **5.** 3 **6.** 1 **7.** 2
8. 2 **9.** **Bonus** **10.** 1 **11.** 3 **12.** 4 **13.** 1,2,3,4 **14.** 3
15. 2 **16.** 3 **17.** 4

EXERCISE (JA)

1. A,D **2.** C,D **3.** 2 **4.** C **5.** A,B,D **6.** 4 **7.** D **8.** B **9.** A
10. D **11.** D **12.** B **13.** 2 **14.** 4 **15.** A,D **16.** A,B,C **17.** A,C,D **18.** D

ELLIPSE

EXERCISE (O-1)

1. D **2.** B **3.** C **4.** D **5.** A **6.** A **7.** C **8.** C

9. (a) D; (b) A,B,C,D **10.** A **11.** A **12.** A,B,D **13.** A,B **14.** A,B **15.** C

16. B **17.** D **18.** (A) S, (B) R, (C) P, (D) Q

EXERCISE (O-2)

1. B **2.** A **3.** C **4.** A,B **5.** A,C **6.** A,B,C **7.** A,D

EXERCISE (S-1)

1. (a) $20x^2 + 45y^2 - 40x - 180y - 700 = 0$; (b) $3x^2 + 5y^2 = 32$ **2.** 16 **3.** 65

4. $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **5.** $a^2p^2 + b^2q^2 = r^2\sec^2\frac{\pi}{8} = (4 - 2\sqrt{2})r^2$ **7.** $x + y - 5 = 0,\ x + y + 5 = 0$

8. 24 sq. units **9.** 54 **10.** $bx + a\sqrt{3}\,y = 2ab$ **12.** $(x - 1)^2 + y^2 = \frac{11}{3}$

13. $12x + 5y = 48$; $12x - 5y = 48$

EXERCISE (S-2)

1. 80 **5.** (a) $\frac{3}{5}$; (b) $240\sqrt{3}$; (c) 36

EXERCISE (JM)

1. 4 **2.** 3 **3.** 1 **4.** 3 **5.** 1 **6.** 4 **7.** 3 **8.** 4

9. 1 **10.** 4 **11.** 3 **12.** 2 **13.** 3 **14.** 3 **15.** 3 **16.** 1

17. 4

EXERCISE (JA)

1. D **2.** C **3.** A **4.** C **5.** 9 **6.** A **7.** 4 **8.** A,B

9. A **10.** C **11.** A,C **12.** 3,4

HYPERBOLA

EXERCISE (O-1)

1. **(a)** $(-4,-1)$; **(b)** $\frac{5}{4}$; **(c)** $(1,-1)$, $(-9,-1)$; **(d)** $5x+4=0$, $5x+36=0$, **(e)** $\frac{9}{2}$; **(f)** $(x+4)^2+(y+1)^2=16$;

(g) $(x+4)^2+(y+1)^2=7$

2. A **3.** B **4.** B **5.** B **6.** A **7.** C **8.** B **9.** A

10. B **11.** B **12.** A **13.** B **14.** D **15.** B **16.** A **17.** B

18. A,B,C,D **19.** C,D **20.** B,C **21.** C,D **22.** A,B,C,D **23.** D

24. C **25.** D

EXERCISE (O-2)

1. C **2.** D **3.** B **4.** C **5.** A,C,D **6.** B,C,D **7.** B,C

8. A,B,C **9.** B,C,D

EXERCISE (S-1)

1. $\sqrt{\frac{48}{5}}$ **2.** $a^2=25/2$; $b^2=16$

4. $(-1,2)$; $(4,2)$ & $(-6,2)$; $5x-4=0$ & $5x+14=0$; $\frac{32}{3}$; 6; 8; $y-2=0$; $x+1=0$

5. $x+y\pm3\sqrt{3}=0$ **6.** $3x+2y-5=0$; $3x-2y+5=0$ **7.** (a) 2; (b) $\frac{x^2}{3}+\frac{y^2}{1}=1$; (c) $x^2+y^2=4$

8. $x+1=0$ and $y+1=0$ **9.** 81 **11.** $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$ **13.** 25 **14.** $-\left(\frac{a^2+b^2}{b}\right)$

EXERCISE (S-2)

1. 45 **2.** $y=\frac{5}{12}x+\frac{3}{4}$; $x-3=0$; 8 sq. unit **3.** $\frac{x^2}{49}+\frac{y^2}{36}=1$; $\frac{x^2}{9}-\frac{y^2}{4}=1$ **4.** 9

EXERCISE (JM)

1. 3 **2.** 2 **3.** 3 **4.** 4 **5.** 3 **6.** 4 **7.** 4 **8.** 1

9. 4 **10.** 3 **11.** 1 **12.** 1

EXERCISE (JA)

1. B **2.** D **3.** A,B **4.** (A) p, (B) s,t; (C) r; (D) q,s **5.** B

6. A **7.** 2 **8.** B,D **9.** B **10.** A,B **11.** A,B,D **12.** B,C,D

13. D **14.** A **15.** D **16.** B