COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

Purely real	Purely imaginary	Imaginary
if b = 0	if a = 0	if b ≠ 0

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

In general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where $n \in I$

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

***Illustration* 1 :** The value of $i^{57} + 1/i^{125}$ is :-

(A) 0 (B) –2i (C) 2i (D) 2

Solution :

$$i^{57} + 1/i^{125} = i^{56}.\ i + \frac{1}{i^{124}.i}$$

$$= \left(i^4\right)^{14} i + \frac{1}{\left(i^4\right)^{31} i}$$

$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

Ans. (A)

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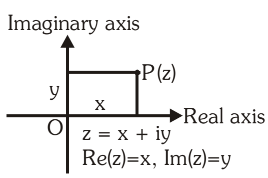
2. ARGAND DIAGRAM :

Master Argand had done a systematic study on complex numbers and represented every complex number $z = x + iy$ as a set of ordered pair (x, y) on a plane called complex plane (Argand Diagram) containing two perpendicular axes. Horizontal axis is known as Real axis & vertical axis is known as Imaginary axis.



All complex numbers lying on the real axis are called as purely real and those lying on imaginary axis as purely imaginary.

3. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

(a) Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$

(c) Multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(d) Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$

Note :

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Illustration 2 : $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

(A) $2n\pi \pm \frac{\pi}{3}, n \in I$ (B) $n\pi + \frac{\pi}{3}, n \in I$ (C) $n\pi \pm \frac{\pi}{3}, n \in I$ (D) none of these

Solution : $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)} = \frac{(3-4\sin^2\theta) + i(8\sin\theta)}{(1+4\sin^2\theta)}$$

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \quad \Rightarrow \quad 3 - 4\sin^2\theta = 0 \text{ (only if } \theta \text{ be real)}$$

$$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sin\frac{\pi}{3}\right)^2$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$$

Ans. (C)

Do yourself - 1 :

(i) Determine least positive value of n for which $\left(\frac{1+i}{1-i}\right)^n = 1$

(ii) Find the value of the sum $\sum_{n=1}^{5}(i^n + i^{n+2})$, where $i = \sqrt{-1}$.

4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts are respectively equal.

Illustration 3 : The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ are

(A) x = –1, y = 3 (B) x = 3, y = –1 (C) x = 0, y = 1 (D) x = 1, y = 0

Solution : $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i \Rightarrow (4+2i)x+(9-7i)y-3i-3=10i$

Equating real and imaginary parts, we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence x = 3 and y = –1. **Ans.(B)**

Illustration 4 : Find the square root of 7 + 24 i.

Solution : Let $\sqrt{7+24i} = a + ib$

Squaring $a^2 - b^2 + 2iab = 7 + 24i$

Compare real & imaginary parts $a^2 - b^2 = 7$ & $2ab = 24$

By solving these two equations

We get $a = \pm 4$, $b = \pm 3$

$\sqrt{7+24i} = \pm(4+3i)$

Illustration 5 : If $x = -5+2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution : We have, $x = -5 + 2\sqrt{-4}$

$\Rightarrow x + 5 = 4i \Rightarrow (x+5)^2 = 16i^2$

$\Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow x^2(0) - x(0) + 4(0) - 160 \Rightarrow -160$ **Ans.**

Do yourself - 2 :

(i) Find the value of $x^3 + 7x^2 - x + 16$, where $x = 1 + 2i$.

(ii) If $a + ib = \frac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\frac{b}{a}=\frac{2c}{c^2-1}$.

(iii) Find square root of –15 – 8i

5. THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

(a) CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

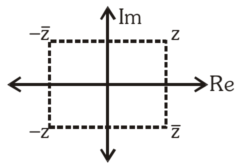
(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$, which is purely real

(iv) If z is purely real, then $z - \bar{z} = 0$.

(v) If z is purely imaginary, then $z + \bar{z} = 0$

(vi) If z lies in the 1^{st} quadrant, then $\bar{z}$ lies in the 4^{th} quadrant and $-\bar{z}$ lies in the 2^{nd} quadrant.



(b) Modulus :

If P denotes complex number z = x + iy, then the length OP is called modulus of complex number z. It is denoted by |z|.

$$OP = |z| = \sqrt{x^2 + y^2}$$

Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

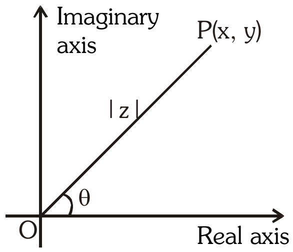
Note : Unlike real numbers, $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$

is not correct.

(c) Argument or Amplitude :

If P denotes complex number z = x + iy and if OP makes an angle θ with real axis, then θ is called one of the arguments of z.

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive real axis)



Note :

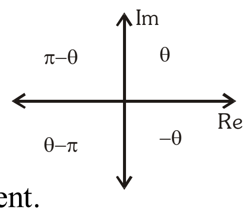
(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \leq \pi$ is called **Amplitude (principal value of the argument).**

(iii) Principal argument of a complex number z = x + iy can be found out using method given below :

(a) Find $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$ such that $\theta \in \left(0, \dfrac{\pi}{2}\right)$.

(b) Use given figure to find out the principal argument according as the point lies in respective quadrant.



(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) The unique value of $\theta = \tan^{-1}\dfrac{y}{x}$ such that $0 < \theta \leq 2\pi$ is called **least positive argument.**

(vi) If z = 0, arg(z) is not defined

(vii) If z is real & negative, arg(z) = π.

(viii) If z is real & positive, arg(z) = 0

(ix) If $\theta = \dfrac{\pi}{2}$, z lies on the positive side of imaginary axis.

(x) If $\theta = -\dfrac{\pi}{2}$, z lies on the negative side of imaginary axis.

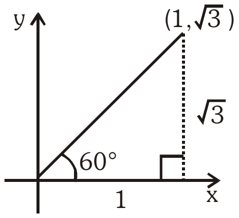
By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number 0 + 0i the argument is not defined and this is the only complex number which is given by its modulus only.

Illustration 6 : Find the modulus, argument, principal value of argument, least positive argument of complex numbers (a) $1 + i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $1 - i\sqrt{3}$ (d) $-1 - i\sqrt{3}$

Solution :

(a) For $z = 1 + i\sqrt{3}$



$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

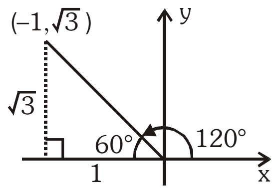
$\arg(z) = 2n\pi + \dfrac{\pi}{3}, \quad n \in I$

Least positive argument is $\dfrac{\pi}{3}$

If the point is lying in first or second quadrant then amp(z) is taken in anticlockwise direction.

In this case amp(z) $= \dfrac{\pi}{3}$

(b) For $z = -1 + i\sqrt{3}$



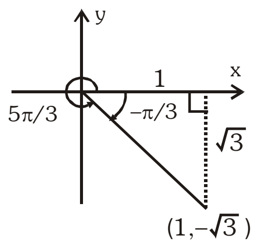
$|z| = 2$

$\arg(z) = 2n\pi + \dfrac{2\pi}{3}, \; n \in I$

Least positive argument $= \dfrac{2\pi}{3}$

$\text{amp}(z) = \dfrac{2\pi}{3}$

(c) For $z = 1 - i\sqrt{3}$



$|z| = 2$

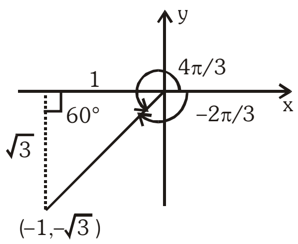
$\arg(z) = 2n\pi - \dfrac{\pi}{3}, \; n \in I$

Least positive argument $= \dfrac{5\pi}{3}$

If the point lies in third or fourth quadrant then consider amp(z) in clockwise direction.

In this case amp(z) $= -\dfrac{\pi}{3}$

(d) For $z = -1 - i\sqrt{3}$



$|z| = 2$

$\arg(z) = 2n\pi - \dfrac{2\pi}{3}, \; n \in I$

Least positive argument $= \dfrac{4\pi}{3}$

$\text{amp}(z) = -\dfrac{2\pi}{3}$

Illustration 7 : Find modulus and argument for $z = 1 - \sin\alpha + i\cos\alpha$, $\alpha \in (0,2\pi)$

Solution : $|z| = \sqrt{(1-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{2-2\sin\alpha} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$

Case (i) For $\alpha \in \left(0,\frac{\pi}{2}\right)$, z will lie in I quadrant.

$$\text{amp}(z) = \tan^{-1}\frac{\cos\alpha}{1-\sin\alpha} \Rightarrow \text{amp}(z) = \tan^{-1}\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \tan^{-1}\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$\Rightarrow \arg z = \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Since $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$\therefore$ amp $(z) = \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$, $|z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

Case (ii) at $\alpha = \frac{\pi}{2}$: $z = 0 + 0i$

$|z| = 0$

amp (z) is not defined.

Case (iii) For $\alpha \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, z will lie in IV quadrant

so amp $(z) = -\tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\frac{\pi}{2},\pi\right)$

$\therefore$ amp $(z) = -\left(\frac{\alpha}{2} + \frac{\pi}{4} - \pi\right) = \frac{3\pi}{4} - \frac{\alpha}{2}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Case (iv) at $\alpha = \frac{3\pi}{2}$: $z = 2 + 0i$

$|z| = 2$

amp $(z) = 0$

Case (v) For $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

z will lie in I quadrant

arg $(z) = \tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\pi,\frac{5\pi}{4}\right)$

$\therefore$ arg $z = \frac{\alpha}{2} + \frac{\pi}{4} - \pi = \frac{\alpha}{2} - \frac{3\pi}{4}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Do yourself - 3 :

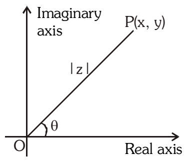
Find the modulus and amplitude of following complex numbers :

(i) $-2+2\sqrt{3}i$ **(ii)** $-\sqrt{3}-i$ **(iii)** $-2i$ **(iv)** $\dfrac{1+2i}{1-3i}$ **(v)** $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

6. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + i y$ can be represented by a point on the cartesian plane known as complex plane by the ordered pair (x , y). There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.



For $z = x + iy$; $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$ and $\theta = \tan^{-1}\dfrac{y}{x}$

Note :

(i) Distance between the two complex numbers z_1 & z_2 is given by $|z_1 - z_2|$.

(ii) $|z - z_0| = r$, represents a circle, whose centre is z_0 and radius is r.

Illustration 8 : Find the locus of :

(a) $|z-1|^2 + |z+1|^2 = 4$ (b) $\text{Re}(z^2) = 0$

Solution : (a) Let $z = x + iy$

$\Rightarrow (|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$

$\Rightarrow (x-1)^2 + y^2 + (x+1)^2 + y^2 = 4$

$\Rightarrow x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4 \Rightarrow x^2 + y^2 = 1$

Above represents a circle on complex plane with center at origin and radius unity.

(b) Let $z = x + iy$

$\Rightarrow z^2 = x^2 - y^2 + 2xyi$

$\therefore \text{Re}(z^2) = 0$

$\Rightarrow x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $\text{Re}(z^2) = 0$ represents a pair of straight lines passing through origin.

Illustration 9 : If z is a complex number such that $z^2 = (\bar{z})^2$, then

(A) z is purely real (B) z is purely imaginary

(C) either z is purely real or purely imaginary (D) none of these

Solution : Let $z = x + iy$, then its conjugate $\bar{z} = x - iy$

Given that $z^2 = (\bar{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy \Rightarrow 4ixy = 0$

If $x \neq 0$ then $y = 0$ and if $y \neq 0$ then $x = 0$. **Ans. (C)**

Illustration 10 : Among the complex number z which satisfies $|z - 25i| \leq 15$, find the complex numbers z having

(a) least positive argument (b) maximum positive argument

(c) least modulus (d) maximum modulus

Solution : The complex numbers z satisfying the condition

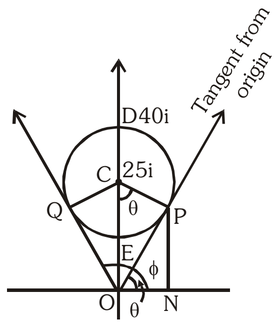
$|z - 25i| \leq 15$

are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle

Here θ = least positive argument

and ϕ = maximum positive argument



$\therefore$ In $\Delta OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$

$\therefore$ $\tan\theta = \dfrac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\therefore$ $z_p = 20(\cos\theta + i\sin\theta) = 20\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\therefore$ $z_p = 12 + 16i$

Similarly $z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

Do yourself - 4 :

(i) Find the distance between two complex numbers $z_1 = 2 + 3i$ & $z_2 = 7 - 9i$ on the complex plane.

(ii) Find the locus of $|z - 2 - 3i| = 1$.

(iii) If z is a complex number, then $z^2 + \bar{z}^2 = 2$ represents -

(A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

(c) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(d) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

Illustration 11 : Express the following complex numbers in polar and exponential form :

(i) $\dfrac{1+3i}{1-2i}$ (ii) $\dfrac{i-1}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$

Solution : (i) Let $z=\dfrac{1+3i}{1-2i}=\dfrac{1+3i}{1-2i}\times\dfrac{1+2i}{1+2i}=-1+i$

$|z|=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\tan\alpha=\left|\dfrac{1}{-1}\right|=1=\tan\dfrac{\pi}{4}\Rightarrow\alpha=\dfrac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ $\theta=\arg(z)=\pi-\alpha=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

and exponential form is $z=\sqrt{2}\,e^{\frac{3\pi}{4}i}$

(ii) Let $z=\dfrac{i-1}{\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}}=\dfrac{i-1}{\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}}=\dfrac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z=\dfrac{2(i-1)}{(1+i\sqrt{3})}\times\dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}\Rightarrow z=\left(\dfrac{\sqrt{3}-1}{2}\right)+i\left(\dfrac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z|=\sqrt{\left(\dfrac{\sqrt{3}-1}{2}\right)^2+\left(\dfrac{\sqrt{3}+1}{2}\right)^2}=\sqrt{\dfrac{2(3+1)}{4}}=\sqrt{2}$.

$\tan\theta=\left|\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right|=\tan\dfrac{5\pi}{12}\Rightarrow\alpha=\dfrac{5\pi}{12}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}\right)$

and exponential form is $z=\sqrt{2}e^{\frac{5\pi}{12}i}$

Illustration 12 : If $x_n=\cos\left(\dfrac{\pi}{2^n}\right)+i\sin\left(\dfrac{\pi}{2^n}\right)$, then $x_1x_2x_3.........\infty$ is equal to -

(A) –1 (B) 1 (C) 0 (D) ∞

Solution : $x_n=\cos\left(\dfrac{\pi}{2^n}\right)+i\sin\left(\dfrac{\pi}{2^n}\right)=1\times e^{i\frac{\pi}{2^n}}$

$x_1x_2x_3.........\infty$

$=e^{i\frac{\pi}{2^1}}.e^{i\frac{\pi}{2^2}}---e^{i\frac{\pi}{2^n}}=e^{i\left(\frac{\pi}{2}+\frac{\pi}{2^2}+---+\frac{\pi}{2^n}\right)}$

$=\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........\right)+i\sin\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........\right)=-1$

$\left(\text{as }\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........=\dfrac{\pi/2}{1-1/2}=\pi\right)$ **Ans. (A)**

Do yourself - 5 :

Express the following complex number in polar form and exponential form :

(i) $-2+2i$ **(ii)** $-1-\sqrt{3}i$ **(iii)** $\frac{(1+7i)}{(2-i)^2}$ **(iv)** $(1-\cos\theta+i\sin\theta),\ \theta\in(0,\pi)$

7. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $z+\bar{z}=2\,\text{Re}(z)$ **(b)** $z-\bar{z}=2i\,\text{Im}(z)$ **(c)** $\overline{(\bar{z})}=z$

(d) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$ **(e)** $\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2$

(f) $\overline{z_1 z_2}=\bar{z}_1.\bar{z}_2$. In general $\overline{z_1z_2.........z_n}=\bar{z}_1.\bar{z}_2.........\bar{z}_n$

(g) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$; $z_2\neq 0$ **(h)** If $f(\alpha+i\beta)=x+iy \Rightarrow f(\alpha-i\beta)=x-iy$

8. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z|\geq 0$ **(b)** $|z|\geq \text{Re}(z)$ **(c)** $|z|\geq \text{Im}(z)$

(d) $|z|=|\bar{z}|=|-z|=|-\bar{z}|$ **(e)** $z\bar{z}=|z|^2$

(f) $|z_1 z_2|=|z_1|.|z_2|$. In general $|z_1z_2.........z_n|=|z_1|.|z_2|.........|z_n|$

(g) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|},\ z_2\neq 0$

(h) $|z^n|=|z|^n,\ n\in I$

(i) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}(z_1\bar{z}_2)$

(j) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\alpha-\beta)$, where α, β are $\arg(z_1)$, $\arg(z_2)$ respectively.

(k) $|z_1+z_2|^2+|z_1-z_2|^2=2\left[|z_1|^2+|z_2|^2\right]$

(l) $||z_1|-|z_2||\leq|z_1+z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

(m) $||z_1|-|z_2||\leq|z_1-z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

9. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}(z_1.z_2)=\text{amp}\,z_1+\text{amp}\,z_2+2k\pi$; $k\in I$

(b) $\text{amp}\left(\frac{z_1}{z_2}\right)=\text{amp}\,z_1-\text{amp}\,z_2+2k\pi$; $k\in I$

(c) $\text{amp}(z^n)=n\,\text{amp}(z)+2k\pi$; $n,k\in I$

where proper value of k must be chosen so that RHS lies in $(-\pi,\pi]$.

Illustration 13 : Find amp z and |z| if $z=\left[\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$.

Solution : $\text{amp}\,z=2\left[\text{amp}(3+4i)+\text{amp}(1+i)+\text{amp}(1+\sqrt{3}i)-\text{amp}(1-i)-\text{amp}(4-3i)-\text{amp}(2i)\right]+2k\pi$

where $k\in I$ and k chosen so that amp z lies in $(-\pi,\pi]$.

$$\Rightarrow \quad \text{amp}\, z = 2\left[\tan^{-1}\frac{4}{3}+\frac{\pi}{4}+\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)-\tan^{-1}\left(-\frac{3}{4}\right)-\frac{\pi}{2}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp}\ z = 2\left[\tan^{-1}\frac{4}{3}+\cot^{-1}\frac{4}{3}+\frac{\pi}{3}\right]+2k\pi \Rightarrow \text{amp}\, z = 2\left[\frac{\pi}{2}+\frac{\pi}{3}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp}\ z = -\frac{\pi}{3} \quad [\text{at } k=-1]$$ **Ans.**

Also,

$$|z| = \left|\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right|^2 \Rightarrow |z| = \left(\frac{|3+4i||1+i||1+\sqrt{3}i|}{|1-i||4-3i||2i|}\right)^2$$

$$\Rightarrow \quad |z| = \left(\frac{5\times\sqrt{2}\times 2}{\sqrt{2}\times 5\times 2}\right)^2 = 1$$ **Ans.**

Aliter

$$z = \left[\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2 \Rightarrow z = \left[-\frac{\sqrt{3}+i}{2}\right]^2 \Rightarrow z = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2}-\frac{\sqrt{3}i}{2}$$

Hence $|z| = 1$, $\text{amp}(z) = -\frac{\pi}{3}$.

Illustration 14 : If $\left|\frac{z-i}{z+i}\right| = 1$, then locus of z is -

(A) x-axis (B) y-axis (C) x = 1 (D) y = 1

Solution : We have, $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = 1$

$$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2 \Rightarrow 4y=0; y=0$$

which is x-axis **Ans. (A)**

Illustration 15: If $|z_1 + z_2|^2 = |z_1|^2+|z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is -

(A) zero or purely imaginary (B) purely imaginary

(C) purely real (D) none of these

Solution : Here let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $|z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $|z_2| = r_2$

$$\therefore \quad |(z_1+z_2)|^2 = |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|^2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2) = |z_1|^2 + |z_2|^2 \text{ if } \cos(\theta_1-\theta_2) = 0$$

$\therefore \quad \theta_1 - \theta_2 = \pm\frac{\pi}{2}$

$\Rightarrow$ amp(z_1) – amp$(z_2) = \pm\frac{\pi}{2} \Rightarrow$ amp$\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1}{z_2}$ is purely imaginary **Ans. (B)**

Illustration 16 : z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

Solution : Here $\left|\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}\right| = 1 \Rightarrow \frac{|z_1 - 2z_2|}{|2 - z_1\overline{z_2}|} = 1$

$\Rightarrow \quad |z_1 - 2z_2| = |2 - z_1\overline{z_2}| \Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z_1\overline{z_2}|^2$

$\Rightarrow \quad (z_1 - 2z_2)\left(\overline{z_1 - 2z_2}\right) = \left(2 - z_1\overline{z_2}\right)\left(\overline{2 - z_1\overline{z_2}}\right)$

$\Rightarrow \quad (z_1 - 2z_2)\left(\overline{z_1} - 2\overline{z_2}\right) = \left(2 - z_1\overline{z_2}\right)\left(2 - \overline{z_1}z_2\right)$

$\Rightarrow \quad z_1\overline{z_1} - 2z_1\overline{z_2} - 2z_2\overline{z_1} + 4z_2\overline{z_2} = 4 - 2\overline{z_1}z_2 - 2z_1\overline{z_2} + z_1\overline{z_1}z_2\overline{z_2}$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad \left(|z_1|^2 - 4\right)\left(1 - |z_2|^2\right) = 0$

But $|z_2| \neq 1$ (given)

$\therefore \quad |z_1|^2 = 4$

Hence, $\quad |z_1| = 2$.

Illustration 17 : The locus of the complex number z in argand plane satisfying the inequality

$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 \left(\text{where } |z-1| \neq \frac{2}{3}\right)$ is -

(A) a circle (B) interior of a circle (C) exterior of a circle (D) none of these

Solution : We have, $\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2} \qquad [\because \log_a x \text{ is a decreasing function if } a < 1]$

$\Rightarrow \quad 2|z-1| + 8 < 3|z-1| - 2$ as $|z - 1| > 2/3$

$\Rightarrow \quad |z-1| > 10$

which is exterior of a circle. **Ans. (C)**

Illustration 18 : If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is -

(A) $1 + \sqrt{2}$ (B) $2 + \sqrt{2}$ (C) $\sqrt{3} + 1$ (D) $\sqrt{5} + 1$

Solution : We have $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \frac{4}{|z|} = 2 + \frac{4}{|z|}$

$\Rightarrow \quad |z|^2 \le 2|z| + 4 \qquad \Rightarrow \quad (|z| - 1)^2 \le 5$

$\Rightarrow \quad |z| - 1 \le \sqrt{5} \qquad \Rightarrow \quad |z| \le \sqrt{5} + 1$

Therefore, the greatest value of $|z|$ is $\sqrt{5} + 1$. **Ans. (D)**

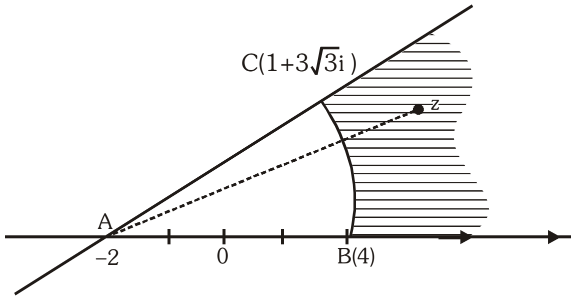
Illustration 19: Shaded region is given by -

(A) $|z + 2| \ge 6,\ 0 \le \arg(z) \le \frac{\pi}{6}$

(B) $|z + 2| \le 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(C) $|z + 2| \ge 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(D) None of these



Solution : Note that AB = 6 and $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore \quad \angle BAC = \frac{\pi}{3}$

Thus, shaded region is given by $|z + 2| \ge 6$ and $0 \le \arg(z + 2) \le \frac{\pi}{3}$ **Ans. (C)**

Do yourself - 6 :

(i) The inequality $|z - 4| < |z - 2|$ represents region given by -

(A) $Re(z) > 0$ (B) $Re(z) < 0$ (C) $Re(z) > 3$ (D) none

(ii) If $z = re^{i\theta}$, then the value of $|e^{iz}|$ is equal to -

(A) $e^{-r\cos\theta}$ (B) $e^{r\cos\theta}$ (C) $e^{r\sin\theta}$ (D) $e^{-r\sin\theta}$

10. SECTION FORMULA AND COORDINATES OF ORTHOCENTRE, CENTROID, CIRCUMCENTRE, INCENTRE OF A TRIANGLE :

If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m + n}$ divides the join of z_1 & z_2 in the ratio m : n.

Note :

(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$

- Orthocentre of the Δ ABC =

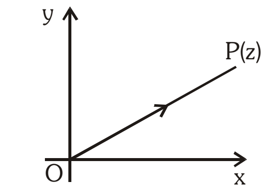
$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \quad \text{or} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

- Incentre of the $\Delta ABC = \dfrac{(az_1 + bz_2 + cz_3)}{(a+b+c)}$

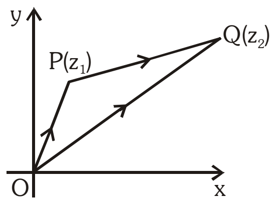
- Circumcentre of the $\Delta ABC = \dfrac{(z_1\sin 2A + z_2\sin 2B + z_3\sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$

11. VECTORIAL REPRESENTATION OF A COMPLEX NUMBER :

(a) In complex number every point can be represented in terms of position vector. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

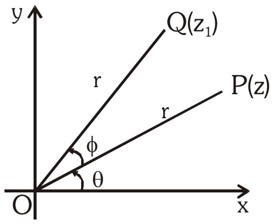


(b) If $P(z_1)$ & $Q(z_2)$ be two complex numbers on argand plane then $\vec{PQ}$ represents complex number $z_2 - z_1$.



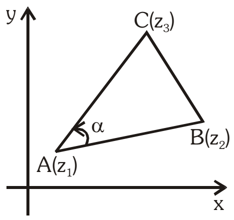
Note :

(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z.e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\,e^{i\phi}$ i.e. $\dfrac{z_1}{|z_1|} = \dfrac{z}{|z|}e^{i\phi}$



(ii) In general, if z_1, z_2, z_3 be the three vertices of ΔABC then

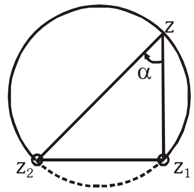
$\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$. Here $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$.



(iii) Note that the locus of z satisfying $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ is:

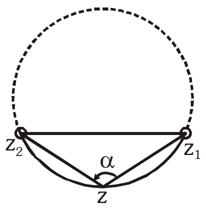
Case (a) $0 < \alpha < \pi/2$

Locus is major arc of circle as shown excluding z_1 & z_2



Case (b) $\frac{\pi}{2} < \alpha < \pi$

Locus is minor arc of circle as shown excluding z_1 & z_2



(iv) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary.

(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(1) $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ (2) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Illustration 20 : Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Solution : In the isosceles triangle ABC, AC = BC and BC⊥AC. It means that AC is rotated through angle π/2 to occupy the position BC.

Hence we have, $\frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -\left(z_1^2 + z_3^2 - 2z_1z_3\right)$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

$\Rightarrow \left(z_1 - z_2\right)^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

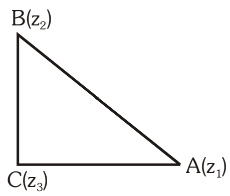
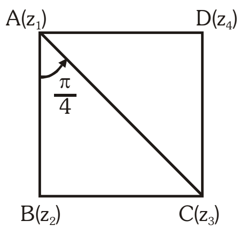


Illustration 21: If the vertices of a square ABCD are z_1, z_2, z_3 & z_4 then find z_3 & z_4 in terms of z_1 & z_2.

Solution : Using vector rotation at angle A

$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$



$\because |z_3 - z_1| = AC$ and $|z_2 - z_1| = AB$

Also AC = $\sqrt{2}$ AB

$\therefore |z_3 - z_1| = \sqrt{2}|z_2 - z_1|$

$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

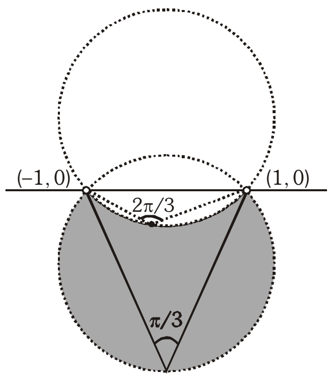
$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$

$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

Illustration 22 : Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Solution : Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$



Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)).

Do yourself - 7 :

(i) A complex number z = 3 + 4i is rotated about another fixed complex number $z_1 = 1 + 2i$ in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

(ii) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

(iii) If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a and b are equal to :-

(A) a = b = 1/2 (B) $a = b = 2 - \sqrt{3}$ (C) $a = b = -2 + \sqrt{3}$ (D) $a = b = \sqrt{2} - 1$

(v) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, find locus of z.

12. DE'MOIVRE'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of the roots of a complex quantity should be determined by using theory of equations.

Illustration 23: If cosα + cosβ + cosγ = 0 and also sinα + sinβ + sinγ = 0, then prove that

(a) cos2α + cos2β + cos2γ = sin2α + sin2β + sin2γ = 0

(b) sin3α + sin3β + sin3γ = 3sin(α + β + γ)

(c) cos3α + cos3β + cos3γ = 3cos(α + β + γ)

Solution : Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ & $z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i \cdot 0 = 0$ .... (i)

(a) Also $\frac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\frac{1}{z_2} = \cos\beta - i\sin\beta,\ \frac{1}{z_3} = \cos\gamma - i\sin\gamma$

$\therefore$ $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ .... (ii)

$= 0 - i \cdot 0 = 0$

Now $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2}\right) = 0 - 2z_1z_2z_3 \cdot 0 = 0$ {using (i) and (ii)}

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0 + i.0$

Equating real and imaginary parts on both sides,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(b) If $z_1 + z_2 + z_3 = 0$ then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$

Equating imaginary parts on both sides, $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) Equating real parts on both sides, $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

Do yourself - 8 :

(i) If $z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$, $r = 0,1,3,4,.........$, then $z_1z_2z_3z_4z_5$ is equal to -

(A) –1 (B) 0 (C) 1 (D) none of these

(ii) If $(x - 1)^4 - 16 = 0$, then the sum of nonreal complex values of x is -

(A) 2 (B) 0 (C) 4 (D) none of these

(iii) If $(\sqrt{3} - i)^n = 2^n$, $n \in Z$, then n is a multiple of -

(A) 6 (B) 10 (C) 9 (D) 12

13. CUBE ROOT OF UNITY :

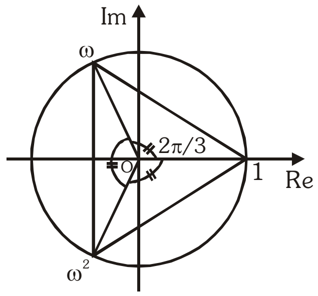
(a) The cube roots of unity are 1, $\frac{-1 + i\sqrt{3}}{2}(\omega)$, $\frac{-1 - i\sqrt{3}}{2}(\omega^2)$.

(b) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda$; $\lambda \in I$

(c) In polar form the cube roots of unity are :

$1 = \cos 0 + i\sin 0$; $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.



(e) The following factorisation should be remembered :

(a, b, c $\in$ R & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Illustration 24 : If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to (where $n \in I$) -

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

Solution :

$$\alpha = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3};$$

$$\beta = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$\alpha^n + \beta^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^n$$

$$= \left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right) + \left(\cos\frac{2n\pi}{3} - i\sin\left(\frac{2n\pi}{3}\right)\right) = 2\cos\left(\frac{2n\pi}{3}\right)$$

Ans. (A)

Illustration 25 : If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$.

Solution : We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore$ $(x - 1)^3 + 8 = 0$

$\therefore$ $(x - 1)^3 = (-2)^3$

$\Rightarrow$ $\left(\frac{x-1}{-2}\right)^3 = 1$ $\Rightarrow$ $\frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\therefore$ $x = -1, 1 - 2\omega, 1 - 2\omega^2$

Here $\alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$

$\therefore$ $\alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$

Then $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right) = \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2$

Therefore $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$. **Ans.**

Do yourself - 9 :

(i) If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^2$ equals : -

(A) ω (B) -4ω (C) ω^2 (D) 4ω

(ii) If ω is a non real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to :-

(A) 0 (B) 3 (C) 1 (D) 2

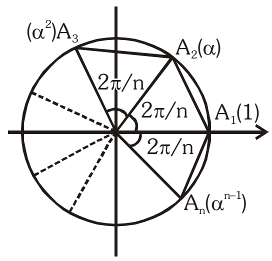
14. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 .... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(a) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(b) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(c) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(d) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(e) $(1-\alpha_1)(1-\alpha_2)......(1-\alpha_{n-1}) = n$

(f) $(1+\alpha_1)(1+\alpha_2).......(1+\alpha_{n-1}) = 0$ if n is even and

$= 1$ if n is odd.

(g) $1.\alpha_1.\alpha_2.\alpha_3.........\alpha_{n-1} = 1$ or -1 according as n is odd or even.

Illustration 26: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-\cos\frac{2\pi k}{7}\right)$

Solution :

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right)-\sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right)=\sum_{k=1}^{6}\sin\frac{2\pi k}{7}-\sum_{k=0}^{6}\cos\frac{2\pi k}{7}+1$$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) $+1 = 0 - 0 + 1 = 1$

15. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

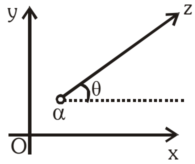
(a) $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta.$

(b) $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

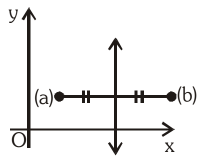
16. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(a) $\text{amp}(z-\alpha) = \theta$ is a ray emanating from the complex



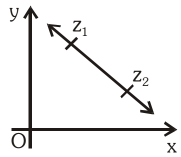
point α and inclined at an angle θ to the x– axis.

(b) $|z-a| = |z-b|$ is the perpendicular bisector of the segment joining a & b.

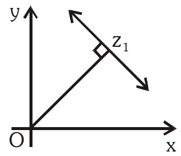


(c) The equation of a line joining z_1 & z_2 is given by ;

$z = z_1 + t\,(z_1 - z_2)$ where t is a parameter.



(d) $z = z_1\,(1 + it)$ where t is a real parameter, is a line through the point z_1 & perpendicular to z_1.



(e) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers to be collinear.

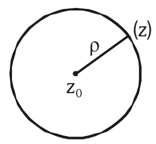
(f) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z\left(\bar{z}_1 - \bar{z}_2\right) - \bar{z}\left(z_1 - z_2\right) + \left(z_1\bar{z}_2 - \bar{z}_1 z_2\right) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(g) The equation of circle having centre z_0 & radius ρ is :

$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form

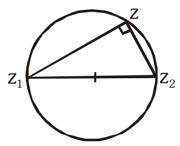
$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$, r is real centre $= -\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - r}$.

Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.



(h) $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \pm\dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

this equation represents the circle described on the line segment joining z_1 & z_2 as diameter.



(i) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\dfrac{z_3 - z_1}{z_3 - z_2}\cdot\dfrac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as

$\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow \dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$

Miscellaneous Illustration :

Illustration 27 : If z is a point on the Argand plane such that |z – 1| = 1, then $\frac{z-2}{z}$ is equal to -

(A) tan (arg z) (B) cot (arg z) (C) i tan (arg z) (D) none of these

Solution : Since |z – 1| = 1,

$\therefore$ let $z-1=\cos\theta+i\sin\theta$

Then, $z-2=\cos\theta+i\sin\theta-1$

$$=-2\sin^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \quad .... \text{(i)}$$

and $z=1+\cos\theta+i\sin\theta$

$$=2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \quad .... \text{(ii)}$$

From (i) and (ii), we get $\frac{z-2}{z}=i\tan\frac{\theta}{2}=i\tan(\arg z)\left(\because \arg z=\frac{\theta}{2} \text{ from (ii)}\right)$ **Ans. (C)**

Illustration 28 : Let a be a complex number such that |a| < 1 and $z_1, z_2, ... , z_n$ be the vertices of a polygon such that $z_k = 1+ a + a^2 + ... a^k$, then show that vertices of the polygon lie within the circle $\left|z-\frac{1}{1-a}\right|=\frac{1}{|1-a|}$.

Solution : We have, $z_k=1+a+a^2+.....+a^k=\frac{1-a^{k+1}}{1-a}$

$$\Rightarrow \quad z_k-\frac{1}{1-a}=\frac{-a^{k+1}}{1-a} \quad \Rightarrow \quad \left|z_k-\frac{1}{1-a}\right|=\frac{|a|^{k+1}}{|1-a|}<\frac{1}{|1-a|} \qquad (\because |a|<1)$$

$\therefore$ Vertices of the polygon $z_1,z_2,.....,z_n$ lie within the circle $\left|z-\frac{1}{1-a}\right|=\frac{1}{|1-a|}$

Illustration 29 : If z_1 and z_2 are two complex numbers and C > 0, then prove that

$$|z_1+z_2|^2 \le (1+C)|z_1|^2+(1+C^{-1})|z_2|^2$$

Solution : We have to prove that : $|z_1+z_2|^2 \le (1+C)|z_1|^2+(1+C^{-1})|z_2|^2$

i.e. $|z_1|^2+|z_2|^2+z_1\overline{z}_2+\overline{z}_1z_2 \le (1+C)|z_1|^2+(1+C^{-1})|z_2|^2$

or $z_1\overline{z}_2+\overline{z}_1z_2 \le C|z_1|^2+C^{-1}|z_2|^2$

or $C|z_1|^2+\frac{1}{C}|z_2|^2-z_1\overline{z}_2-\overline{z}_1z_2 \ge 0$ (using Re $(z_1\overline{z}_2)\le|z_1\overline{z}_2|$)

or $\left(\sqrt{C}|z_1|-\frac{1}{\sqrt{C}}|z_2|\right)^2 \ge 0$ which is always true.

Illustration 30 : If $\theta \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5 and $z^4\cos\theta_1+z^3\cos\theta_2+z^2\cos\theta_3+z\cos\theta_4+\cos\theta_5=2\sqrt{3}$, then show that $|z|>\frac{3}{4}$

Solution : Given that $\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5=2\sqrt{3}$

or $\left|\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5\right|=2\sqrt{3}$

$$2\sqrt{3}\le\left|\cos\theta_1.z^4\right|+\left|\cos\theta_2.z^3\right|+\left|\cos\theta_3.z^2\right|+\left|\cos\theta_4.z\right|+\left|\cos\theta_5\right|$$

$\because$ $\theta_i\in[\pi/6,\ \pi/3]$

$\therefore$ $\frac{1}{2}\le\cos\theta_i\le\frac{\sqrt{3}}{2}$

$$2\sqrt{3}\le\frac{\sqrt{3}}{2}|z|^4+\frac{\sqrt{3}}{2}|z|^3+\frac{\sqrt{3}}{2}|z|^2+\frac{\sqrt{3}}{2}|z|+\frac{\sqrt{3}}{2}$$

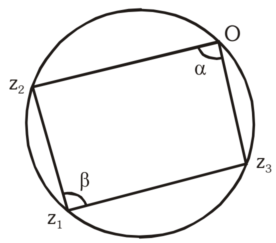
$\Rightarrow 3\le|z|^4+|z|^3+|z|^2+|z|$ $\Rightarrow 3<|z|+|z|^2+|z|^3+|z|^4+|z|^5+..........\infty$

$\Rightarrow 3<\frac{|z|}{1-|z|}$ $\Rightarrow 3-3|z|<|z|$ $\Rightarrow 4|z|>3$

$\therefore$ $|z|>\frac{3}{4}$

Illustration 31 : If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Solution : We have, $\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}\Rightarrow\frac{1}{z_1}-\frac{1}{z_2}=\frac{1}{z_3}-\frac{1}{z_1}\Rightarrow\frac{z_2-z_1}{z_1z_2}=\frac{z_1-z_3}{z_1z_3}$

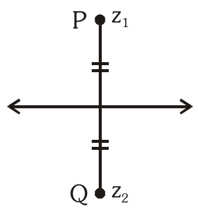


$\Rightarrow\frac{z_2-z_1}{z_3-z_1}=\frac{-z_2}{z_3}$ $\Rightarrow$ $\arg\left(\frac{z_2-z_1}{z_3-z_1}\right)=\arg\left(\frac{-z_2}{z_3}\right)$

$\arg\left(\frac{z_2-z_1}{z_3-z_1}\right)=\pi+\arg\left(\frac{z_2}{z_3}\right)$ $\Rightarrow$ or $\beta=\pi-\arg\frac{z_3}{z_2}=\pi-\alpha$ $=\alpha+\beta=\pi$

Thus the sum of a pair of opposite angle of a quadrilateral is 180°. Hence, the points 0, z_1, z_2 and z_3 are the vertices of a cyclic quadrilateral i.e. lie on a circle.

Illustration 32 : Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$, where r is real and α is non zero complex constant.



Solution : Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$$\bar{\alpha}z+\alpha\bar{z}+r=0 \qquad .... \text{(i)}$$

Now perpendicular bisector of z_1 & z_2 is, $|z-z_1|=|z-z_2|$

or $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$-z\bar{z}_1-z_1\bar{z}+z_1\bar{z}_1=-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$ ($z\bar{z}$ cancels on either side)

or $(\bar{z}_2-\bar{z}_1)z+(z_2-z_1)\bar{z}+z_1\bar{z}_1-z_2\bar{z}_2=0$ .... (ii)

Comparing (i) & (ii) $\dfrac{\bar{\alpha}}{\bar{z}_2-\bar{z}_1}=\dfrac{\alpha}{z_2-z_1}=\dfrac{r}{z_1\bar{z}_1-z_2\bar{z}_2}=\lambda$

$\therefore \quad \bar{\alpha}=\lambda\left(\bar{z}_2-\bar{z}_1\right)$ .... (iii) $\quad \alpha=\lambda\left(z_2-z_1\right)$ .... (iv)

$r=\lambda\left(z_1\bar{z}_1-z_2\bar{z}_2\right)$ .... (v)

Multiplying (iii) by z_1; (iv) by $\bar{z}_2$ and adding

$\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Note that we could also multiply (iii) by z_2 & (iv) by $\bar{z}_1$ & add to get the same result.

Hence $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Again, let $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$ is true w.r.t. the line $\bar{\alpha}z+\alpha\bar{z}+r=0$.

Subtracting $\bar{\alpha}\left(z-z_1\right)+\alpha(\bar{z}-\bar{z}_2)=0$

or $\quad \left|(z-z_1)\right|\left|\bar{\alpha}\right|=\left|\alpha\right|\left|(\bar{z}-\bar{z}_2)\right| \quad$ or $\quad \left|z-z_1\right|=\left|\overline{z-z_2}\right|=\left|z-z_2\right|$

Hence 'z' lies on the perpendicular bisector of joins of z_1 & z_2.

ANSWERS FOR DO YOURSELF

1 : **(i)** $n = 4$ **(ii)** 0

2 : **(i)** $-17 + 24i$ **(iii)** $\pm(1 - 4i)$

3 : **(i)** $|z| = 4;\ \text{amp}(z) = \dfrac{2\pi}{3}$ **(ii)** $|z|=2;\ \text{amp}(z)=-\dfrac{5\pi}{6}$ **(iii)** $|z|=2;\ \text{amp}(z)=-\dfrac{\pi}{2}$

(iv) $|z|=\dfrac{1}{\sqrt{2}};\ \text{amp}(z)=\dfrac{3\pi}{4}$ **(v)** $|z|=2;\ \text{amp}(z)=\dfrac{\pi}{3}$

4 : **(i)** 13 units **(ii)** locus is a circle on complex plane with center at (2,3) and radius 1 unit. **(iii)** C

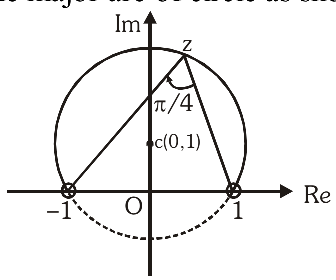
5 : **(i)** $2\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right);\ 2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(ii)** $2\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right);\ 2e^{i\left(\frac{4\pi}{3}\right)}$

(iii) $\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right);\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(iv)** $2\sin\left(\dfrac{\theta}{2}\right)\left(\cos\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)+i\sin\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\right);\ 2\sin\left(\dfrac{\theta}{2}\right)e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$

6 : **(i)** C **(ii)** D

7 : **(i)** $1+(2+2\sqrt{2})i$ **(ii)** 0 **(iii)** $z_2 = z_3 + i(z_1 - z_3)$ **(iv)** B

(v) Locus is all the points on the major arc of circle as shown excluding points 1 & –1.



8 : **(i)** C **(ii)** A **(iii)** D

9 : **(i)** D **(ii)** B

ELEMENTARY EXERCISE

1. Simplify and express the result in the form of a + bi

(a) $\left(\frac{1+2i}{2+i}\right)^2$ **CN0001**

(b) $-i(9+6i)(2-i)^{-1}$ **CN0002**

(c) $\left(\frac{4i^3-i}{2i+1}\right)^2$ **CN0003**

(d) $\frac{(2+i)^2}{2-i}-\frac{(2-i)^2}{2+i}$ **CN0004**

(e) A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$. **CN0005**

2. Given that x, y ∈ R, solve :

(a) $(x + 2y) + i(2x - 3y) = 5 - 4i$ **CN0006**

(b) $(x + iy) + (7 - 5i) = 9 + 4i$ **CN0007**

(c) $x^2 - y^2 - i(2x + y) = 2i$ **CN0008**

3. Find the square root of :

(a) $9 + 40i$ **CN0009**

(b) $-11 - 60i$ **CN0010**

(c) $50i$ **CN0011**

4. (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$ **CN0012**

(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$ **CN0013**

5. Solve the following equations over C and express the result in the form a + ib, a, b ∈ R.

(a) $ix^2 - 3x - 2i = 0$ **CN0014**

(b) $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$ **CN0015**

6. Locate the points representing the complex number z on the Argand plane :

(a) $|z+1-2i| = \sqrt{7}$ **CN0016**

(b) $|z-1|^2 + |z+1|^2 = 4$ **CN0017**

(c) $\left|\frac{z-3}{z+3}\right| = 3$ **CN0018**

(d) $|z-3| = |z-6|$ **CN0019**

7. If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'. **CN0020**

8. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1z_2z_3$ is equilateral. **CN0021**

9. For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex ? **CN0022**

10. If $(x + iy)^{1/3} = a + bi$, then prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

CN0023

11. (a) Prove the identity, $|1-z_1\bar{z}_2|^2 - |z_1 - z_2|^2 = \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$

CN0024

(b) Prove the identity, $|1+z_1\bar{z}_2|^2 + |z_1 - z_2|^2 = \left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$

CN0025

(c) For any two complex numbers, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. Also give the geometrical interpretation of this identity.

CN0026

12. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z-4| + |z+4| = 16$.

CN0027

Paragraph for question nos. 13 to 15

Consider a complex number $w = \frac{z-i}{2z+1}$, where $z = x + iy$ and $x, y \in R$.

13. If the complex number w is purely imaginary then locus of z is -

(A) a straight line

(B) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$.

(C) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin.

(D) neither a circle nor a straight line.

CN0028

14. If the complex number w is purely real then locus of z is

(A) a straight line passing through origin

(B) a straight line with gradient 3 and y intercept (–1)

(C) a straight line with gradient 2 and y intercept 1.

(D) none

CN0028

15. If $|w| = 1$ then the locus of P(z) is

(A) a point circle

(B) an imaginary circle

(C) a real circle

(D) not a circle.

CN0028

EXERCISE (O-1)

1. If $z + z^3 = 0$ then which of the following must be true on the complex plane?

(A) $\text{Re}(z) < 0$ (B) $\text{Re}(z) = 0$ (C) $\text{Im}(z) = 0$ (D) $z^4 = 1$

CN0029

2. Let $i = \sqrt{-1}$. The product of the real part of the roots of $z^2 - z = 5 - 5i$ is

(A) – 25 (B) – 6 (C) – 5 (D) 25

CN0031

3. The complex number z satisfying $z + |z| = 1 + 7i$ then the value of $|z|^2$ equals

(A) 625 (B) 169 (C) 49 (D) 25

CN0033

4. If $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i$ where $x, y \in R$ then

(A) $x = 2$ & $y = -8$ (B) $x = -2$ & $y = 8$ (C) $x = -2$ & $y = -6$ (D) $x = 2$ & $y = 8$

CN0035

5. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

(A) 1 (B) 2 (C) 4 (D) 5

CN0036

6. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(A) i (B) $i - 1$ (C) -1 (D) 0

CN0038

7. If $z = x + iy$ & $\omega = \frac{1 - iz}{z - i}$ then $|\omega| = 1$ implies that, in the complex plane

(A) z lies on the imaginary axis (B) z lies on the real axis

(C) z lies on the unit circle (D) none

CN0040

8. On the complex plane locus of a point z satisfying the inequality $2 \le |z - 1| < 3$ denotes

(A) region between the concentric circles of radii 3 and 1 centered at (1, 0)

(B) region between the concentric circles of radii 3 and 2 centered at (1, 0) excluding the inner and outer boundaries.

(C) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner and outer boundaries.

(D) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner boundary and excluding the outer boundary.

CN0041

9. The locus of z, for $\arg z = -\pi/3$ is

(A) same as the locus of z for $\arg z = 2\pi/3$

(B) same as the locus of z for $\arg z = \pi/3$

(C) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

CN0044

10. If z_1 & $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin & if $\frac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to :

(A) 8 (B) 12 (C) 16 (D) 24

CN0045

11. For $Z_1 = \sqrt[6]{\dfrac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\dfrac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\dfrac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \dfrac{3}{2}$ (B) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$ (D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

CN0047

12. Number of real or purely imaginary solution of the equation, $z^3 + iz - 1 = 0$ is :

(A) zero (B) one (C) two (D) three

CN0048

13. A point 'z' moves on the curve $|z-4-3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

CN0049

14. If z is a complex number satisfying the equation $|z+i| + |z-i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B) 4 (C) 6 (D) 8

CN0050

15. Let Z be a complex number satisfying the equation $(Z^3+3)^2 = -16$ then $|Z|$ has the value equal to

(A) $5^{1/2}$ (B) $5^{1/3}$ (C) $5^{2/3}$ (D) 5

CN0052

16. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\dfrac{3}{|z_2-z_3|} = \dfrac{4}{|z_3-z_1|} = \dfrac{5}{|z_1-z_2|}$,

then the value of $\dfrac{9}{z_2-z_3} + \dfrac{16}{z_3-z_1} + \dfrac{25}{z_1-z_2}$ equals

(A) 0 (B) 3 (C) 4 (D) 5

CN0053

17. The area of the triangle whose vertices are the roots $z^3 + iz^2 + 2i = 0$ is

(A) 2 (B) $\dfrac{3}{2}\sqrt{7}$ (C) $\dfrac{3}{4}\sqrt{7}$ (D) $\sqrt{7}$

CN0054

18. Consider two complex numbers α and β as

$\alpha = \left(\dfrac{a+bi}{a-bi}\right)^2 + \left(\dfrac{a-bi}{a+bi}\right)^2$, where a, b $\in$ R and $\beta = \dfrac{z-1}{z+1}$, where $|z| = 1$, then

(A) Both α and β are purely real (B) Both α and β are purely imaginary

(C) α is purely real and β is purely imaginary (D) β is purely real and α is purely imaginary

CN0055

19. Let Z is complex satisfying the equation $z^2 - (3+i)z + m + 2i = 0$, where m $\in$ R. Suppose the equation has a real root. The additive inverse of non real root, is

(A) 1 – i (B) 1 + i (C) – 1 – i (D) –2

CN0056

20. The minimum value of $|z-1+2i|+|4i-3-z|$ is

(A) $\sqrt{5}$ (B) 5 (C) $2\sqrt{13}$ (D) $\sqrt{15}$

CN0057

21. If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(A) $1-i\sqrt{3}$ (B) $-1+i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

CN0058

22. $z_1=\frac{a}{1-i}$; $z_2=\frac{b}{2+i}$; $z_3=a-bi$ for $a, b \in R$
if $z_1-z_2=1$ then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane is given by

(A) $\frac{1}{9}(1+7i)$ (B) $\frac{1}{3}(1+7i)$ (C) $\frac{1}{3}(1-3i)$ (D) $\frac{1}{9}(1-3i)$

CN0062

23. If P and Q are respectively by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_1}+\frac{1}{z_2}\right|=\left|\frac{1}{z_1}-\frac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\frac{z_1-z_2}{2}$ (B) $\frac{z_1+z_2}{2}$ (C) $\frac{z_1+z_2}{3}$ (D) z_1+z_2

CN0066

24. A particle starts from a point $z_0=1+i$, where $i=\sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i}+\hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1}2$ (B) $\cot^{-1}0$ (C) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\frac{-1}{2}\right)$

CN0068

25. If the complex number z satisfies the condition $|z|\geq 3$, then the least value of $\left|z+\frac{1}{z}\right|$ is equal to

(A) 5/3 (B) 8/3 (C) 11/3 (D) none of these

CN0070

26. Given $z_p=\cos\left(\frac{\pi}{2^P}\right)+i\sin\left(\frac{\pi}{2^P}\right)$, then $\lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n)=$

(A) 1 (B) -1 (C) i (D) $-i$

CN0071

27. The maximum & minimum values of $|z+1|$ when $|z+3|\leq 3$ are :

(A) (5 , 0) (B) (6 , 0) (C) (7 , 1) (D) (5 , 1)

CN0072

28. If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z-1|^2 + |2\omega - 1|^2$ equals

(A) [1, 9] (B) [2, 6] (C) [2, 12] (D) [2, 18]

CN0073

29. If $\text{Arg}(z+a) = \frac{\pi}{6}$ and $\text{Arg}(z-a) = \frac{2\pi}{3}$; $a \in R^+$, then

(A) z is independent of a (B) $|a| = |z+a|$ (C) $z = a\,\text{Cis}\,\frac{\pi}{6}$ (D) $z = a\,\text{Cis}\,\frac{\pi}{3}$

CN0074

30. If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

CN0075

31. The locus represented by the equation $|z-1| + |z+1| = 2$ is :

(A) an ellipse with focii $(1, 0)$; $(-1, 0)$

(B) one of the family of circles passing through the points of intersection of the circles $|z-1| = 1$ & $|z+1| = 1$

(C) the radical axis of the circles $|z-1| = 1$ and $|z+1| = 1$

(D) the portion of the real axis between the points $(1, 0)$; $(-1, 0)$ including both.

CN0076

[MATCH THE COLUMN]

32. Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

Column-I (equations in z)		**Column-II** (principal value of arg (z))	
(A)	$z^2 - z + 1 = 0$	(P)	$-2\pi/3$
(B)	$z^2 + z + 1 = 0$	(Q)	$-\pi/3$
(C)	$2z^2 + 1 + i\sqrt{3} = 0$	(R)	$\pi/3$
(D)	$2z^2 + 1 - i\sqrt{3} = 0$	(S)	$2\pi/3$

CN0078

EXERCISE (O-2)

1. Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is :

(A) an isosceles right angled triangle (B) a right angled triangle which is not isosceles

(C) an equilateral triangle (D) an isosceles triangle which is not right angled.

CN0079

2. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ \text{CiS}\left(\frac{\pi}{4} + \theta\right)$ where $\theta = \text{amp } z$. If 'O' is the origin, then the ΔOPQ is :

(A) isosceles but not right angled (B) right angled but not isosceles

(C) right isosceles (D) equilateral.

CN0080

3. If $z=\frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i}+\frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{\text{amp } z}\right)$ equals

(A) 1 (B) π (C) 3π (D) 4

CN0081

4. z is a complex number such that $z+\frac{1}{z}=2\cos 3°$, then the value of $z^{2000}+\frac{1}{z^{2000}}+1$ is equal to

(A) 0 (B) – 1 (C) $\sqrt{3}+1$ (D) $1-\sqrt{3}$

CN0082

5. If $z^4+1=\sqrt{3}\,i$

(A) z^3 is purely real

(B) z represents the vertices of a square of side $2^{1/4}$

(C) z^9 is purely imaginary

(D) z represents the vertices of a square of side $2^{3/4}$.

CN0084

6. Let z is a complex number satisfying the equation $Z^6+Z^3+1=0$. If this equation has a root $re^{i\theta}$ with $90° < \theta < 180°$ then the value of 'θ' is

(A) 100° (B) 110° (C) 160° (D) 170°

CN0085

7. If A and B be two complex numbers satisfying $\frac{A}{B}+\frac{B}{A}=1$. Then the two points represented by A and B and the origin form the vertices of

(A) an equilateral triangle (B) an isosceles triangle which is not equilateral

(C) an isosceles triangle which is not right angled (D) a right angled triangle

CN0086

8. If $1, \alpha_1, \alpha_2 ...., \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r+\alpha_{2009-r})$ equals

(A) 2009 (B) 2008 (C) 0 (D) – 2009

CN0087

9. If $x=\frac{1+\sqrt{3}\,i}{2}$ then the value of the expression, $y=x^4-x^2+6x-4$, equals

(A) $-1+2\sqrt{3}\,i$ (B) $2-2\sqrt{3}\,i$ (C) $2+2\sqrt{3}\,i$ (D) none

CN0088

10. (a) If $w(\neq 1)$ is a cube root of unity and $(1+w)^7=A+Bw$, then A & B are respectively the numbers

(A) 0, 1 (B) 1, 1 (C) 1, 0 (D) –1, 1

(b) If $(w \neq 1)$ is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix}=$

(A) 0 (B) 1 (C) i (D) w

CN0089

11. The straight line $(1+2i)z+(2i-1)\bar{z}=10i$ on the complex plane, has intercept on the imaginary axis equal to

(A) 5 (B) 5/2 (C) – 5/2 (D) – 5

CN0091

One ore more than one is/are correct :

12. If z is a complex number which simultaneously satisfies the equations $3|z-12|=5|z-8i|$ and $|z-4|=|z-8|$ then the Im(z) can be

(A) 15 (B) 16 (C) 17 (D) 8

CN0097

13. Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$.
Which of the following statement(s) is/are correct?

(A) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.

(B) $\sum_{k=1}^{3} \text{Amp}(z_k)=\frac{\pi}{2}$

(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}},\frac{-1}{3}\right)$

(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$

CN0098

14. If $z \in C$, which of the following relation(s) represents a circle on an Argand diagram?

(A) $|z-1|+|z+1|=3$ (B) $(z-3+i)(\bar{z}-3-i)=5$

(C) $3|z-2+i|=7$ (D) $|z-3|=2$

CN0099

15. If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots., \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i=\sqrt{-1}$) can take the value equal to

(A) 0 (B) 1 (C) i (D) (1 + i)

CN0105

16. Let point z moves on $|z-1|=1$ such that minimum & maximum value of $|z-2\sqrt{6}i|$ are m & M respectively, then-

(A) $m+M=10$ (B) $m^2+M^2=52$

(C) $m+M=8$ (D) $m^2+M^2=16$

CN0107

Paragraph for question nos. 17 to 19

Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z+1| \le 2 + \text{Re}(z)\}$, $B = \{z : |z-1| \ge 1\}$ and $C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

17. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B) 5 (C) 6 (D) 10

CN0077

18. The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

CN0077

19. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

(A) –1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) – 2

CN0077

EXERCISE (S-1)

1. Find the modulus, argument and the principal argument of the complex numbers.

(i) $6(\cos 310° - i\sin 310°)$ (ii) $-2(\cos 30° + i\sin 30°)$ (iii) $\frac{2+i}{4i+(1+i)^2}$

CN0109

2. a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$

If two roots of the equation $P(z) = 0$ are 2 and i, then find the value of 'a'.

CN0111

3. (a) Solve the following equation $z^2 - (3-2i)z = (5i-5)$ expressing your answer in the form of (a + ib).

CN0113

(b) If $(1-i)$ is a root of the equation $z^3 - 2(2-i)z^2 + (4-5i)z - 1 + 3i = 0$, then find the other two roots.

CN0114

4. (a) If $iz^3 + z^2 - z + i = 0$, then find $|z|$.

CN0115

(b) Let z_1 and z_2 be two complex numbers such that $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \ne 1$, find $|z_1|$.

CN0116

(c) Find the minimum value of the expression $E = |z|^2 + |z-3|^2 + |z-6i|^2$ (where $z = x + iy$, $x, y \in R$)

CN0117

5. Let z be a complex number such that $z \in c \backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

CN0119

6. Let $z = (0, 1) \in C$. Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

CN0120

7. Among the complex numbers z satisfying the condition $\left|z + 3 - \sqrt{3}\,i\right| = \sqrt{3}$, find the number having the least positive argument.

CN0121

8. If A, B and C are the angles of a triangle $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$, where $i = \sqrt{-1}$, then find the value of D.

CN0122

9. Dividing f(z) by $z - i$, we get the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

CN0123

10. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where $z, w \in C$ (where C is the set of complex numbers). If M and n respectively be the greatest and least modulus of w, then find the value of (2010m + M).

CN0126

EXERCISE (S-2)

1. Find the sum of the series $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) \ldots\ldots (n-1)(n-\omega)(n-\omega^2)$ where ω is one of the imaginary cube root of unity.

CN0128

2. Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

CN0131

3. Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ **CN0133**

(b) $\text{Re}\left(\frac{z+2i}{iz+2}\right) \le 4 \quad (z \ne 2i)$ **CN0134**

(c) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$ **CN0135**

(d) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$ **CN0136**

4. If the equation $(z+1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots z_7$, find the value of

(a) $\sum_{r=1}^{7} \text{Re}(z_r)$ and **CN0137**

(b) $\sum_{r=1}^{7} \text{Im}(z_r)$ **CN0138**

5. Let $z_i(i = 1, 2, 3, 4)$ represent the vertices of a square all of which lie on the sides of the triangle with vertices (0, 0), (2, 1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is $\frac{m}{n}$ (where m and n are in their lowest form). Find the value of (m + n).

CN0140

6. Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1 + i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$. (where $i = \sqrt{-1}$)

CN0141

7. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

CN0142

8. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i. Find the value of 'b'.

CN0144

EXERCISE (JM)

1. If $\left|Z - \frac{4}{Z}\right| = 2$, then the maximum value of $|Z|$ is equal to :- **[AIEEE-2009]**

(1) 2 (2) $2 + \sqrt{2}$ (3) $\sqrt{3} + 1$ (4) $\sqrt{5} + 1$

CN0188

2. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals :- **[AIEEE-2010]**

(1) 0 (2) 1 (3) 2 (4) ∞

CN0189

3. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$ **[AIEEE-2010]**

(1) –2 (2) –1 (3) 1 (4) 2

CN0190

4. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that :- **[AIEEE-2011]**

(1) $|\beta| = 1$ (2) $\beta \in (1, \infty)$ (3) $\beta \in (0, 1)$ (4) $\beta \in (-1, 0)$

CN0191

5. If $\omega(\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals :- **[AIEEE-2011]**

(1) (1, 0) (2) (–1, 1) (3) (0, 1) (4) (1, 1)

CN0192

6. If $z \neq 1$ and $\frac{z^2}{z - 1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.

(2) either on the real axis or on a circle passing through the origin.

(3) on a circle with centre at the origin.

(4) either on the real axis or on a circle not passing through the origin.

CN0145

7. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals **[JEE (Main)-2013]**

(1) –θ (2) $\frac{\pi}{2}-\theta$ (3) θ (4) $\pi-\theta$∴

CN0146

8. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$: **[JEE(Main)-2014]**

(1) is equal to $\frac{5}{2}$

(2) lies in the interval (1, 2)

(3) is strictly greater than $\frac{5}{2}$

(4) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

CN0147

9. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a : **[JEE(Main)-2015]**

(1) circle of radius 2

(2) circle of radius $\sqrt{2}$

(3) straight line parallel to x-axis

(4) straight line parallel to y-axis

CN0148

10. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main)-2016]**

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

CN0149

11. Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to :- **[JEE(Main)-2017]**

(1) 1 (2) –z (3) z (4) –1

CN0150

12. If α, β ∈ C are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to- **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

CN0151

13. Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$). Then $|z|$ is equal to :- **[JEE(Main)-2019]**

(1) $\frac{5}{4}$ (2) $\frac{\sqrt{41}}{4}$ (3) $\frac{\sqrt{34}}{3}$ (4) $\frac{5}{3}$

CN0152

14. If $\frac{z-\alpha}{z+\alpha}(\alpha \in R)$ is a purely imaginary number and |z| = 2, then a value of α is : **[JEE(Main)-2019]**

(1) 1 (2) 2 (3) $\sqrt{2}$ (4) $\frac{1}{2}$

CN0153

15. Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1| = 9$ and $|Z_2-3-4i|=4$. Then the minimum value of $|Z_1-Z_2|$ is : **[JEE(Main)-2019]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

CN0154

16. If $z = \frac{\sqrt{3}}{2}+\frac{i}{2}\left(i=\sqrt{-1}\right)$, then $\left(1+iz+z^5+iz^8\right)^9$ is equal to **[JEE(Main)-2019]**

(1) –1 (2) 1 (3) 0 (4) $\left(-1+2i\right)^9$

CN0155

17. Let z∈C be such that |z| < 1. If $\omega = \frac{5+3z}{5(1-z)}$, then :- **[JEE(Main)-2019]**

(1) $5Im(\omega) < 1$ (2) $4Im(\omega) > 5$ (3) $5Re(\omega) > 1$ (4) $5Re(\omega) > 4$

CN0156

18. If z and w are two complex numbers such that |zw| = 1 and arg(z) – arg(w) = $\frac{\pi}{2}$, then :

[JEE(Main)-2019]

(1) $\bar{z}w = i$ (2) $\bar{z}w = -i$ (3) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (4) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

CN0157

19. The equation |z–i| = |z–1|, i = $\sqrt{-1}$, represents: **[JEE(Main)-2019]**

(1) the line through the origin with slope –1. (2) a circle of radius 1.

(3) a circle of radius $\frac{1}{2}$. (4) the line through the origin with slope 1.

CN0158

EXERCISE (JA)

1. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with 0 < t < 1. If Arg(w) denotes the principal argument of a nonzero complex number w, then

[JEE 2010, 3M]

(A) $|z-z_1|+|z-z_2|=|z_1-z_2|$ (B) $Arg(z-z_1) = Arg(z-z_2)$

(C) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = 0$ (D) $Arg(z-z_1) = Arg(z_2-z_1)$

CN0164

2. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to **[JEE 2010, 3M]**

CN0165

3. Match the statements in **Column-I** with those in **Column-II.**
[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I	**Column II**
(A) The set of points z satisfying $\|z-i\|z\|\|=\|z+i\|z\|\|$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\|z+4\|+\|z-4\|=10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $\|w\|=2$, then the set of points $z=w-\frac{1}{w}$ is contained in or equal to	(t) the set of points z satisfying $\|\text{Im } z\|\le 1$
(D) If $\|w\|=1$, then the set of points $z=w+\frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\|\text{Re } z\|\le 2$
	(t) the set of points z satisfying $\|z\|\le 3$

[JEE 10, 3+3+8]
CN0166

4. **Comprehension (3 questions together)**
Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad ...(E)$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a + b + c is
(A) 0 (B) 12 (C) 7 (D) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to -
(A) –2 (B) 2 (C) 3 (D) –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is -
(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

[JEE 2011, 3+3+3]
CN0167

5. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011, 4M]**

CN0168

6. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z.$

Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011, 4M]**

CN0169

7. Match the statements given in **Column I** with the values given in **Column II**

Column I	Column II
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for	(s) π
$|z| = 1, z \neq 1$ is given by	(t) $\frac{\pi}{2}$

[JEE 2011, 2+2+2+2M]

CN0170

8. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

Column I	Column II
(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number, } |z|=1, z \neq \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set	(r) $[2, \infty)$
$\left\{f(\theta) : 0 \leq \theta < \frac{\pi}{2}\right\}$ is	(s) $(-\infty, -1] \cup [1, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10), x \geq 0$, then $f(x)$ is increasing in	(t) $(-\infty, 0] \cup [2, \infty)$

[JEE 2011, 2+2+2+2M]

CN0171

9. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - **[JEE 2012, 3M, –1M]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

CN0172

10. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ **[JEE(Advanced) 2013, 2M]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

CN0173

11. Let ω be a complex cube root of unity with ω ≠ 1 and $P = [p_{ij}]$ be a n × n matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE(Advanced) 2013, 3, (–1)]**

(A) 57 (B) 55 (C) 58 (D) 56

CN0174

12. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

CN0175

Paragraph for Question 13 and 14

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \text{Re}\, z > 0\}$.

13. $\min\limits_{z \in S} |1-3i-z| =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

CN0176

14. Area of S = **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

CN0176

15. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has

(A) only purely imaginary roots (B) all real roots

(C) two real and two purely imaginary roots (D) neither real nor purely imaginary roots.

[JEE(Advanced) 2014, 3(–1)]

CN0177

16. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9.

	List-I		List-II
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert \lvert 1-z_2\rvert ... \lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	4.	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

[JEE(Advanced) 2014, 3(–1)]

CN0178

17.

	Column-I		Column-II
(A)	In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \ge 1 \end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R)	3

(D) Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of |q – a| is (are) (S) 4

(T) 5

[JEE 2015, 8(Each 2M, –1M)]

CN0179

18. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$$ is **[JEE 2015, 4M, –0M]**

CN0180

19. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s ∈ {1, 2, 3}. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

[JEE(Advanced)-2016, 3(0)]

CN0181

20. Let a,b ∈ ℝ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where $i = \sqrt{-1}$.
If z = x + iy and z ∈ S, then (x,y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for a > 0, b ≠ 0

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for a < 0, b ≠ 0

(C) the x-axis for a ≠ 0, b = 0

(D) the y-axis for a = 0, b ≠ 0

[JEE(Advanced)-2016, 4(–2)]

CN0182

21. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?

[JEE(Advanced)-2017, 4(–2)]

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

CN0183

22. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) **FALSE** ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

CN0184

23. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

CN0185

24. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

[JEE(Advanced)-2019, 3(–1)]

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

CN0186

25. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$$

equals ____ **[JEE(Advanced)-2019, 3(0)]**

CN0187

ANSWER KEY

ELEMENTARY EXERCISE

1. **(a)** $\frac{7}{25}+\frac{24}{25}$i; **(b)** $\frac{21}{5}-\frac{12}{5}$i; **(c)** 3 + 4i; **(d)** $\frac{22}{5}$i; **(e)** 15

2. **(a)** x = 1, y = 2; **(b)** (2,9); **(c)** (–2,2) or $\left(-\frac{2}{3},-\frac{2}{3}\right)$ **3.** **(a)** ± (5 + 4i) ; **(b)** ±(5 – 6i); **(c)** ± 5(1 + i)

4. **(a)** –160 ; **(b)** – (77 +108 i) **5.** **(a)** – i, – 2i **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

6. **(a)** on a circle of radius $\sqrt{7}$ with centre (–1, 2); **(b)** on a unit circle with centre at origin
(c) on a circle with centre (–15/4, 0) & radius 9/4; **(d)** a straight line.

7. a = b = 2 $-\sqrt{3}$ **8.** $z_3=\sqrt{3}(1-i)$ and $z'_3=\sqrt{3}(-1+i)$

9. x = 1, y = –4 or x = –1, y = –4 **12.** $\frac{x^2}{64}+\frac{y^2}{48}=1$ **13.** B **14.** C **15.** C

EXERCISE (O-1)

1. B	**2.** B	**3.** A	**4.** B	**5.** D	**6.** B	**7.** B	**8.** D
9. C	**10.** A	**11.** B	**12.** A	**13.** D	**14.** B	**15.** B	**16.** A
17. A	**18.** C	**19.** C	**20.** C	**21.** C	**22.** A	**23.** B	**24.** B
25. B	**26.** B	**27.** A	**28.** D	**29.** D	**30.** A	**31.** D	

32. (A) Q,R; (B) P,S; (C) Q,S; (D) P,R

EXERCISE (O-2)

1. A **2.** C **3.** D **4.** A **5.** D **6.** C **7.** A **8.** D
9. A **10.** **(a)** B; **(b)** A **11.** A **12.** C,D **13.** A,B **14.** B,C,D
15. A,B,C,D **16.** A,B **17.** B **18.** A **19.** B

EXERCISE (S-1)

1. (i) Modulus = 6, Arg = $2k\pi+\frac{5\pi}{18}$ (k ∈ I), Principal Arg = $\frac{5\pi}{18}$

(ii) Modulus = 2, Arg = $2k\pi+\frac{7\pi}{6}$ (k ∈ I), Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\frac{\sqrt{5}}{6}$, Arg = $2k\pi-\tan^{-1}2$ (k ∈ I), Principal Arg = $-\tan^{-1}2$

2. –11/2 **3.** (a) z = (2 + i) or (1 – 3i); (b) z = 1 or 2 – i **4.** (a) 1, (b) 2, (c) 30

6. $\begin{cases}(1,\ 0) & \text{for } n=4k\\(1,\ 1) & \text{for } n=4k+1\\(0,\ 1) & \text{for } n=4k+2\\(0,\ 0) & \text{for } n=4k+3\end{cases}$ **7.** $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ **8.** – 4 **9.** $\frac{iz}{2}+\frac{1}{2}+i$ **10.** 673

EXERCISE (S-2)

1. $\left[\frac{n(n+1)}{2}\right]^2 - n$ **2.** $\pi - 2$,

3. **(a)** The region between the concentric circles with centre at (0, 2) & radii 1 & 3 units.

(b) region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$.

(c) semicircle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$.

(d) a ray emanating from the point (3 + 4i) directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

4. (a) $-\frac{7}{2}$, (b) zero **5.** 41 **6.** 26 **7.** (3 + 7i) **8.** 51

EXERCISE (JM)

1. 4 **2.** 2 **3.** 3 **4.** 2 **5.** 4 **6.** 2 **7.** 3 **8.** 2
9. 1 **10.** 1 **11.** 2 **12.** 2 **13.** 4 **14.** 2 **15.** 1 **16.** 1
17. 3 **18.** 2 **19.** 4

EXERCISE (JA)

1. A,C,D **2.** 1 **3.** (A) $\rightarrow$ (q,r), (B) $\rightarrow$ (p), (C) $\rightarrow$ (p,s,t), (D) $\rightarrow$ (q,r,s,t) **4.** **(i)** D, **(ii)** A, **(iii)** B

5. 5 **6.** Bonus **7.** (A) $\rightarrow$ (q); (B) $\rightarrow$ (p) or (p,q,r,s,t); (C) $\rightarrow$ (s); (D) $\rightarrow$ (t)

8. (A) $\rightarrow$ (s); (B) $\rightarrow$ (t); (C) $\rightarrow$ (r); (D) $\rightarrow$ (r) **9.** D **10.** C **11.** B,C,D

12. C,D **13.** C **14.** B **15.** D **16.** C

17. (A) $\rightarrow$ (P,Q); (B) $\rightarrow$ (P,Q); (C) $\rightarrow$ (P,Q,S,T); (D) $\rightarrow$ (Q,T) **18.** 4 **19.** 1 **20.** A,C,D

21. A,D **22.** A,B,D **23.** A,C,D **24.** 2 **25.** 3.00