FUNDAMENTALS OF MATHEMATICS

1. NUMBER SYSTEM :

Natural Numbers : (N) = $\{1, 2, 3....\infty\}$

Whole Numbers : (W) = $\{0, 1, 2, 3.....\infty\}$

Integers : (I) = $\{-\infty,........-3, -2, -1, 0, 1, 2, 3.....\infty\}$

Positive Integers : (I^+) = $\{1, 2, 3 ... \infty\}$

Negative Integers : (I^-) = $\{-\infty, ... -3, -2, -1\}$

Non-negative Integers : $\{0, 1, 2, 3........\}$

Non-positive Integers : $\{-\infty, ... -3, -2, -1, 0\}$

Even Integers = $\{....-6, -4, -2, 0, 2, 4, 6 ...\}$

Odd Integers = $\{-5, -3, -1, 1, 3, 5 ......\}$

Note :

(i) Zero is neither positive nor negative.

(ii) Zero is even number.

(ii) Positive means > 0.

(iv) Non-negative means $\geq$ 0.

2. FRACTION $\left(\frac{p}{q}\right)$:

(a) Proper Fraction = $\frac{3}{5}: N^r < D^r$

(b) Improper Fraction $= \frac{5}{3}: N^r > D^r$

(c) Mixed Fraction : $2+\frac{3}{5}$

(d) Compound Fraction : $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

(e) Complex Fraction : $2\frac{1}{3}$

(f) Continued Fraction : $2+\cfrac{2}{2+\cfrac{2}{+.....}}$

This is usually written in the more compact form $2+\frac{1}{2+}\;\frac{1}{2+}$ ....

3. RATIONAL NUMBERS (Q) :

All the numbers that can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers. Integers, Fractions, Terminating decimal numbers, Non-terminating but repeating decimal numbers are all rational numbers. Q = $\left\{\frac{p}{q}: p, q \in I \text{ and } q \neq 0\right\}$

Note :

(i) Integers are rational numbers, but converse need not be true.

(ii) A rational number always exists between two distinct rational numbers, hence infinite rational numbers exist between two rational numbers.

4. IRRATIONAL NUMBERS (Q^C) :

There are real numbers which can not be expressed in p/q form. Non-Terminating non repeating decimal numbers are irrational number e.g. $\sqrt{2}, \sqrt{5}, \sqrt{3}, \sqrt[3]{10}$; e, π.

e $\approx$ 2.71 is called Napier's constant and $\pi \approx 3.14$

Note :

(i) Sum of a rational number and an irrational number is an irrational number e.g. $2 + \sqrt{3}$

(ii) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.

(iii) Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.

5. REAL NUMBERS (R) :

The complete set of rational and irrational number is the set of real numbers, $R = Q \cup Q^C$. The real numbers can be represented as a position of a point on the real number line.

6. COMPLEX NUMBERS. (C) :

A number of the form a + ib, where a, b $\in$ R and $i = \sqrt{-1}$ is called a complex number. Complex number is usually denoted by z and the set of all complex numbers is represented by $C = \{(x + iy) : x, y \in R, i = \sqrt{-1}\}$

$$N \subset W \subset I \subset Q \subset R \subset C$$

7. EVEN NUMBERS :

Numbers divisible by 2, unit's digit 0, 2, 4, 6, 8 & represented by 2n.

8. ODD NUMBERS :

Not divisible by 2, last digit 1, 3, 5, 7, 9 represented by $(2n \pm 1)$

(a) even $\pm$ even = even

(b) even $\pm$ odd = odd

(c) odd $\pm$ odd = even

(d) even $\times$ any number = even number

(e) odd $\times$ odd = odd

9. PRIME NUMBERS :

Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct positive integral factors, namely 1 and itself. e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 .....

10. COMPOSITE NUMBERS :

A number that has more than two divisors

Note :

(i) '1' is neither prime nor composite.

(ii) '2' is the only even prime number.

(iii) '4' is the smallest composite number.

(iv) Natural numbers which are not prime are composite numbers (except 1)

11. CO-PRIME NUMBERS/ RELATIVELY PRIME NUMBERS :

Two natural numbers (not necessarily prime) are coprime, if their H.C.F. is one
e.g. (1, 2), (1, 3), (3, 4) (5, 6) etc.

Note :

(i) Two distinct prime number(s) are always co-prime but converse need not be true.

(ii) Consecutive natural numbers are always co-prime numbers.

12. TWIN PRIME NUMBERS :

If the difference between two prime numbers is two, then the numbers are twin prime numbers.

e.g. {3, 5}, {5, 7}, {11, 13} etc.

13. NUMBERS TO REMEMBER :

Number	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
Cube	8	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859	8000
Sq. Root	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16										

Note :

(i) Square of a real number is always non negative (i.e. $x^2 \geq 0$)

(ii) Square root of a positive number is always positive e.g. $\sqrt{4} = 2$

(iii) $\sqrt{x^2} = |x|$

14. DIVISIBILITY RULES :

Divisible by	**Remark.**
2	Last digit of number is 0, 2, 4, 6 or 8
3	Sum of digits of number divisible by 3 (Remainder will be same when number is divided by 3 or sum of digits is divided by 3.)
4	Number formed by last two digits divisible by 4 (Remainder will be same whether we divide the number or its last two digits)
5	Last digit 0 or 5
6	Divisible by 2 and 3 simultaneously.
8	Number formed by last three digits is divisible by 8 (Remainder will be same whether we divide the number or its last three digits)
9	Sum of digits divisible by 9. (Remainder will be same when number is divided by 9 or sum of digit is divided by 9)
10	Last digit 0
11	(Sum of digits at even places) – (sum of digits at odd places) = 0 or divisible by 11

15. LCM AND HCF :

(a) HCF is the highest common factor between any two or more numbers or algebraic expressions. When dealing only with numbers, it is also called "Greatest common divisor" (GCD).

(b) LCM is the lowest common multiple of two or more numbers or algebraic expressions.

(c) The product of HCF and LCM of two numbers (or expressions) is equal to the product of the numbers.

16. FACTORIZATION :

Formulae :

(a) $(a \pm b)^2 = a^2 \pm 2ab + b^2 = (a \mp b)^2 \pm 4ab$

(b) $a^2 - b^2 = (a+b)(a-b)$

- If $a^2 - b^2 = 1$ then $a + b = \frac{1}{a-b}$

For example : $\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}$ or $\sqrt{3} + \sqrt{2} = \frac{1}{\sqrt{3} - \sqrt{2}}$

(c) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(d) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(e) $a^3 + b^3 = (a+b)(a^2 - ab + b^2) = (a+b)^3 - 3ab(a+b)$

(f) $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)^3 + 3ab(a-b)$

(g) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(h) $\mathbf{a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}$

$$\mathbf{= \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}}$$

(i) $\mathbf{(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)}$

(j) $\mathbf{a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)}$

17. CYCLIC FACTORS :

If an expression remain same after replacing a by b, b by c & c by a, then it is called cyclic expression and its factors are called cyclic factors. e.g. $a(b - c) + b(c - a) + c(a - b)$

18. REMAINDER THEOREM :

If a polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + ... + a_n$ is divided by $x-p$, then the remainder is obtained by putting $x = p$ in the polynomial.

19. FACTOR THEOREM :

A polynomial $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + ... + a_n$ is divisible by $x-p$, if the remainder is zero

i.e. if $a_1p^n + a_2p^{n-1} + ... + a_n = 0$ then $x - p$ will be a factor of polynomial.

20. RATIO AND PROPORTION :

(a) If $\frac{a}{b} = \frac{c}{d}$, then : $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo); $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo);

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo); $\frac{a}{c} = \frac{b}{d}$ (alternendo); $\frac{b}{a} = \frac{d}{c}$ (invertendo)

(b) $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=......=\frac{\lambda_1 a+\lambda_2 c+\lambda_3 e...}{\lambda_1 b+\lambda_2 d+\lambda_3 f...}$, where $\lambda_1,\lambda_2,\lambda_3$......... are real numbers

(c) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=......$, then each ratio $=\left(\frac{a^n+c^n+e^n}{b^n+d^n+f^n}\right)^{\frac{1}{n}}$

Example : $\frac{a}{b}=\frac{c}{d}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$

21. INDICES AND SURDS

Important Results :

1. $a \times a \times a \times .... \times a$ (m times) $= a^m$

2. $a^m \times a^n = a^{m+n}$

3. $a^m \div a^n = a^{m-n}$

4. $(a^m)^n = a^{mn}$

5. $a^{-m}=\frac{1}{a^m}$

6. $\left(\frac{x}{y}\right)^m=\frac{x^m}{y^m}$

7. $(xy)^m = x^m.y^m$

8. $\sqrt[n]{x}=x^{1/n}$ **; $n \geq 2$, $n \in N$**

9. $a^0 = 1$

10. $a^x = a^y \Rightarrow x = y$ or $a = 1$ or $a = 0$ if $x > 0$ & $y > 0$

11. $a^x = b^x \Rightarrow a = b$ or $x = 0$

12. $a^{p/q} = (a^p)^{1/q} = (a^{1/q})^p$

13. $(x^a)^b \neq x^{a^b}$ but $= x^{ab}$ e.g. $(2^3)^2 = 2^6 = 64$ & $2^{3^2} = 2^9 = 512$

22. INTERVALS :

Intervals are basically subsets of R. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

(a) Open interval : $(a, b) = \{x : a < x < b\}$ i.e. end points are not included.

(b) Closed interval : $[a, b] = \{x : a \leq x \leq b\}$ i.e. end points are also included.

This is possible only when both a and b are finite.

(c) Semi open or semi closed interval : $(a, b] = \{x : a < x \leq b\}$; $[a, b) = \{x : a \leq x < b\}$

(d) The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$

(ii) $[a, \infty) = \{x : x \geq a\}$

(iii) $(-\infty, b) = \{x : x < b\}$

(iv) $(-\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = R$

Note :

(i) For some particular values of x, we use symbol { } e.g. If x = 1, 2 we can write it as $x \in \{1, 2\}$

(ii) If there is no values of x, then we say $x \in \phi$ (null set)

23. BASIC CONCEPTS OF GEOMETRY :

(A) BASIC THEOREMS & RESULTS OF TRIANGLES :

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both conditions are independent & necessary)
In case of a triangle, any one of the conditions is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem) :** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.
Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem :**

(i) **AAA similarity** : If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS similarity** : If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular

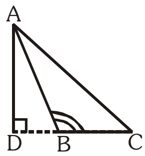
(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

(d) **Pythagoras theorem :**

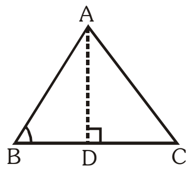
(i) In a right triangle the square of hypotenuse is equal to the sum of square of the other two sides.
Converse : In a triangle if square of one side is equal to sum of the squares of the other two sides. then the angle opposite to the first side is a right angle.

(ii) In obtuse Δ



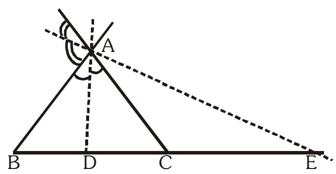
$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$

(iii) In Acute Δ



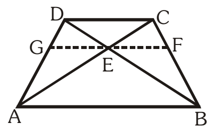
$AC^2 = AB^2 + BC^2 - 2BC \cdot BD$

(e) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true) i.e. $\frac{AB}{AC} = \frac{BD}{DC} = \frac{BE}{CE}$



(f) The line joining the mid points of two sides of a triangle is parallel & half of the third side. (It's converse is also true)

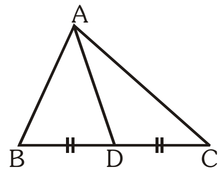
(g) (i) The diagonals of a trapezium divided each other proportionally. (converse is also true) i.e. $\frac{AE}{EC}=\frac{BE}{ED}$



(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally i.e. $\frac{DG}{GA}=\frac{CF}{FB}$

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on transversals are proportional.

(h) In any triangle the sum of squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side. i.e. $AB^2 + AC^2 = 2\left(\frac{1}{2}BC\right)^2 + 2\,(AD)^2$

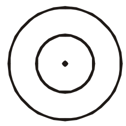


$= 2(AD^2 + BD^2)$

(i) In any triangle the three times the sum of squares of the sides of a triangle is equal to four times the sum of the square of the medians of the triangle.

(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

(B) BASIC THEOREMS & RESULTS OF CIRCLES :



(a) **Concentric circles** : Circles having same centre.

(b) **Congruent circles :** Iff their radii are equal.

(c) **Congruent arcs :** Iff they have same degree measure at the centre.

Theorem 1 :

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.

Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Converse : If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2 :

(i) The perpendicular from the centre of a circle to a chord bisects the chord.

Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3 :

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4 :

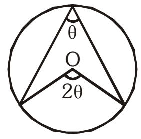
(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.

Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal.

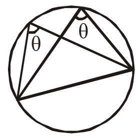
(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

(iii) Of any two chords of a circle larger will be near to centre.

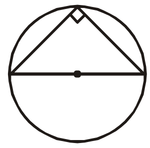
Theorem 5 :



(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.



(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.

Converse : The arc of a circle subtending a right angle in alternate segment is semi circle.

Theorem 6 :

Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7 :

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals :

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1 :

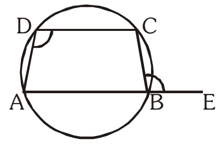
The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

OR

The opposite angles of a cyclic quadrilateral are supplementary.

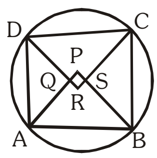
Converse : If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2 :



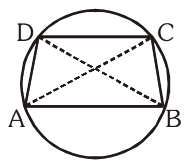
If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle. i.e. $\angle CBE = \angle ADC$

Theorem 3 :



The quadrilateral PQRS formed by angle bisectors of a cyclic quadrilateral is also cyclic.

Theorem 4 :



If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal. i.e. $AB \parallel CD \Leftrightarrow AC = BD \ \& \ AD = BC$

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse : If two non-parallel sides of a trapezium are equal, then it is cyclic.

OR

An isosceles trapezium is always cyclic.

Theorem 5 :

The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided that they are not parallel), intersect at right angle.

(C) TANGENTS TO A CIRCLE :

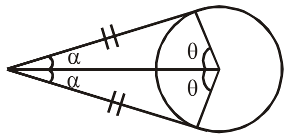
Theorem 1 :

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Theorem 2 :

If two tangents are drawn to a circle from an external point, then :



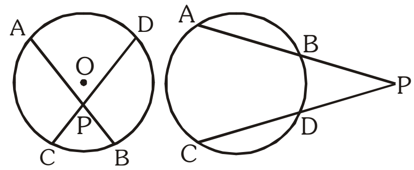
(i) they are equal.

(ii) they subtend equal angles at the centre,

(iii) they are equally inclined to the segment, joining the centre to that point.

Theorem 3 :

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord. PA × PB = PC × PD

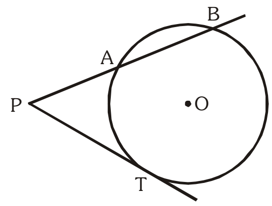


Theorem 4 :

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then PA × PB = PT^2

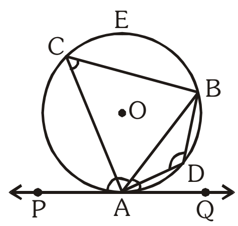
OR

Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.



Theorem 5 :

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.



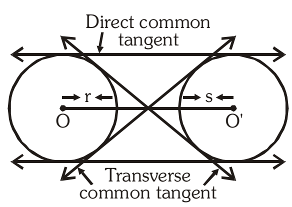
$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$

Converse :

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

(D) COMMON TANGENTS OF TWO CIRCLES :

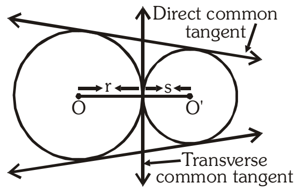
A common tangent is called direct tangent if both centres of circle lie on same side of it and called transverse tangent if centres lie on opposite side of it.



(a) When OO' > r + s i.e. the distance between the centres is greater than the sum of the radii.

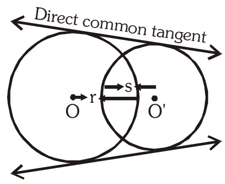
In this case, the two circles do not intersect with each other and four common tangents can be drawn to two circles. Two of them are called direct (external) common tangents and the other two are known as transverse (internal or indirect) common tangents

(b) When $OO' = r + s$ i.e. the distance between the centres is equal to the sum of the radii.



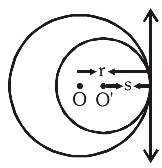
In this case, the two circles touch each other externally the common point of the two circles is called the point of contact and three common tangents can be drawn to the two circles. Two of them are direct common tangents and one transverse common tangent.

(c) When $|r - s| < OO' < r + s$ i.e. the distance between the centres is less than the sum of the radii and greater than their absolute difference.



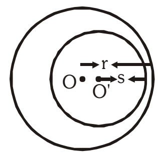
In this case, the two circles intersect in two points and there are two direct common tangents only.

(d) When $OO' = r - s$, $r > s$ i.e. the distance between the centres is equal to the difference of the radii.



In this case the two circles touch internally. The common point of the two circles is called their point of contact and there is only one common tangent to the two circles.

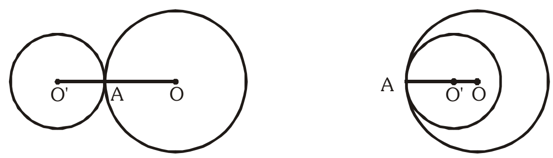
(e) When $OO' < r - s$, $r > s$ i.e. the distance between the centres is less than the difference of the radii.



In this case one circle lies inside the other and they do not touch. In such a case there is no common tangent.

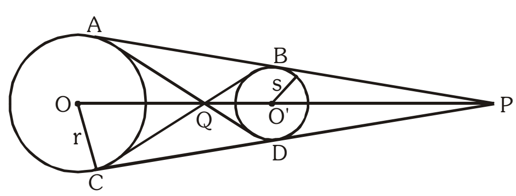
Theorem 1 :

If two circles touch each other (internally or externally) the point of contact lies on the line through the centres.



Theorem 2 :

The points of intersection of direct common tangents and transverse common tangents to two circles divide the line segment joining the two centres externally and internally respectively in the ratio of their radii.



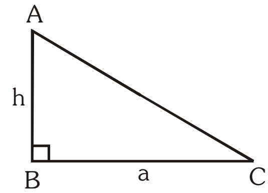
(i) P divides OO' externally in the ratio r : s i.e. $\dfrac{OP}{O'P} = \dfrac{r}{s}$

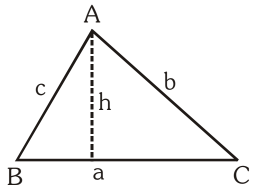
(ii) Q divides OO' internally in the ratio r : s i.e. $\dfrac{OQ}{O'Q} = \dfrac{r}{s}$

24. BASIC CONCEPT OF MENSURATION

PLANE

(A) TRIANGLE :





(a) Sum of three angle is 180°

(b) Perimeter = Sum of three sides = a + b + c = 2s

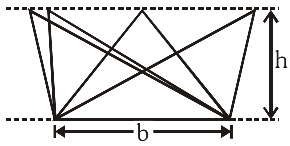
Semi perimeter s = (a + b + c)/2

(c) Area = 1/2 (Base × Height)

$= \frac{1}{2}$ (Any side × Altitude over it) $= \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Note : Area of triangles formed between two same parallel lines and on the same base is same

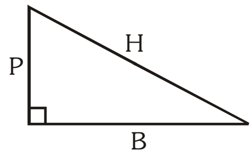
Area $= \frac{1}{2}bh$



(d) **Right Angle Triangle** : One angle 90° (Right angle)

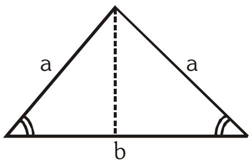
& Hypotenuse^2 = Perpendicular^2 + Base^2 (Pythagoras theorem)

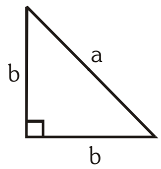
Area $= \frac{1}{2}PB$



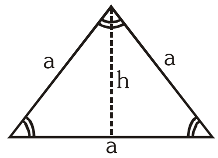
(e) **Isosceles Triangle :** Two sides equal hence two angle are equal.

Special case : Isosceles Right Triangle : Two sides equal and Base = Perpendicular.





(f) **Equilateral Triangle :** All three sides and angles (60°) are equal; $h = \left(\frac{\sqrt{3}}{2}\right)a$;

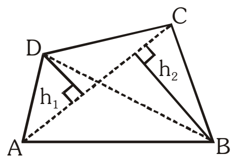


Area $= \left(\frac{1}{2}\right)$base × height $= \left(\frac{1}{2}\right)(a) \times \left(\frac{\sqrt{3}}{2}\right)a = \left(\frac{\sqrt{3}}{4}\right)a^2 = \frac{h^2}{\sqrt{3}}$

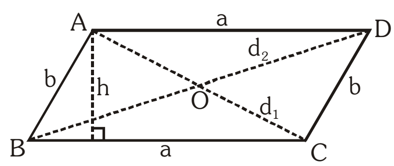
(B) QUADRILATERAL :

(a) Sum of all angles is 360°

Area $= \frac{1}{2}(AC)(h_1 + h_2)$ i.e. sum of areas of $\Delta ACD + \Delta ABC = \frac{1}{2}d_1 d_2 \sin\theta$



(b) **Parallelogram :**



(i) Opposite sides are parallel and equal.

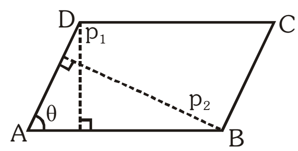
(ii) Opposite angles are equal. ($\angle B = \angle D$ and $\angle A = \angle C$)

(iii) Diagonals bisects each other. AO = OC & BO = OD

(iv) Perimeter = 2(a + b) ;

(v) Area $= \frac{1}{2}(ah) + \frac{1}{2}(ah) = ah$ i.e. sum of area of

$\Delta ACD + \Delta ABC$ also, Area $= \frac{p_1 p_2}{\sin\theta}$



(c) Special cases of parallelogram :

(i) **Rhombus :** All sides are equal and opposite angles are equal.

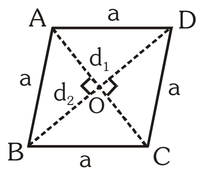
AB = BC = CD = DA = a

$\angle A = \angle C$ & $\angle B = \angle D$

Diagonals are not equal ($d_1 \neq d_2$) but bisects each other at 90°

$AC \neq BD$ but $AC \perp BD$

Area $= \frac{1}{2}(d_1 \times d_2)$ i.e. sum of areas of Δ ACD + ΔABC

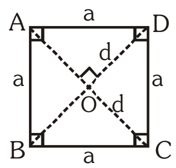


(ii) **Square :** All sides are equal and all angle are equal (90°)

Diagonals are equal and perpendicular bisectors of each other

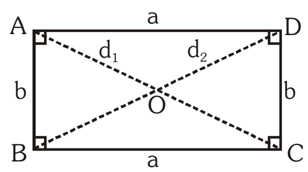
Area $= a^2 = \frac{d^2}{2}$

$AC \perp BD$ & AO = OC, BO = OD



(iii) **Rectangle :** Opposite sides are equal and parallel, all angles are equal (90°) and diagonal are equal and bisects each other but not at 90°.

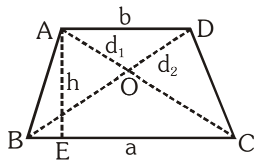
Area = a × b; Perimeter = 2(a + b)



(iv) **Trapezium** : Any two opposite sides are parallel but not equal. Diagonals cuts in same proportion. AD || BC ; AD ≠ BC ; $d_1 \neq d_2$

Area $= \left(\frac{1}{2}\right)(a + b)\ h$ i.e. sum of area of $\Delta ABC + \Delta ACD$

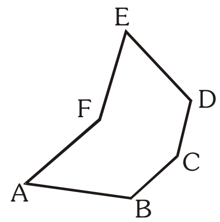
$\frac{AO}{OC} = \frac{OD}{OB}$ $(\because \ \Delta BOC \sim \Delta DOA)$



(C) POLYGON :

A plane figure enclosed by line segments (sides of polygon).

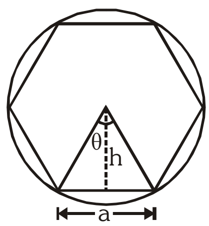
(a) n sides polygon have n sides : Triangle and quadrilaterals are polygon of three and four sides respectively. The polygons having 5 to 10 sides are called, PENTAGON, HEXAGON, HEPTAGON, OCTAGON, NANOGON and DECAGON respectively.



(b) Regular polygon : Polygon which has all equal sides and equal angles and can be inscribed in a circle whose center coincides with the center of polygon. Therefore the center is equidistant from all its vertices.

(i) A regular polygon can also circumscribe a circle.

(ii) **A 'n' sided regular polygon can be divided into 'n' Isosceles Congruent Triangles with a common vertex i.e. centre of polygon.**



(iii) Area $= n \times \left(\frac{1}{2}\right) \times a \times h$

(iv) Perimeter = na

(v) Each interior angle of polygon = $\left(\frac{n-2}{n}\right)\times 180°$

(vi) Angle subtended at the centre of inscribed/circumscribed circle by one side = 360°/n

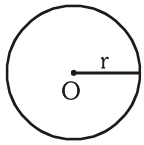
(vii) Each exterior angle $=\left(\frac{360}{n}\right)^{o}$

(viii) Sum of all interior angle = (n – 2) × 180°

(ix) Sum of all exterior angles = 360°

(x) **Convex polygon :** If any two consecutive vertices are joined then remaining all other vertices will lie on same side.

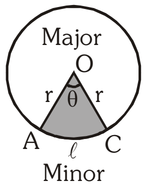
(D) CIRCLE :



Area A = πr^2 ; Circumference (perimeter) = $2\pi r$

(a) Sector of a circle : Bounded by arc of circle (subtending angle 'θ' at center) and two radii. Circle is divided into minor (containing 'θ') and major sectors

(i) Arc length of sector : $\ell = \left(\frac{\theta°}{360°}\right)2\pi r$



(ii) Area : $A = \left(\frac{\theta°}{360°}\right)\pi r^2 = \left(\frac{1}{2}\right)\ell r$

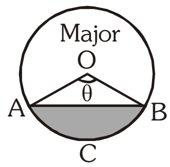
(iii) Perimeter of sector AOC = $2r + \ell$

(b) Segment of a circle : Bounded by arc of the circle and the chord (determining the segment).

(i) Circle is divided into two segments minor segment and major segment.

(ii) When chord is diameter, sector coincides with segment.

(iii) Area (segment ACB) = Area of sector OACB - Area of ΔAOB



$=\left(\frac{\theta°}{360°}\right)\times \pi r^2 - \left(\frac{1}{2}\right)\times\left(2r\sin\frac{\theta}{2}\right)\times\left(r\cos\frac{\theta}{2}\right)$

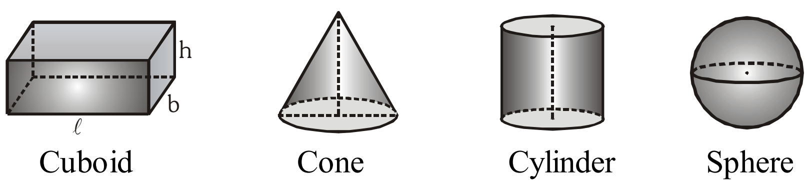
$\text{Area} = \left(\frac{\theta°}{360°}\right)\pi r^2 - \left(\frac{1}{2}\right) r^2 \sin\theta$

SOLIDS

Require three dimension to describe

(a) Surfaces of solids : Plane areas bounding the solid e.g. six rectangle faces bounding a brick. Surface area is measured in square units.

(b) Volume of solids : Space occupied by a solid and is measured in cubic units.



(A) CUBOID :

Rectangular shaped solid also known as rectangular parallelopiped (e.g. match box, brick)

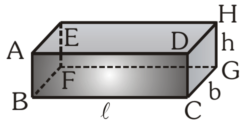
(a) Have six rectangular faces with opposite faces parallel and congruent.

(b) Have twelve edges (Edge - The line segment where two adjacent faces meets).

(c) Three adjacent faces meet at a point called vertex and cuboid have eight vertices

(d) **Surface area :** $A = 2[\ell \times b + b \times h + h \times \ell]$ square unit.

(e) **Volume :** $V = l \times b \times h$ cubic unit.

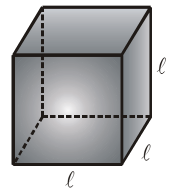


(B) CUBE :

Special case of cuboid having all sides equal.

Area = $6\ell^2$; Volume = ℓ^3 Unit cube : Side $\ell = 1$

Volume is 1 cubic unit (From this cubic unit is derived)



(C) CYLINDER :

Having a lateral (curved) surface and two congruent circular cross section.

(e.g. Jar, Circular Pillars, Drums, Pipes etc.)

(a) **Axis :** Line joining the centers of two circular cross section.

(b) **Right circular cylinder :** When axis is perpendicular to circular cross section.

(c) **Generators :** Lines parallel to axis and lying on the lateral surface.

(d) **Base :** With cylinder in vertical position, the lower circular end is base.

(e) **Height (h) :** Distance between two circular faces.

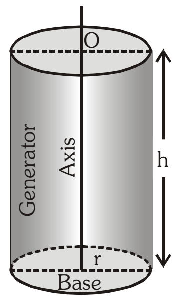
(f) **Radius (r) :** Radius of base or top circle.

(g) **Total surface area :** Base area + curved surface area

$= 2\pi r^2 + 2\pi rh = 2\pi r(h + r)$(including two circular ends).

Without circular ends (Hollow cylinder) = $2\pi rh$

(h) **Volume :** $V = \pi r^2 h$



(D) CONE :

Have a curved surface with a vertex (V) and circular base radius : r and center O)

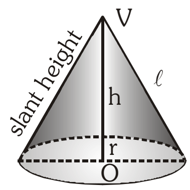
(a) **Axis :** Line joining vertex and center of base circle (VO)

(b) **Height of cone (h) :** Length of VO

(c) **Slant height (Q) :** Distance of vertex from any point of base circle

$\ell = \sqrt{r^2 + h^2}$

(d) **Right circular cone :** When axis is perpendicular to base.



(e) The cross section of a cone parallel to base is a circle and perpendicular to base is an isosceles triangle.

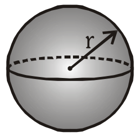
(f) **Volume :** $(1/3)\pi r^2 h$ (volume of a cone is 1/3rd of volume of a cylinder with same height and base radius).

(g) Curved surface Area : $\pi r\ell$

(h) Total surface Area : $\pi r\ell + \pi r^2 = \pi r(\ell + r)$

(i) A right circular cone can be generated by rotating a right angled triangle about its right angle forming side.

(E) SPHERE :

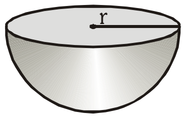


All point on its surface are equidistant from its center, the distance is called radius (r) and any line passing through center with end points on surface is called diameter.

(a) **Volume :** $(4/3)\pi r^3$

(b) **Surface area :** $4\pi r^2$

(F) HEMISPHERE :



A sphere is divided into two hemi spheres by a plane passing through center.

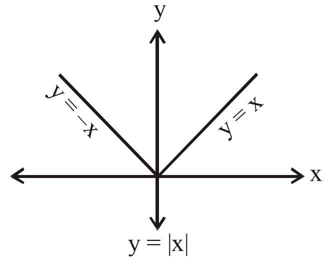
(a) **Volume :** $(2/3)\pi r^3$

(b) **Curved surface area :** $= 2\pi r^2$

(c) **Total surface area :** $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

MODULUS

1. ABSOLUTE VALUE FUNCTION/MODULUS FUNCTION :



The symbol of modulus function is |x|

and is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Properties of Modulus :

For any a, b ∈ R

(a) $|a| \geq 0$ (b) $|a| = |-a|$

(c) $|ab| = |a||b|$ (d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(e) $|a + b| \leq |a|+|b|$ (f) $|a|-|b| \leq |a - b|$

(g) $|a + b| = |a| + |b| \Rightarrow ab \geq 0$ (h) $\sqrt{x^2} = |x|$

Illustration 1 : If $||x-1| - 2| = 5$, then find x.

Solution : $|x - 1| - 2 = \pm 5$

$|x - 1| = 7, -3$

Case-I : When $|x - 1| = 7 \Rightarrow x - 1 = \pm 7 \Rightarrow x = 8, -6$

Case-II : When $|x - 1| = -3$ (reject)

Illustration 2 : If $|x - 1| + |x + 1| = 2$, then find x.

Solution : **Case-I :** If $x \leq -1$

$-(x - 1) - (x + 1) = 2$

$\Rightarrow -x + 1 - x - 1 = 2$

$\Rightarrow -2x = 2 \Rightarrow x = -1$ ....(i)

Case-II : If $-1 < x < 1$

$-(x - 1) + (x + 1) = 2$

$\Rightarrow -x + 1 + x + 1 = 2$

$\Rightarrow 2 = 2 \Rightarrow -1 < x < 1$ ....(ii)

Case-III : If $x \geq 1$

$x - 1 + x + 1 = 2$

$\Rightarrow x = 1$ ....(iii)

Thus from (i), (ii) and (iii) $-1 \leq x \leq 1$

Do yourself - 1 :

(i) Solve : $|x + 3| = 2(5 - x)$

(ii) Solve : $x|x| + 7x - 8 = 0$

2. INEQUALITIES INVOLVING MODULUS FUNCTION :

Properties of modulus function :

(i) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.

(ii) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $\left||a|-|b|\right| \leq |a \pm b| \leq |a|+|b|$

(v) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Illustration 3 : If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

(A) $0 \leq x \leq 4$ (B) $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ (D) none of these

Solution : **Case I :** $x \leq 1$, then

$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$

Hence $x \leq 0$ ...(i)

Case II : $1 < x \leq 2$, then

$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$

But $1 < x \leq 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \leq 3$, then

$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$

But $2 < x \leq 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$.

Illustration 4 : Solve for x : (a) $||x - 1| + 2| \leq 4$. (b) $\frac{x-4}{x+2} \leq \left|\frac{x-2}{x-1}\right|$

Solution : (a) $||x - 1| + 2| \leq 4 \Rightarrow -4 \leq |x - 1| + 2 \leq 4$

$\Rightarrow -6 \leq |x - 1| \leq 2$

$\Rightarrow |x - 1| \leq 2 \Rightarrow -2 \leq x - 1 \leq 2$

$\Rightarrow -1 \leq x \leq 3 \Rightarrow x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\frac{x-4}{x+2} \leq 0 \Rightarrow x \in (-2, 4] - \{1\}$ ....(i)

(Note : {1} is not in domain of RHS)

Case 2 : $\frac{x-4}{x+2} > 0 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty)$ ....(ii)

Given inequation becomes

$\frac{x-2}{x-1} \geq \frac{x-4}{x+2}$ or $\frac{x-2}{x-1} \leq -\frac{x-4}{x+2}$

on solving we get | on solving we get

$x \in (-2, 4/5) \cup (1, \infty)$ | $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get | taking intersection with (ii) we get

$x \in (4, \infty)$ ....(iii) | $x \in \phi$

Hence, solution of the original inequation : $x \in (-2,\infty) - \{1\}$ (taking union of (i) & (iii))

Illustration 5: The equation $|x| + \left|\frac{x}{x-1}\right| = \frac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\frac{x^2}{|x-1|} = \left|x + \frac{x}{x-1}\right|$

$\because |x| + \left|\frac{x}{x-1}\right| = \left|x + \frac{x}{x-1}\right|$ is true only if $\left(x.\frac{x}{x-1}\right) \geq 0 \Rightarrow x \in \{0\} \cup (1, \infty)$. **Ans (A,B)**

3. IRRATIONAL INEQUALITIES :

Illustration 6 : Solve for x, if $\sqrt{x^2-3x+2} > x-2$

Solution :

$$\left[\begin{array}{l} \begin{cases} x^2-3x+2 \geq 0 \\ x-2 \geq 0 \\ (x^2-3x+2) > (x-2)^2 \end{cases} \\ \text{or} \\ \begin{cases} x^2-3x+2 \geq 0 \\ x-2 < 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} (x-1)(x-2) \geq 0 \\ (x-2) \geq 0 \\ x-2 > 0 \end{cases} \Rightarrow x > 2 \\ \text{or} \\ \begin{cases} (x-1)(x-2) \geq 0 \\ x-2 < 0 \end{cases} \Rightarrow x \leq 1 \end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself - 2 :

(i) Solve for x if $\frac{|x^2-4|}{x^2+x-2} > 1$

(ii) Solve for x if $\sqrt{x^2-x} > (x-1)$

LOGARITHM

4. DEFINITION :

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent. We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$

Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number.

Limitations of logarithm: $\log_a N$ is defined only when

(i) $N > 0$ (ii) $a > 0$ (iii) $a \neq 1$

Note :

(i) For a given value of N, $\log_a N$ will give us a unique value.

(ii) Logarithm of zero does not exist.

(iii) Logarithm of negative reals are not defined in the system of real numbers.

Illustration 7 : If $\log_4 m = 1.5$, then find the value of m.

Solution : $\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 8 : If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Illustration 9 : The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : $1 + \log_b\{1 + \log_c(1 + \log_p N)\} = a^0 = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0 \Rightarrow 1 + \log_c(1 + \log_p N) = 1$

$\Rightarrow \log_c(1 + \log_p N) = 0 \Rightarrow 1 + \log_p N = 1$

$\Rightarrow \log_p N = 0 \Rightarrow N = 1$ **Ans. (D)**

Do yourself - 3 :

(i) Express the following in logarithmic form :

(a) $81 = 3^4$ (b) $0.001 = 10^{-3}$ (c) $2 = 128^{1/7}$

(ii) Express the following in exponential form :

(a) $\log_2 32 = 5$ (b) $\log_{\sqrt{2}} 4 = 4$ (c) $\log_{10} 0.01 = -2$

(iii) If $\log_{2\sqrt{3}} 1728 = x$, then find x.

5. FUNDAMENTAL IDENTITIES :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

Note : $N = (a)^{\log_a N}$ e.g. $2^{\log_2 7} = 7$

Do yourself - 4 :

(i) Find the value of the following :

(a) $\log_{1.4\bar{3}} \frac{43}{30}$ **(b)** $\left(\frac{1}{2}\right)^{\log_2 5}$

(ii) If $4^{\log_2 2x} = 36$, then find x.

6. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where a > 0, a ≠ 1 and x is any real number, then-

(a) $\log_a mn = \log_a m + \log_a n$

(b) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(c) $\log_a m^x = x \log_a m$

Illustration 10 : Find the value of $2\log\frac{2}{5} + 3\log\frac{25}{8} - \log\frac{625}{128}$

Solution :

$$2\log\frac{2}{5} + 3\log\frac{25}{8} + \log\frac{128}{625}$$

$$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$$

$$= \log\frac{2^2}{5^2}\cdot\frac{5^6}{2^9}\cdot\frac{2^7}{5^4} = \log 1 = 0$$

Illustration 11 : If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of

$$\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$$

Solution :

$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$

$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$

$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$

$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$

$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$

Illustration 12 : If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

L.H.S. $= \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) =$ R.H.S.

Illustration 13 : If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$)-

(A) $\frac{1}{p} + \frac{1}{q}$ (B) $\frac{1}{2p} + \frac{1}{q}$ (C) $\frac{1}{p} + \frac{1}{2q}$ (D) $\frac{1}{2p} + \frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p} + \frac{1}{q}$

Do yourself - 5 :

(i) Show that $\frac{1}{2}\log 9 + 2\log 6 + \frac{1}{4}\log 81 - \log 12 = 3\log 3$

7. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Note :

(i) $\log_b a \cdot \log_a b = \frac{\log a}{\log b} \cdot \frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}}$.

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) **Base power formula :** $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are called Natural or Napierian logarithm. **We will consider logx as $\log_e x$ or ℓnx.**

(v) Conversion of base e to base 10 & viceversa :

$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a$; $\log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$

Illustration 14 : If a, b, c are distinct positive real numbers different from 1 such that $(\log_b a \cdot \log_c a - \log_a a) + (\log_a b \cdot \log_c b - \log_b b) + (\log_a c \cdot \log_b c - \log_c c) = 0$, then abc is equal to -

(A) 0 (B) e (C) 1 (D) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \cdot \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}\cdot\frac{\log a}{\log c}+\frac{\log b}{\log a}\cdot\frac{\log b}{\log c}+\frac{\log c}{\log a}\cdot\frac{\log c}{\log b}=3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 15 : Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$

$= 625 + 216 + 49 = 890.$

Do yourself - 6 :

(i) Evaluate : $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$

(ii) Evaluate : $\log_9 27 - \log_{27} 9$

(iii) Evaluate : $2^{\log_3 5} - 5^{\log_3 2}$

(iv) Evaluate : $\log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9$

(v) If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, then x can be -

(A) 2 (B) 3 (C) 3.5 (D) π

(vi) If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

8. POINTS TO REMEMBER :

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

$$\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$$

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10}\sqrt[3]{10}=\frac{1}{3}$; $\log_{\sqrt{7}}49=4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)=3$; $\log_2\left(\frac{1}{32}\right)=-5$; $\log_{10}(0.001) = -3$

(iii) $x+\frac{1}{x}\ge 2$ if x is positive real number and $x+\frac{1}{x}\le -2$ if x is negative real number

(iv) $n \ge 2,\ n \in N$

$\sqrt[n]{a}=a^{1/n} \Rightarrow$ n^{th} root of 'a' ('a' is a non negative number)

Some important values : $\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$; $\ell n2 = 0.693$, $\ell n10 = 2.303$

9. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \ge 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \le m < 1)$

(ii) If the characteristic of $\log_{10}N$ be n, then the number of digits in N is (n + 1)

(iii) If the characteristic of $\log_{10}N$ be (–n), then there exist (n – 1) zeros after decimal in N.

10. ANTILOGARITHM :

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, **log n = m ⇔ n = antilog m**

Do yourself - 7 :

(i) Evaluate : $\log_{10}(0.06)^6$

(ii) Find number of digits in 18^{20}

(iii) Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$

(iv) Find antilog of $\frac{5}{6}$ to the base 64.

11. LOGARITHMIC INEQUALITIES :

Points to remember :

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x<y & \text{if} \quad a>1 \\ x>y & \text{if} \quad 0<a<1 \end{cases}$

(ii) If a > 1, then (a) $\log_a x < p \Rightarrow 0 < x < a^p$ (b) $\log_a x > p \Rightarrow x > a^p$

(iii) If 0 < a < 1, then (a) $\log_a x < p \Rightarrow x > a^p$ (b) $\log_a x > p \Rightarrow 0 < x < a^p$

Illustration 16 : Solve for x : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1$ (b) $\log_{1/3}(\log_4(x^2 - 5)) > 0$

Solution : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow \quad 0 < x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2 - 5x + 6 > 0 \\ x^2 - 5x + 6 \leq 2 \end{cases} \Rightarrow x \in [1,2) \cup (3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

(b) $\log_{1/3}(\log_4(x^2 - 5)) > 0 \Rightarrow 0 < \log_4 (x^2 - 5) < 1$

$$\begin{cases} 0 < \log_4(x^2 - 5) \Rightarrow x^2 - 5 > 1 \\ \log_4(x^2 - 5) < 1 \Rightarrow 0 < x^2 - 5 < 4 \end{cases} \Rightarrow \quad 1 < (x^2 - 5) < 4$$

$\Rightarrow \quad 6 < x^2 < 9 \quad \Rightarrow \quad x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$

Hence, solution set of original inequation : $x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$

Illustration 17 : Solve for x : $\log_2 x \leq \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \leq \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \leq 0$$

$$\Rightarrow \quad \frac{t^2 - t - 2}{t-1} \leq 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \leq 0$$

$\Rightarrow \quad t \in (-\infty, -1] \cup (1,2]$

or $\quad \log_2 x \in (-\infty, -1] \cup (1,2]$

or $\quad x \in \left(0, \dfrac{1}{2}\right] \cup (2,4]$

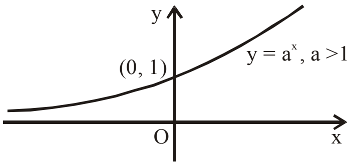
Illustration 18 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

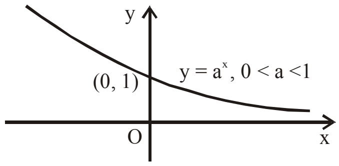
Solution : This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x+3>1 \\ 0<x^2<2x+3 \end{cases} \\ \text{or} \\ \begin{cases} 0<2x+3<1 \\ x^2>2x+3>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ (x-3)(x+1)<0 \ \& \ x \neq 0 \end{cases} \\ \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ (x-3)(x+1)>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ -1<x<3 \end{cases} \Rightarrow -1<x<3 \ \& \ x \neq 0 \\ \text{or} \qquad\qquad\qquad\qquad \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ x<-1 \text{ or } x>3 \end{cases} \Rightarrow -\frac{3}{2}<x<-1 \end{array}\right.$$

Hence, solution of the original inequation is $x \in \left(-\dfrac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

12. EXPONENTIAL INEQUATIONS :





$$\text{If } a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b \text{ when } a > 1 \\ f(x) < \log_a b \text{ when } 0 < a < 1 \end{cases}$$

Illustration 19 : Solve for x : $2^{x+2} > \left(\frac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base 2 > 1, we have $x + 2 > -\frac{2}{x}$

(the sign of the inequality is retained).

Now $x + 2 + \frac{2}{x} > 0 \quad \Rightarrow \quad \frac{x^2 + 2x + 2}{x} > 0$

$\Rightarrow \quad \frac{(x+1)^2 + 1}{x} > 0 \quad \Rightarrow \quad x \in (0, \infty)$

Illustration 20 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0 < \frac{4}{5} < 1$, the inequality is equivalent to the inequality $x - 1 > 4(1+\sqrt{x})$

$\Rightarrow \quad \frac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \quad \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5 \qquad$ ....(i)

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \quad \frac{(x-5)^2}{16} > x \quad$ or $\quad \frac{(x-5)^2}{16} - x > 0$

or $\quad x^2 - 26x + 25 > 0 \quad$ or $\quad (x - 25)(x - 1) > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty) \qquad$ ....(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

Do yourself-8 :

(i) Solve for x : (a) $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ (b) $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

(ii) Solve for x : $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

Miscellaneous Illustrations :

Illustration 21 : Show that $\log_4 18$ is an irrational number.

Solution : $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow 3 = 2^{p/q} \Rightarrow 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 22 : If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$\log_{c+b}a + \log_{c-b}a = 2\log_{c+b}a \cdot \log_{c-b}a$.

Solution : We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ .... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2-b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \quad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)}a \cdot \log_{(c-b)}a = \text{RHS}$$

ANSWERS FOR DO YOURSELF

1 : **(i)** $\dfrac{7}{3}$ **(ii)** $x = 1$

2 : **(i)** $x \in (-\infty, -2) \cup (1, 3/2)$ **(ii)** $x \in R - (0,1]$

3 : **(i)** **(a)** $\log_3 81 = 4$ **(b)** $\log_{10}(0.001) = -3$ **(c)** $\log_{128} 2 = 1/7$

(ii) **(a)** $32 = 2^5$ **(b)** $4 = (\sqrt{2})^4$ **(c)** $0.01 = 10^{-2}$

(iii) 6

4 : **(i)** **(a)** 1 **(b)** $\dfrac{1}{5}$ **(ii)** 3

6 : **(i)** 3 **(ii)** 5/6 **(iii)** 0 **(iv)** 2 **(v)** (A) **(vi)** (C)

7 : **(i)** $\bar{8}.6686$ **(ii)** 26 **(iii)** 155 **(iv)** 32

8 : **(i)** (a) $x \in (1,8)$ (b) $x \in (-\infty, 1/2)$ **(ii)** $x \in (-1,1)$

EXERCISE (O-1)

1. Sum of roots of the equation $(x+3)^2 - 4|x+3| + 3 = 0$ is -

(A) 4 (B) 12 (C) –12 (D) –4 LG0111

2. The solution of the inequation $|x^2 - 2x - 3| < |x^2 - x + 5|$ is -

(A) $(-\infty, 5)$ (B) $(-\infty, 2) \cup (3, 8) \cup (8, \infty)$

(C) $(-8, \infty)$ (D) $(3, 8)$ LG0112

3. The minimum value of $f(x) = |x-1| + |x-2| + |x-3|$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0 LG0113

4. The complete solution set of the inequation $\sqrt{x+18} < 2 - x$ is -

(A) $[-18, -2]$ (B) $(-\infty, -2) \cup (7, \infty)$ (C) $(-18, 2) \cup (7, \infty)$ (D) $[-18, -2)$ LG0114

5. Solution of the inequality, $x - 3 < \sqrt{x^2 + 4x - 5}$ is -

(A) $(-\infty, -5] \cup [1, \infty)$ (B) $(-5, 3]$ (C) $(-\infty, -5] \cup \left(\frac{7}{5}, \infty\right)$ (D) $\left(\frac{7}{5}, \infty\right)$ LG0115

6. If $\log_y x + \log_x y = 7$, then the value of $(\log_y x)^2 + (\log_x y)^2$, is

(A) 43 (B) 45 (C) 47 (D) 49 LG0116

7. If $\log_2(\log_3(\log_4(x))) = 0$, $\log_3(\log_4(\log_2(y))) = 0$ and $\log_4(\log_2(\log_3(z))) = 0$, then the sum of x, y and z is-

(A) 89 (B) 58 (C) 105 (D) 50 LG0117

8. $\log_{10}(\log_2 3) + \log_{10}(\log_3 4) + \log_{10}(\log_4 5) + ... + \log_{10}(\log_{1023} 1024)$ simplifies to

(A) a composite (B) a prime number

(C) rational which is not an integer (D) an integer LG0118

9. If $\log_9 x + \log_4 y = \frac{7}{2}$ and $\log_9 x - \log_8 y = -\frac{3}{2}$, then x + y equals

(A) 35 (B) 41 (C) 67 (D) 73 LG0119

10. If $\log_a b + \log_b c + \log_c a$ vanishes where a, b and c are positive reals different than unity then the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$ is LG0120

(A) an odd prime (B) an even prime (C) an odd composite (D) an irrational number

11. Let $x = 2^{\log 3}$ and $y = 3^{\log 2}$ where base of the logarithm is 10, then which one of the following holds good ?

(A) $2x < y$ (B) $2y < x$ (C) $3x = 2y$ (D) $y = x$ LG0121

12. The sum of all the solutions to the equation $2\log_{10} x - \log_{10}(2x - 75) = 2$

(A) 30 (B) 350 (C) 75 (D) 200 LG0122

13. The value of 'a' for which $\frac{\log_a 7}{\log_6 7} = \log_\pi 36$ holds good, is

(A) $1/\pi$ (B) π^2 (C) $\sqrt{\pi}$ (D) 2 LG0123

14. Let W,X,Y and Z be positive real numbers such that

$\log(W.Z) + \log(W.Y) = 2; \log(Y.Z) + \log(Y.X) = 3; \log(X.W) + \log(X.Z) = 4.$

The value of the product (WXYZ) equals (base of the log is 10)

(A) 10^2 (B) 10^3 (C) 10^4 (D) 10^9 **LG0124**

15. If $\sum_{r=0}^{n-1} \log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99} \log_r(r+1)$, then 'n' is equal to

(A) 4 (B) 3 (C) 5 (D) 6 **LG0125**

16. The number $N = 6\log_{10}2 + \log_{10}31$, lies between two successive integers whose sum is equal to

(A) 5 (B) 7 (C) 9 (D) 10 **LG0126**

17. Number of real solution(s) of the equation $|x-3|^{3x^2-10x+3} = 1$ is - **LG0127**

(A) exactly four (B) exactly three (C) exactly two (D) exactly one

18. The solution set of the inequality $\log_{\frac{5}{8}}\left(2x^2 - x - \frac{3}{8}\right) \geq 1$ is -

(A) $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ (B) $\left[-\frac{1}{2}, 1\right]$

(C) $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ (D) $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$ **LG0128**

19. Solution set of the inequality, $2 - \log_2(x^2 + 3x) \geq 0$ is - **LG0129**

(A) $[-4, 1]$ (B) $[-4, -3) \cup (0, 1]$ (C) $(-\infty, -3) \cup (1, \infty)$ (D) $(-\infty, -4) \cup [1, \infty)$

20. If $\log_{1/3}\left(\frac{3x-1}{x+2}\right)$ is less than unity then x must lie in the interval -

(A) $(-\infty, -2) \cup (5/8, \infty)$ (B) $(-2, 5/8)$

(C) $(-\infty, -2) \cup (1/3, 5/8)$ (D) $(-2, 1/3)$ **LG0130**

EXERCISE (S-1)

1. Solve the following equations where $x \in R$.

(a) $(x-1)|x^2-4x+3|+2x^2+3x-5=0$ **LG0131**

(b) $|x^2+4x+3|+2x+5=0$ **LG0132**

(c) $|x+3|(x+1)+|2x+5|=0$ **LG0133**

MATCH THE COLUMN

2. Consider the function $f(x) = |x-1| - 2|x+2| + |x+3|$

Column-I	Column-II
(A) If $f(x) = k$ has no solution, then $k \in$	(p) $(2, 4)$
(B) If $f(x) = k$ has one solution, then $k \in$	(q) $(-\infty, -2) \cup (4, \infty)$
(C) If $f(x) = k$ has two solution, then $k \in$	(r) $(-2, 2) \cup \{4\}$
(D) If $f(x) = k$ has more then two solution, then $k \in$	(s) $\{-2, 2\}$

LG0134

3. Find the square of the sum of the roots of the equation

$\log_3 x . \log_4 x . \log_5 x = \log_3 x . \log_4 x + \log_4 x . \log_5 x + \log_5 x . \log_3 x$. **LG0048**

4. **(a)** Calculate : $4^{5\log_{4\sqrt{2}}\left(3-\sqrt{6}\right)-6\log_8\left(\sqrt{3}-\sqrt{2}\right)}$ **LG0049**

(b) Simplify : $\dfrac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}.\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}}-\left(125\right)^{\log_{25} 6}\right)$ **LG0050**

(c) Find the value of $49^{(1-\log_7 2)}+5^{-\log_5 4}$. **LG0053**

(d) Find the value of the expression $\dfrac{2}{\log_4 (2000)^6}+\dfrac{3}{\log_5 (2000)^6}$ **LG0072**

5. If a,b and c are positive real numbers such that $a^{\log_3 7}=27; b^{\log_7 11}=49$ and $c^{\log_{11} 25}=\sqrt{11}$. Find the value of $\left(a^{(\log_3 7)^2}+b^{(\log_7 11)^2}+c^{(\log_{11} 25)^2}\right)$. **LG0074**

6. Solve for x :

(a) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)}=\dfrac{1}{2}$ **LG0135**

(b) $\log(\log x)+\log(\log x^3-2)=0$; where base of log is 10. **LG0136**

(c) $\log_x 2 . \log_{2x} 2 = \log_{4x} 2$ **LG0137**

(d) $5^{\log x}+5\,x^{\log 5}=3$ $(a>0)$; where base of log is a. **LG0138**

(e) If $9^{1+\log x}-3^{1+\log x}-210=0$; where base of log is 3. **LG0139**

7. If $x,y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2} = \sqrt{N}$ where N is a natural number, find the value of N. **LG0102**

8. Solve the system of equations :

$$\log_a x \log_a(xyz) = 48$$
$$\log_a y \log_a(xyz) = 12,\ a > 0,\ a \neq 1$$
$$\log_a z \log_a(xyz) = 84$$

LG0088

9. Solve the equation for x : $x^{\log x + 4} = 32$, where base of logarithm is 2. **LG0140**

10. Find the product of the positive roots of the equation $\sqrt{(2008)}(x)^{\log_{2008} x} = x^2$. **LG0141**

11. Solve the inequality : $\log_{1/2}(x+1) > \log_2(2-x)$. **LG0067**

12. Solve the inequality : $\log_x 2 \cdot \log_{2x} 2 \cdot \log_2 4x > 1$. **LG0068**

EXERCISE (JA)

1. Let (x_0, y_0) be the solution of the following equations **[JEE 2011, 3 (–1)]**

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$
$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6 **LG0107**

2. The value of $6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$ is **[JEE 2012, 4M]**

LG0108

3. If $3^x = 4^{x-1}$, then $x =$ **[JEE-Advanced 2013, 4(–1)]**

(A) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (B) $\frac{2}{2 - \log_2 3}$ (C) $\frac{1}{1 - \log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3 - 1}$ **LG0109**

4. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is ______ **[JEE(Advanced)-2018, 3(0)]**

LG0110

ANSWER KEY

EXERCISE (O-1)

1. C	**2.** C	**3.** B	**4.** D	**5.** A	**6.** C	**7.** A	**8.** D	**9.** C	**10.** A
11. D	**12.** D	**13.** C	**14.** B	**15.** B	**16.** B	**17.** B	**18.** A	**19.** B	**20.** A

EXERCISE (S-1)

1. (a) 1; (b) $-4, -\sqrt{3}-1$; (c) $-4, -2, -\sqrt{3}-1$ **2.** (A)→(q), (B)→(r), (C)→(p), (D)→(s)

3. 3721 **4.** (a) 9 (b) 1 (c) $\frac{25}{2}$ (d) 1/6 **5.** 469

6. **(a)** x = 5, **(b)** x = 10 **(c)** $x = 2^{\sqrt{2}}$ or $2^{-\sqrt{2}}$ **(d)** $x = 2^{-\log a}$ where base of log is 5 **(e)** x = 5

7. 507 **8.** (a^4, a, a^7) or $\left(\frac{1}{a^4}, \frac{1}{a}, \frac{1}{a^7}\right)$ **9.** x = 2 or 1/32

10. $(2008)^2$ **11.** $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$ **12.** $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$

EXERCISE (JA)

1. C **2.** 4 **3.** A,B,C **4.** 8

TRIGONOMETRIC RATIOS & IDENTITIES

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) Measurement of angles : Commonly two systems of measurement of angles are used.

(i) Sexagesimal or English System : Here 1 right angle = 90° (degrees)

1° = 60' (minutes)

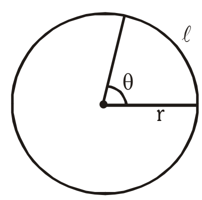
1' = 60" (seconds)

(ii) Circular system : Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r ' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the these systems : $\frac{D}{90}=\frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r',

by an arc of length 'ℓ' then $\frac{\ell}{r}=\theta$.



Note that here ℓ, r are in the same units and θ is always in radians.

Illustration 1 : If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and $75° = \left(75\times\frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \quad \frac{\pi}{3}=\frac{s}{r_1}$ and $\frac{5\pi}{12}=\frac{s}{r_2}$

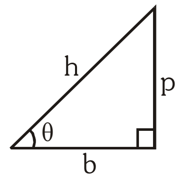
$\Rightarrow \quad \frac{\pi}{3}r_1 = s$ and $\frac{5\pi}{12}r_2 = s \Rightarrow \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5 : 4$ **Ans.**

Do yourself - 1 :

(i) The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

2. T-RATIOS (or Trigonometric functions) :

In a right angle triangle



$\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\text{cosec}\,\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$

'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin\theta . \text{cosec}\,\theta = 1$ **(2)** $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$ **(4)** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(8) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(9) $\text{cosec}\theta + \cot\theta = \frac{1}{\text{cosec}\theta - \cot\theta}$

(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\text{cosec}\,\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\text{cosec}^2\theta-1}}{\text{cosec}\,\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\text{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\text{cosec}\theta}{\sqrt{\text{cosec}^2\theta-1}}$
$\text{cosec}\,\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\text{cosec}\,\theta$

Illustration 2 : If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3 : $4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$ **Ans.(C)**

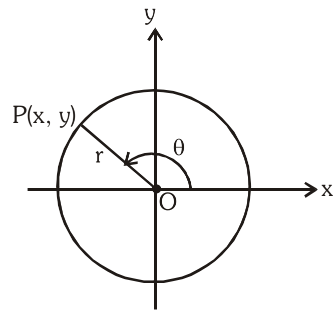
Do yourself - 2 :

(i) If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\text{cosec}\theta$ in first quadrant.

(ii) If $\sin\theta + \text{cosec}\theta = 2$, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

4. NEW DEFINITION OF T-RATIOS :

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.



$\sin\theta = y/r$,

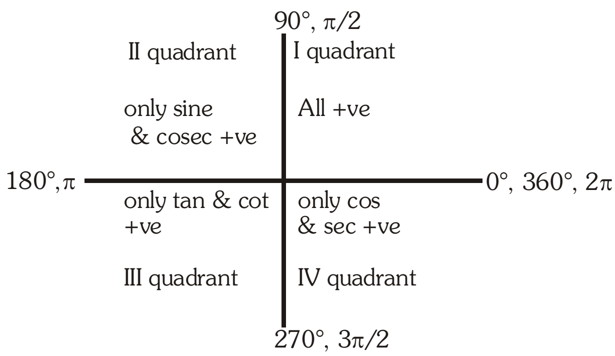
$\cos\theta = x/r$

$\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$

7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles / T-ratio	0°	30°	45°	60°	90°	180°	270°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
$\mathrm{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. → Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

Illustration 4 : If $\sin\theta = -\dfrac{1}{2}$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) none of these

Solution : Let us first find out θ lying between 0 and 360°.

Since $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = 210°$ or $330°$ and $\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30°$ or $210°$

Hence, $\theta = 210°$ or $\dfrac{7\pi}{6}$ is the value satisfying both. **Ans. (C)**

Do yourself - 3 :

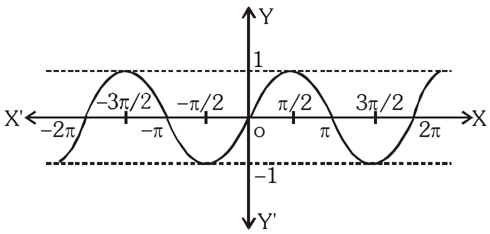
(i) If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\text{cosec}^2\theta$.

(ii) Prove that : (a) $\cos 570° \sin 510° + \sin(-330°)\cos(-390°) = 0$

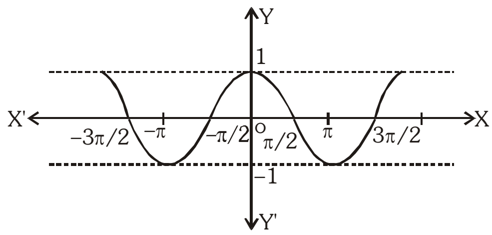
(b) $\tan\frac{11\pi}{3} - 2\sin\frac{9\pi}{3} - \frac{3}{4}\text{cosec}^2\frac{\pi}{4} + 4\cos^2\frac{17\pi}{6} = \frac{3-2\sqrt{3}}{2}$

8. GRAPH OF TRIGONOMETRIC FUNCTIONS :

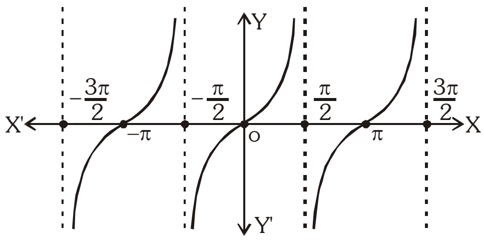
(i) y = sinx



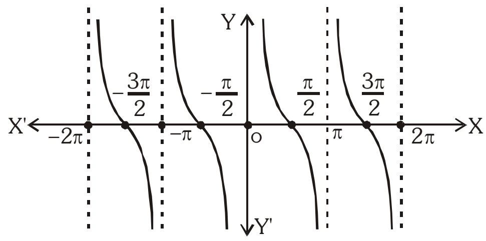
(ii) y = cosx



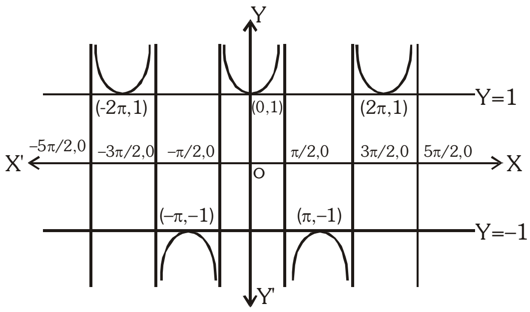
(iii) y = tanx



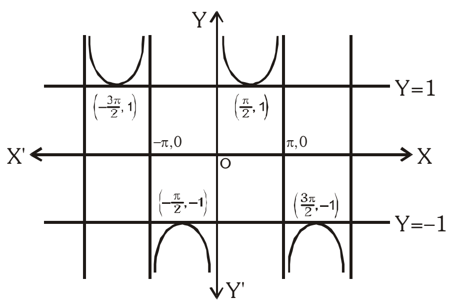
(iv) y = cotx



(v) y = secx



(vi) y = cosecx



9. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	$R-\{(2n+1)\pi/2 ; n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

10. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(i) sin (A + B) = sin A cos B + cos A sin B. **(ii)** sin (A – B) = sin A cos B – cos A sin B.

(iii) cos (A + B) = cos A cos B – sin A sin B **(iv)** cos (A – B) = cos A cos B + sin A sin B

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ **(vi)** $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A+B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$ **(viii)** $\cot(A-B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results :

(i) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

Illustration 5 : Prove that $\sqrt{3}$ cosec20° – sec20° = 4.

Solution : L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20°\cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration 6: Prove that tan70° = cot70° + 2cot40° **.**

Solution : L.H.S. $= \tan 70° = \tan(20° + 50°) = \dfrac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or tan70° – tan20° tan50° tan70° = tan20° + tan50°

or tan70° = tan70° tan50° tan20° + tan20° + tan50° = 2 tan 50° + tan20°

= cot70° + 2cot40° = R.H.S.

Do yourself - 4 :

(i) If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A \,\&\, B < \dfrac{\pi}{2}$, then find the value of the following :

(a) sin(A + B) (b) sin(A – B) (c) cos(A + B) (d) cos(A – B)

(ii) If x + y = 45°, then prove that :

(a) (1 + tanx)(1 + tany) = 2 (b) (cotx – 1)(coty – 1) = 2

(Remember these results)

11. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE :

(i) 2 sin A cos B = sin (A+ B) + sin (A – B). **(ii)** 2 cos A sin B = sin (A + B) – sin (A – B).

(iii) 2 cos A cos B = cos (A + B) + cos (A – B) **(iv)** 2 sin A sin B = cos (A – B) – cos (A + B)

Illustration 7: If sin2A = λ sin2B, then prove that $\frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$.

Solution : Given sin2A = λ sin2B

$\Rightarrow \frac{\sin 2A}{\sin 2B} = \frac{\lambda}{1}$

Applying componendo & dividendo,

$\frac{\sin 2A + \sin 2B}{\sin 2B - \sin 2A} = \frac{\lambda+1}{1-\lambda}$

$$\Rightarrow \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}} = \frac{\lambda+1}{1-\lambda} \quad \Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times-\sin(A-B)} = \frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1} \quad \Rightarrow \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

12. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustration 8 : $\frac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) tan θ (B) cos θ (C) cot θ (D) none of these

Solution : L.H.S.= $\frac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta.\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta} = \frac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)]}$

$$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(2\cos^2\theta) + \cos^2\theta]} = \frac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$$

Ans. (A)

Illustration 9 : Show that sin12°.sin48°.sin54° = 1/8

Solution : L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36° \sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36° \sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18° \cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18° \cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36° \cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36° \cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

Do yourself - 5 :

(i) Simplify $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

(ii) Prove that

(a) (sin3A + sinA)sinA + (cos3A – cosA)cosA = 0

(b) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

(c) $\frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

13. TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES :

(i) sin (A+B+C) = sinAcosBcosC + sinBcosAcosC + sinCcosAcosB – sinAsinBsinC

= ΣsinA cosB cosC – Πsin A

= cosA cosB cosC [tanA + tanB + tanC – tanA tanB tanC]

(ii) cos (A+B+C) = cosA cosB cosC – sinA sinB cosC – sinA cosB sinC – cosA sinB sinC

= Πcos A – Σsin A sin B cos C

= cos A cos B cos C [1 – tan A tan B – tan B tan C – tan C tan A]

(iii) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$

14. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

(a) Trigonometrical ratios of an angle 2θ in terms of the angle θ :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$ **(iv)** $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$ **(vi)** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Illustration 10: Prove that : $\dfrac{2\cos 2A + 1}{2\cos 2A - 1} = \tan(60° + A)\tan(60° - A)$.

Solution : R.H.S. = tan(60° + A) tan(60° – A)

$$= \left(\frac{\tan 60° + \tan A}{1 - \tan 60° \tan A}\right)\left(\frac{\tan 60° - \tan A}{1 + \tan 60° \tan A}\right) = \left(\frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A}\right)\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$$

$$= \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = \frac{3 - \dfrac{\sin^2 A}{\cos^2 A}}{1 - 3\dfrac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A - \sin^2 A}{\cos^2 A - 3\sin^2 A} == \frac{2\cos^2 A + \cos^2 A - 2\sin^2 A + \sin^2 A}{2\cos^2 A - 2\sin^2 A - \sin^2 A - \cos^2 A}$$

$$= \frac{2(\cos^2 A - \sin^2 A) + \cos^2 A + \sin^2 A}{2(\cos^2 A - \sin^2 A) - (\sin^2 A + \cos^2 A)} = \frac{2\cos 2A + 1}{2\cos 2A - 1} = \text{L.H.S.}$$

Do yourself - 6 :

(i) Prove that :

(a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (b) $\dfrac{1+\sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot\theta$

(b) Trigonometrical ratios of an angle 3θ in terms of the angle θ :

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **(ii)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Illustration 11 : Prove that : tanA + tan(60° + A) + tan(120° + A) = 3tan3A

Solution : L.H.S. = tanA + tan(60° + A) + tan(120° + A)

= tanA + tan(60° + A) + tan{180° –(60° – A)}

= tanA + tan(60° + A) – tan(60° – A) [∵ tan(180° – θ) = –tanθ]

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

Do yourself - 7 :

(i) Prove that :

(a) $\cot\theta \cot(60° - \theta)\cot(60° + \theta) = \cot 3\theta$ (b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

15. TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

(iv) $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(v) $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$

(vi) $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

(vii) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ **(viii)** $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ **(x)** $2\sin\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$ **(xii)** $\tan\frac{\theta}{2} = \frac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

for **(vii)** to **(xii)**, we decide the sign of ratio according to value of θ.

Illustration 12: $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution : $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ = \sqrt{1+\sin 135^\circ} = \sqrt{1+\frac{1}{\sqrt{2}}}$ (using $\cos A + \sin A = \sqrt{1+\sin 2A}$)

$= \frac{1}{2}\sqrt{4+2\sqrt{2}}$ **Ans.(A)**

Do yourself - 8 :

(i) Find the value of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

16. TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES :

(i) $\sin 18^\circ = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \cos\frac{2\pi}{5}$

(ii) $\cos 36^\circ = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \sin\frac{3\pi}{10}$

(iii) $\sin 72^\circ = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18^\circ = \cos\frac{\pi}{10}$

(iv) $\sin 36^\circ = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ = \cos\frac{3\pi}{10}$

(v) $\sin 15^\circ = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \cos\frac{5\pi}{12}$

(vi) $\cos 15^\circ = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \sin\frac{5\pi}{12}$

(vii) $\tan 15° = \tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

(viii) $\tan 75° = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

(ix) $\tan(22.5°) = \tan\frac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8}$

(x) $\tan(67.5°) = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\frac{\pi}{8}$

Illustration 13 : Evaluate sin78° – sin66° – sin42° + sin6°.

Solution : The expression $= (\sin 78° - \sin 42°) - (\sin 66° - \sin 6°) = 2\cos(60°)\sin(18°) - 2\cos 36°.\sin 30°$

$$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

Do yourself - 9 :

(i) Find the value of

(a) $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$ (b) $\cos^2 48° - \sin^2 12°$

17. CONDITIONAL TRIGONOMETRIC IDENTITIES :

If A + B + C = 180°, then

(i) tan A + tan B + tan C = tan A tan B tan C

(ii) cot A cot B + cot B cot C + cot C cot A = 1

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iv) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) sin 2A + sin 2B + sin 2C = 4 sinA sinB sinC

(vi) cos 2A + cos 2B + cos 2C = –1–4 cosA cosB cosC

(vii) sin A + sin B + sin C = 4 $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) cos A + cos B + cos C = 1 + 4 $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 14 : In any triangle ABC, sin A – cos B = cos C, then angle B is

(A) π/2 (B) π/3 (C) π/4 (D) π/6

Solution : We have , sin A – cos B = cos C

sin A = cos B + cos C

$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \because \; A+B+C=\pi$

$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$

$\Rightarrow \quad \cos\frac{A}{2}=\cos\frac{B-C}{2}$ or $A = B - C$; But $A + B + C = \pi$

Therefore $2B = \pi \Rightarrow B = \pi/2$ **Ans.(A)**

Illustration 15: If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to-

(A) $1 - 4\cos A \cos B \cos C$ (B) $4 \sin A \sin B \sin C$

(C) $1 + 2\cos A \cos B \cos C$ (D) $1 - 4 \sin A \sin B \sin C$

Solution : $\cos 2A + \cos 2B + \cos 2C = 2\cos(A + B)\cos(A - B) + \cos 2C$

$= 2\cos\left(\frac{3\pi}{2}-C\right)\cos(A - B) + \cos 2C \qquad \because \; A+B+C=\frac{3\pi}{2}$

$= -2\sin C\cos(A - B) + 1 - 2\sin^2 C = 1 - 2\sin C\,[\cos(A - B) + \sin C]$

$= 1 - 2\sin C\left[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)\right]$

$= 1 - 2\sin C\,[\cos(A - B) - \cos(A + B)] = 1 - 4\sin A\sin B\sin C$

Ans.(D)

Do yourself - 10 :

(i) If ABCD is a cyclic quadrilateral, then find the value of $\sin A + \sin B - \sin C - \sin D$

(ii) If $A + B + C = \frac{\pi}{2}$, then find the value of $\tan A \tan B + \tan B \tan C + \tan C \tan A$

18. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $a, b > 0$

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β areknown angles.

(iv) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

Illustration 16: Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7 + 3 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 7 + 3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Illustration 17: Find the maximum value of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : We have $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

Do yourself - 11 :

(i) Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta + \frac{\pi}{6}\right)$ for all real values of θ.

(ii) Find the minimum value of cosθ + cos2θ for all real values of θ.

(iii) Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

19. IMPORTANT RESULTS :

(i) $\sin\theta \sin(60° - \theta) \sin(60° + \theta) = \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta . \cos(60° - \theta) \cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$

(iv) $\cot\theta \cot(60° - \theta) \cot(60° + \theta) = \cot 3\theta$

(v) (a) $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \frac{3}{2}$

(b) $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \frac{3}{2}$

(c) $\tan\theta + \tan(60° + \theta) + \tan(120° + \theta) = 3\tan 3\theta$

(vi) (a) If tan A + tan B + tan C = tan A tan B tan C, then A + B + C = $n\pi$, $n \in I$

(b) If tan A tan B + tan B tan C + tan C tan A = 1, then A + B + C = $(2n+1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta\cos 2\theta\cos 4\theta \ldots\ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\text{cosec}2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Do yourself - 12 :

(i) Evaluate $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots\ldots\ldots$ to n terms

Miscellaneous Illustration :

Illustration 18: Prove that

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Solution : We know $\tan\theta = \cot\theta - 2\cot 2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha, \ldots\ldots\ldots\ldots$ in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot 2\alpha)$

$2(\tan 2\alpha) = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2(\tan 2^2\alpha) = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

....

$2^{n-1}(\tan 2^{n-1}\alpha) = 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha)$

Adding,

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore$ $\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Illustration 19: If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=\frac{1}{4}$, prove that A = B = C = D = π/2.

Solution :

$$\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right)=1$$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{C+D}{2}\right)\right\}=1$$

Since, A + B = 2π – (C + D), the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right\}=1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}+1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2-4\left\{1-\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\}\ge 0$$

$$\left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2\ge 4$$

$\Rightarrow \cos\frac{A-B}{2}+\cos\frac{C-D}{2}\ge 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2}\le 1$

$$\Rightarrow \cos\frac{A-B}{2}=1\ \&\ \cos\frac{C-D}{2}=1$$

$$\Rightarrow \frac{A-B}{2}=0=\frac{C-D}{2}$$

⇒ A = B, C = D.

Similarly A = C, B = D ⇒ A = B = C = D = π/2

ANSWERS FOR DO YOURSELF

1 :	**(i)**	10π cm	**2 :**	**(i)**	$\frac{3}{5},\frac{4}{5},\frac{5}{3}$		**(ii)**	2	
3 :	**(i)**	8	**4 :**	**(i)**	(a) $\frac{187}{205}$	(b) $\frac{-133}{205}$	(c) $\frac{-84}{205}$	(d) $\frac{156}{205}$	
5 :	**(i)**	$\frac{1}{\sqrt{3}}$	**8:**	**(i)**	(a) $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$	(b) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$	(c) $\sqrt{2}-1$		
9:	**(i)**	(a) $-\frac{1}{2}$	(b) $\frac{\sqrt{5}+1}{8}$	**10 :**	**(i)** 0	**(ii)** 1			
11 :	**(i)**	7 & –7	**(ii)** $-\frac{9}{8}$	**(iii)**	$4+\sqrt{10}$ & $4-\sqrt{10}$	**12 :**	**(i)** 0		

EXERCISE (O-1)

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^2 x + \cos^4 x$ is -

(A) 0 (B) 2 (C) 1 (D) 3 **TR0001**

2. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to -

(A) 2 (B) 0 (C) 4 (D) 6 **TR0002**

3. If $\tan A = -\frac{1}{2}$ and $\tan B = -\frac{1}{3}$, (where A,B > 0), then A + B can be

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{7\pi}{4}$ **TR0101**

4. $\cos^2 48° - \sin^2 12°$ is equal to -

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}+1}{8}$ (C) $\frac{\sqrt{3}-1}{4}$ (D) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ **TR0102**

5. The expression $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$ is equals -

(A) $\tan\theta$ (B) $\tan 2\theta$ (C) $\sin 2\theta$ (D) $\cos 2\theta$ **TR0008**

6. If $3\sin\alpha = 5\sin\beta$, then $\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} =$

(A) 1 (B) 2 (C) 3 (D) 4 **TR0014**

7. $\frac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)}$ is equal to -

(A) $\tan A$ (B) $\frac{\sin A}{\sin B}$ (C) $\frac{\cos A}{\cos B}$ (D) $\frac{\sin C}{\cos B}$ **TR0007**

8. $\frac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} =$

(A) $\frac{1}{2}\tan\theta$ (B) $\frac{1}{2}\cot\theta$ (C) $\tan\theta$ (D) $\cot\theta$ **TR0009**

9. If $A = \tan 6^\circ \tan 42^\circ$ and $B = \cot 66^\circ \cot 78^\circ$, then -

(A) A = 2B (B) A = 1/3B (C) A = B (D) 3A = 2B **TR0011**

10. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$

(A) –1 (B) 0 (C) 1 (D) 2 **TR0003**

11. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha + \beta =$

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$ **TR0013**

12. If tan A + tan B + tan C = tan A. tan B. tan C, then -

(A) A,B,C must be angles of a triangle

(B) the sum of any two of A,B,C is equal to the third

(C) A+B+C must be n integral multiple of π

(D) None of these **TR0023**

13. The value of sin10º + sin20º + sin30º+....+ sin 360º is equal to -

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2 **TR0004**

14. The number of real solutions of the equation $\sin(e^x) = 2^x + 2^{-x}$ is -

(A) 1 (B) 0 (C) 2 (D) Infinite **TR0103**

15. If $f(x) = \dfrac{\sin 3x}{\sin x}$, $x \neq n\pi$, then the range of values of f(x) for real values of x is -

(A) $[-1,3]$ (B) $(-\infty,-1]$ (C) $(3, +\infty)$ (D) $[-1,3)$ **TR0024**

16. If cos x + cos y + cos α = 0 and sin x + sin y + sin α = 0, then $\cot\left(\dfrac{x+y}{2}\right) =$

(A) sin α (B) cos α (C) cot α (D) 2 sin α **TR0006**

17. The value of $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}$ is :-

(A) $\dfrac{1}{16}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{1}{2}$ (D) 1 **TR0028**

18. Maximum and minimum value of $2\sin^2\theta - 3\sin\theta + 2$ is -

(A) $\dfrac{1}{4}, -\dfrac{7}{4}$ (B) $\dfrac{1}{4}, \dfrac{21}{4}$ (C) $\dfrac{21}{4}, -\dfrac{3}{4}$ (D) $7, \dfrac{7}{8}$ **TR0025**

19. For $\theta \in (0, \pi/2)$, the maximum value of $\sin\left(\theta+\dfrac{\pi}{6}\right)+\cos\left(\theta+\dfrac{\pi}{6}\right)$ is attained at $\theta =$

(A) $\dfrac{\pi}{12}$ (B) $\dfrac{\pi}{6}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{4}$ **TR0026**

20. Minimum value of the expression $\cos^2\theta - (6\sin\theta\cos\theta) + 3\sin^2\theta + 2$, is -

(A) $4+\sqrt{10}$ (B) $4-\sqrt{10}$ (C) 0 (D) 4 **TR0027**

EXERCISE (S-1)

1. Prove that : $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta) = \sin^2\beta$ **TR0056**

2. Prove that : $\cos 2\alpha = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$ **TR0072**

3. Prove that : $\sin^4\frac{\pi}{16} + \sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16} + \sin^4\frac{7\pi}{16} = \frac{3}{2}$ **TR0055**

4. If $X = \sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right), Y = \cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$ then prove that $\frac{X}{Y}-\frac{Y}{X} = 2\tan 2\theta$. **TR0053**

5. Find the positive integers p,q,r,s satisfying $\tan\frac{\pi}{24} = \left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{r}-s\right)$. **TR0077**

6. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$. **TR0058**

7. If $\cos(\alpha+\beta) = \frac{4}{5}$; $\sin(\alpha-\beta) = \frac{5}{13}$ & α, β lie between 0 & $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$ **TR0059**

8. If the value of the expression sin25°. sin35°.sin85° can be expressed as $\frac{\sqrt{a}+\sqrt{b}}{c}$ where $a,b,c \in N$ and are in their lowest form, find the value of $(a + b + c)$. **TR0060**

9. Prove that $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$. **TR0057**

10. Prove that $4\cos\frac{2\pi}{7}.\cos\frac{\pi}{7} - 1 = 2\cos\frac{2\pi}{7}$. **TR0104**

11. Let $P(k) = \left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\frac{(2k-1)\pi}{4k}\right)\left(1+\cos\frac{(2k+1)\pi}{4k}\right)\left(1+\cos\frac{(4k-1)\pi}{4k}\right)$ then find the value of **(a)** P(5) and **(b)** P(6). **TR0062**

Calculate without using trigonometric tables (Q.12 to Q.15) :

12. $4\cos 20° - \sqrt{3}\cot 20°$ **TR0063**

13. $\frac{2\cos 40° - \cos 20°}{\sin 20°}$ **TR0064**

14. $\cos^6\frac{\pi}{16}+\cos^6\frac{3\pi}{16}+\cos^6\frac{5\pi}{16}+\cos^6\frac{7\pi}{16}$ **TR0065**

15. $\tan 10° - \tan 50° + \tan 70°$ **TR0066**

16. Given that $(1 + \tan 1°)(1 + \tan 2°)....(1 + \tan 45°) = 2^n$, find n. **TR0067**

17. Prove that from the equality $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b} = \frac{1}{a+b}$ follows the relation; $\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3} = \frac{1}{(a+b)^3}$. **TR0105**

18. (a) If $y = 10\cos^2 x - 6\sin x\cos x + 2\sin^2 x$, then find the greatest & least value of y. **TR0068**

(b) If $y = 1 + 2\sin x + 3\cos^2 x$, find the maximum & minimum values of $y \;\forall\; x \in R$. **TR0069**

(c) If $y = 9\sec^2 x + 16\,\text{cosec}^2 x$, find the minimum value of y for all permissible value of x. **TR0070**

(d) If $a \le 3\cos\left(\theta+\frac{\pi}{3}\right) + 5\cos\theta + 3 \le b$, find a and b, where a is the minimum value & b is the maximum value. **TR0071**

19. Let k = 1°, then prove that $\sum_{n=0}^{88} \frac{1}{\cos nk.\cos(n+1)k} = \frac{\cos k}{\sin^2 k}$ TR0075

20. If x and y are real number such that $x^2 + 2xy - y^2 = 6$, find the minimum value of $(x^2 + y^2)^2$. TR0106

21. If A + B + C = π; prove that $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 1$. TR0076

EXERCISE (JM)

1. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of cos4x is :- **[JEE(Main)-2017]**

(1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$ (3) $\frac{1}{3}$ (4) $\frac{2}{9}$ TR0086

2. For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals : **[JEE(Main)-Jan 19]**

(1) $13 - 4\cos^6\theta$ (2) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

(3) $13 - 4\cos^2\theta + 6\cos^4\theta$ (4) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$ TR0087

3. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot.....\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is : **[JEE(Main)-Jan 19]**

(1) $\frac{1}{256}$ (2) $\frac{1}{2}$ (3) $\frac{1}{512}$ (4) $\frac{1}{1024}$ TR0088

4. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for k = 1, 2, 3, ... Then for all x ∈ R, the value of $f_4(x) - f_6(x)$ is equal to :- **[JEE(Main)-Jan 19]**

(1) $\frac{5}{12}$ (2) $\frac{-1}{12}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$ TR0089

5. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is : **[JEE(Main)-Jan 19]**

(1) $\sqrt{19}$ (2) $\frac{\sqrt{79}}{2}$ (3) $\sqrt{31}$ (4) $\sqrt{34}$ TR0090

6. If $\cos(\alpha+\beta) = \frac{3}{5}, \sin(\alpha-\beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then tan(2α) is equal to : **[JEE(Main)-Apr 19]**

(1) $\frac{21}{16}$ (2) $\frac{63}{52}$ (3) $\frac{33}{52}$ (4) $\frac{63}{16}$ TR0107

7. The value of $\cos^2 10° - \cos 10°\cos 50° + \cos^2 50°$ is : **[JEE(Main)-Apr 19]**

(1) $\frac{3}{2}(1+\cos 20°)$ (2) $\frac{3}{4}$ (3) $\frac{3}{4} + \cos 20°$ (4) $\frac{3}{2}$ TR0108

8. The value of sin 10° sin30° sin50° sin70° is :- **[JEE(Main)-Apr 19]**

(1) $\frac{1}{36}$ (2) $\frac{1}{32}$ (3) $\frac{1}{18}$ (4) $\frac{1}{16}$ TR0109

EXERCISE (JA)

1. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then **[JEE 2009, 4]**

(A) $\tan^2 x=\frac{2}{3}$ (B) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$ (C) $\tan^2 x=\frac{1}{3}$ (D) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

TR0094

2. For $0<\theta<\frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6}\text{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\text{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$ is (are) -

[JEE 2009, 4]

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$ **TR0095**

3. The maximum value of the expression $\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is **[JEE 2010, 3]**

TR0096

4. Let $P=\left\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\right\}$ and $Q=\left\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\right\}$ be two sets. Then

(A) $P\subset Q$ and $Q-P\neq\varnothing$ (B) $Q\not\subset P$ **TR0098**

(C) $P\not\subset Q$ (D) $P=Q$ **[JEE 2011,3]**

5. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$ **TR0099**

6. For non-negative integers n, let $f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$

Assuming $\cos^{-1}x$ takes values in $[0,\pi]$, which of the following options is/are correct ?

[JEE(Advanced)-2019, 4(–1)]

(1) $\sin(7\cos^{-1}f(5))=0$ (2) $f(4)=\frac{\sqrt{3}}{2}$ **TR0110**

(3) $\lim_{n\to\infty}f(n)=\frac{1}{2}$ (4) If $\alpha=\tan(\cos^{-1}f(6))$, then $\alpha^2+2\alpha-1=0$

ANSWER KEY

EXERCISE (O-1)

1. C **2.** B **3.** D **4.** B **5.** B **6.** D **7.** B **8.** D
9. C **10.** B **11.** B **12.** C **13.** A **14.** B **15.** D **16.** C
17. B **18.** D **19.** A **20.** B

EXERCISE (S-1)

5. p = 3, q = 2; r = 2; s = 1 **7.** $\frac{56}{33}$ **8.** 24 **11.** **(a)** $\frac{3-\sqrt{5}}{32}$; **(b)** $\frac{2-\sqrt{3}}{16}$

12. –1 **13.** $\sqrt{3}$ **14.** $\frac{5}{4}$ **15.** $\sqrt{3}$ **16.** n = 23

18. **(a)** $y_{max}=11$, $y_{min}=1$; **(b)** $y_{max}=\frac{13}{3}$, $y_{min}=-1$; **(c)** 49; **(d)** a = –4 & b = 10 **20.** 18

EXERCISE (JM)

1. 1 **2.** 1 **3.** 3 **4.** 4 **5.** 1 **6.** 4 **7.** 2
8. 4

EXERCISE (JA)

1. A,B **2.** C,D **3.** 2 **4.** D **5.** C **6.** 1,2,4

TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) Principal solution :- The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) General solution :- Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) Particular solution :- The solution of the trigonometric equation lying in the given interval.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$ $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n$, $n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, $\cos\frac{n\pi}{2} = 0$,

$$\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$$

Illustration 1 : Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1$.

Solution : We have, $\frac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1 \quad \Rightarrow \quad \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \quad \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4}, n \in I \qquad \{\text{using } \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha)$

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ.

Do yourself-1 :

(i) Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\text{cosec}\left(\frac{\theta}{2}\right) = -1$

4. IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING TRIGONOMETRIC EQUATIONS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $| k | \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(iii) If $\cot\theta$ or $\text{cosec}\,\theta$ is involved in the equation, θ should not be multiple of π or 0.

5. DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS :

(a) Solving trigonometric equations by factorisation.

e.g. $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore \quad (2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore \quad (1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\therefore \quad (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow \quad \cos x = -1 \text{ or } \sin x = \frac{1}{2}$

$\Rightarrow \quad \cos x = -1 = \cos\pi \quad \Rightarrow \quad x = 2n\pi + \pi = (2n + 1)\pi, n \in I$

or $\quad \sin x = \frac{1}{2} = \sin\frac{\pi}{6} \quad \Rightarrow \quad x = k\pi + (-1)^k \frac{\pi}{6}, k \in I$

Illustration 2 : If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Solution : Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \cos^2\theta = \frac{1}{6}\sin\theta \cdot \tan\theta \quad \Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$ **Ans. (A)**

(b) Solving of trigonometric equation by reducing it to a quadratic equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore \quad 6 - 10\cos x = 3 - 3\cos^2 x \quad \Rightarrow \quad 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow \quad (3\cos x - 1)(\cos x - 3) = 0 \quad \Rightarrow \quad \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore \quad \cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \quad \Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Illustration 3 : Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution : The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4} \quad \Rightarrow \quad \cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0 \quad \Rightarrow \quad (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

For the given interval, $n = 0$ and $n = 1$.

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **Ans.**

Illustration 4 : Find the number of solutions of tanx + secx = 2cosx in $[0, 2\pi]$.

Solution : Here, tanx + secx = 2cosx $\Rightarrow$ sinx + 1 = 2 $\cos^2 x$

$\Rightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}, -1$

But sinx = –1 $\Rightarrow x = \frac{3\pi}{2}$ for which tanx + secx = 2 cosx is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of tanx + secx = 2cos x is 2. **Ans.**

Illustration 5 : Solve the equation $5\sin^2 x - 7\sin x \cos x + 16\cos^2 x = 4$

Solution : To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2 x + \cos^2 x = 1$

writing the equation in the form,

$5\sin^2 x - 7\sin x \cdot \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow$ $\sin^2 x - 7\sin x \cos x + 12\cos^2 x = 0$

dividing by $\cos^2 x$ on both side we get,

$\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow$ tanx = 3, 4

i.e., $\tan x = \tan(\tan^{-1} 3)$ or $\tan x = \tan(\tan^{-1} 4)$

$\Rightarrow$ $x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$, $n \in I$. **Ans.**

Illustration 6 : If $x \neq \frac{n\pi}{2}$, $n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution : As $x \neq \frac{n\pi}{2}$ $\Rightarrow$ $\cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$ $\Rightarrow$ $\sin^2 x - 3\sin x + 2 = 0$

$\therefore$ $(\sin x - 2)(\sin x - 1) = 0 \Rightarrow$ sinx = 1, 2

where sinx = 2 is not possible and sinx = 1 which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

Illustration 7 : Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}$ sinx . cosx.

Solution : $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cdot \cos x$ $\Rightarrow$ $(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = \frac{7}{2}\sin x \cdot \cos x$

$\Rightarrow$ $1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$ $\Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow$ $(2\sin 2x - 1)(\sin 2x + 4) = 0$ $\Rightarrow$ $\sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow$ $2x = n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$, $n \in I$ **Ans.**

Do yourself-2 :

(i) Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

(ii) Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

(c) Solving trigonometric equations by introducing an auxilliary argument.

Consider, $a \sin\theta + b\cos\theta = c$ .... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustration 8 : Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

Solution : Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

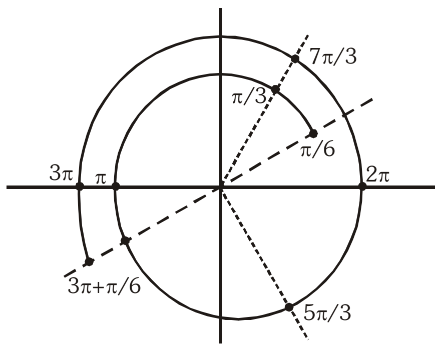
$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \Rightarrow x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$



But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. **Ans.**

Illustration 9 : Prove that the equation $k\cos x - 3\sin x = k + 1$ possess a solution iff $k \in (-\infty, 4]$.

Solution : Here, k cosx – 3sinx = k + 1, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2+9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

$\Rightarrow$ The interval of k for which the equation $(k\cos x - 3\sin x = k + 1)$ has a solution is $(-\infty, 4]$.

Ans.

Do yourself-3 :

(i) Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\operatorname{cosec}\theta = 1 + \cot\theta$.

(d) Solving trigonometric equations by transforming sum of trigonometric functions into product.

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2\sin x \cos 3x = 0$

$\Rightarrow (\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow \cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow 3x = 2n\pi \pm \frac{\pi}{2}$ or $x = m\pi + (-1)^m\frac{\pi}{6}$

$\Rightarrow x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = m\pi + (-1)^m\frac{\pi}{6}$; (n, m $\in$ I)

Illustration 10 : Solve : $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta = 0$

Solution : We have $\cos\theta + \cos7\theta + \cos3\theta + \cos5\theta = 0$

$\Rightarrow \quad 2\cos4\theta\cos3\theta + 2\cos4\theta\cos\theta = 0 \;\Rightarrow\; \cos4\theta(\cos3\theta + \cos\theta) = 0$

$\Rightarrow \quad \cos4\theta(2\cos2\theta\cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\,\pi/2,\; n_1 \in I$

or $\cos2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\dfrac{\pi}{4},\; n_2 \in I$

or $\cos4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\dfrac{\pi}{8},\; n_3 \in I$ **Ans.**

(e) Solving trigonometric equations by transforming a product into sum.

e.g. $\sin5x.\cos3x = \sin6x.\cos2x$

$\sin8x + \sin2x = \sin8x + \sin4x$

$\therefore \quad 2\sin2x\,.\,\cos2x - \sin2x = 0$

$\Rightarrow \quad \sin2x(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin2x = 0$ or $\cos2x = \dfrac{1}{2}$

$\Rightarrow \quad \sin2x = 0 = \sin0$ or $\cos2x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow \quad 2x = n\pi + (-1)^n \times 0,\; n \in I$ or $2x = 2m\pi \pm \dfrac{\pi}{3},\; m \in I$

$\Rightarrow \quad x = \dfrac{n\pi}{2},\; n \in I$ or $x = m\pi \pm \dfrac{\pi}{6},\; m \in I$

Illustration 11 : Solve : $\cos\theta\cos2\theta\cos3\theta = \dfrac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\dfrac{1}{2}(2\cos\theta\cos3\theta)\cos2\theta = \dfrac{1}{4} \;\Rightarrow\; (\cos2\theta + \cos4\theta)\cos2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1}{2}[2\cos^2 2\theta + 2\cos4\theta\cos2\theta] = \dfrac{1}{2} \;\Rightarrow\; 1 + \cos4\theta + 2\cos4\theta\cos2\theta = 1$

$\therefore \quad \cos4\theta\,(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ or $(1 + 2\cos2\theta) = 0$

Now from the first equation : $2\cos4\theta = 0 = \cos(\pi/2)$

$\therefore \quad 4\theta = \left(n + \dfrac{1}{2}\right)\pi \;\Rightarrow\; \theta = (2n+1)\dfrac{\pi}{8},\; n \in I$

for $n = 0,\; \theta = \dfrac{\pi}{8}$; $n = 1,\; \theta = \dfrac{3\pi}{8}$; $n = 2,\; \theta = \dfrac{5\pi}{8}$; $n = 3,\; \theta = \dfrac{7\pi}{8}$ $\quad (\because\; 0 \le \theta \le \pi)$

and from the second equation :

$\cos 2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi-\pi/3) = \cos(2\pi/3)$

$\therefore \quad 2\theta = 2k\pi \pm 2\pi/3 \quad \therefore \quad \theta = k\pi \pm \pi/3,\ k \in I$

again for $k = 0,\ \theta = \frac{\pi}{3};\ k = 1,\ \theta = \frac{2\pi}{3}$ $\quad (\because\ 0 \le \theta \le \pi)$

$\therefore \quad \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

Do yourself-4 :

(i) Solve $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$.

(ii) Solve for x : $\sin x + \sin 3x + \sin 5x = 0$.

(f) Solving equations by a change of variable :

(i) Equations of the form $P(\sin x \pm \cos x, \sin x. \cos x) = 0$, where $P(y,z)$ is a polynomial, can be solved by the substitution :

$\cos x \pm \sin x = t \quad \Rightarrow \quad 1 \pm 2 \sin x. \cos x = t^2$.

e.g. $\sin x + \cos x = 1 + \sin x. \cos x$.

put $\sin x + \cos x = t$

$\Rightarrow \quad \sin^2 x + \cos^2 x + 2\sin x . \cos x = t^2$

$\Rightarrow \quad 2\sin x \cos x = t^2 - 1 \quad (\because\ \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \quad \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2}$

$\Rightarrow \quad 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow \quad (t-1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \quad \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \quad x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2},\ n \in I$

(ii) Equations of the form of asinx + bcosx + d = 0, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

e.g. 3 cos x + 4 sin x = 5

$$\Rightarrow \quad 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) = 5$$

$$\Rightarrow \quad \frac{3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = 5$$

$$\Rightarrow \quad 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2} \quad \Rightarrow \quad 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$$

$$\Rightarrow \quad 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0 \quad \Rightarrow \quad \left(2\tan\frac{x}{2} - 1\right)^2 = 0$$

$$\Rightarrow \quad 2\tan\frac{x}{2} - 1 = 0 \Rightarrow \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right),\ n \in I \quad \Rightarrow \quad x = 2n\pi + 2\tan^{-1}\frac{1}{2},\ n \in I$$

(iii) Many equations can be solved by introducing a new variable.

e.g. $\sin^4 2x + \cos^4 2x = \sin 2x.\cos 2x$

substituting $\sin 2x.\cos 2x = y$ $\because$ $(\sin^2 2x + \cos^2 2x)^2 = \sin^4 2x + \cos^4 2x + 2\sin^2 2x.\cos^2 2x$

$\Rightarrow$ $\sin^4 2x + \cos^4 2x = 1 - 2\sin^2 2x.\cos^2 2x$ substituting above result in given equation :

$1 - 2y^2 = y$

$$\Rightarrow \quad 2y^2 + y - 1 = 0 \quad \Rightarrow \quad 2(y+1)\left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad y = -1 \quad \text{or} \quad y = \frac{1}{2} \Rightarrow \sin 2x.\cos 2x = -1 \quad \text{or} \quad \sin 2x.\cos 2x = \frac{1}{2}$$

$\Rightarrow$ $2\sin 2x.\cos 2x = -2$ or $2\sin 2x.\cos 2x = 1$

$\Rightarrow$ $\sin 4x = -2$ (which is not possible) or $2\sin 2x.\cos 2x = 1$

$$\Rightarrow \quad \sin 4x = 1 = \sin\frac{\pi}{2} \quad \Rightarrow 4x = n\pi + (-1)^n\frac{\pi}{2},\ n \in I \Rightarrow x = \frac{n\pi}{4} + (-1)^n\frac{\pi}{8},\ n \in I$$

Illustration 12 : Find the general solution of equation $\sin^4x + \cos^4x = \sin x \cos x$.

Solution : Using half-angle formulae, we can represent given equation in the form :

$$\left(\frac{1-\cos 2x}{2}\right)^2 + \left(\frac{1+\cos 2x}{2}\right)^2 = \sin x \cos x$$

$\Rightarrow$ $(1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 4\sin x \cos x$

$\Rightarrow$ $2(1 + \cos^2 2x) = 2\sin 2x \Rightarrow \quad 1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow$ $\sin^2 2x + \sin 2x = 2$

$\Rightarrow$ $\sin 2x = 1$ or $\sin 2x = -2$ (which is not possible)

$\Rightarrow$ $2x = 2n\pi + \frac{\pi}{2}, \ n \in I$

$\Rightarrow$ $x = n\pi + \frac{\pi}{4}, \ n \in I$ **Ans.**

(g) Solving trigonometric equations with the use of the boundness of the functions involved.

e.g. $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right).\cos x = 0$

$\therefore$ $\sin x \cos\frac{x}{4} + \cos x \sin\frac{x}{4} + \cos x = 2$

$\therefore$ $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow$ $\sin\left(\frac{5x}{4}\right) = 1$ & $\cos x = 1$ (as $\sin\theta \le 1$ & $\cos\theta \le 1$)

Now consider

$\cos x = 1 \Rightarrow x = 2\pi, 4\pi, 6\pi, 8\pi$ ....

and $\sin\frac{5x}{4} = 1 \Rightarrow x = \frac{2\pi}{5}, \frac{10\pi}{5}, \frac{18\pi}{5}$ ....

Common solution to above APs will be the AP having

First term = 2π

Common difference = LCM of 2π and $\frac{8\pi}{5} = \frac{40\pi}{5} = 8\pi$

$\therefore$ General solution will be general term of this AP i.e. $2\pi + (8\pi)n, \ n \in I$

$\Rightarrow$ $x = 2(4n + 1)\pi, \ n \in I$

Illustration 13 : Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Solution : We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore$ $(\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $(1 + \sin 2x)$ admits the maximum value as 2.

Also $\left(\sqrt{2}\right)^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x = 2n\pi + \pi/4,\ n \in I$ ...... (i)

and $1 + \sin 2x = 2$ $\Rightarrow$ $\sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $2x = m\pi + (-1)^m \frac{\pi}{2},\ m \in I$ $\Rightarrow$ $x = \frac{m\pi}{2} + (-1)^m \frac{\pi}{4}$ ...... (ii)

The value of x in $[0, \pi]$ satisfying equations (i) and (ii) is $x = \frac{\pi}{4}$ (when n = 0 & m = 0)

Ans.

Note : $\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \ge 0$, this solution is not in domain.

Illustration 14 : Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$

Solution :

$2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$ .... (i)

$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \le 1$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\Rightarrow$ $\cos^2 x = 1$ and $y = 1/2$ $\Rightarrow \cos x = \pm 1 \Rightarrow$ $x = n\pi$, where $n \in I$.

Hence $x = n\pi,\ n \in I$ and $y = 1/2$. **Ans.**

Illustration 15 : The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution : Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$ $\Rightarrow$ $y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x = (\text{a number} < 2)(\text{a number} \le 1)$ $\Rightarrow$ $y < 2$ .... (i)

and when $y = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \ge 2$ $\Rightarrow$ $y \ge 2$ .... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. **Ans. (A)**

Note: If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

Do yourself-5 :

(i) If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -

(A) $x = 1, y = 0$ (B) $x = 1, y = \pi/2$ (C) $x = 2, y = 0$ (D) $x = 2, y = \pi/2$

(ii) If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

6. TRIGONOMETRIC INEQUALITIES :

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

Illustration 16 : Find the solution set of inequality $\sin x > 1/2$.

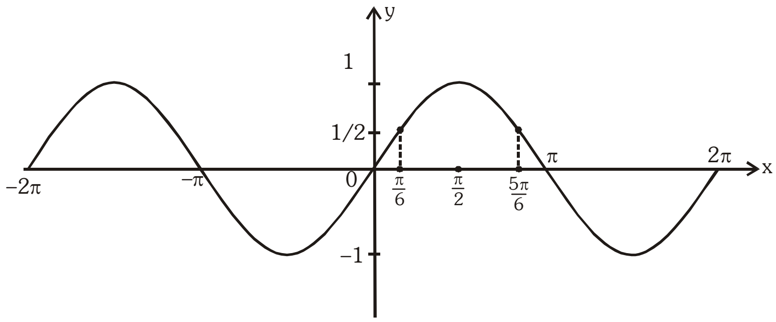
Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow$ $2n\pi + \pi/6 < x < 2n\pi + 5\pi/6,\ n \in I$



Thus, the required solution set is $\underset{n\in I}{\cup}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$ **Ans.**

Illustration 17 : Find the value of x in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for which $\sqrt{2}\sin 2x+1\le 2\sin x+\sqrt{2}\cos x$

Solution : We have, $\sqrt{2}\sin 2x+1\le 2\sin x+\sqrt{2}\cos x \Rightarrow 2\sqrt{2}\sin x\cos x-2\sin x-\sqrt{2}\cos x+1\le 0$

$\Rightarrow \quad 2\sin x(\sqrt{2}\cos x-1)-1(\sqrt{2}\cos x-1)\le 0 \Rightarrow \quad (2\sin x-1)(\sqrt{2}\cos x-1)\le 0$

$\Rightarrow \quad \left(\sin x-\frac{1}{2}\right)\left(\cos x-\frac{1}{\sqrt{2}}\right)\le 0$

Above inequality holds when :

Case-I : $\sin x-\frac{1}{2}\le 0$ and $\cos x-\frac{1}{\sqrt{2}}\ge 0 \Rightarrow \sin x\le\frac{1}{2}$ and $\cos x\ge\frac{1}{\sqrt{2}}$

Now considering the given interval of x :

for $\sin x\le\frac{1}{2}$: $x\in\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]\cup\left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$ and for $\cos x\ge\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

For both to simultaneously hold true : $x\in\left[-\frac{\pi}{4}, \frac{\pi}{6}\right]$

Case-II : $\sin x-\frac{1}{2}\ge 0$ and $\cos x\le\frac{1}{\sqrt{2}}$

Again, for the given interval of x :

for $\sin x\ge\frac{1}{2}$: $x\in\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and for $\cos x\le\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]\cup\left[\frac{\pi}{4}, \frac{3\pi}{2}\right]$

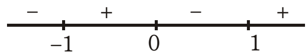
For both to simultaneously hold true : $x\in\left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$

$\therefore$ Given inequality holds for $x\in\left[-\frac{\pi}{4}, \frac{\pi}{6}\right]\cup\left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$ **Ans.**

Illustration 18 : Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

Solution : We have : $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow \quad (\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$



So $\quad \tan\alpha < -1,\ 0 < \tan\alpha < 1$

$\therefore$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ **Ans.**

Do yourself - 6 :

(i) Find the solution set of the inequality : $\cos x \geq -1/2$.

(ii) Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

<u>Miscellaneous Illustration :</u>

Illustration 19 : Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution : We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow \quad \tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow \quad \tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0 \qquad \Rightarrow \quad \tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$\theta = 2n\pi + \frac{\pi}{4},\ n \in I$ **Ans.**

Illustration 20 : Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution : Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\log_5 5 = \log_5(5/2) \quad \Rightarrow \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \log_5 \cos x = -\log_5 2 \Rightarrow \cos x = 1/2 \qquad \Rightarrow x = 2n\pi \pm \pi/3,\ n \in I$ **Ans.**

Illustration 21 : If the set of all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x + \sqrt{2}| < \sqrt{6}$ is $\left(\frac{a\pi}{24}, \frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution : $|4\sin x + \sqrt{2}| < \sqrt{6}$

$\Rightarrow \quad -\sqrt{6} < 4\sin x + \sqrt{2} < \sqrt{6} \qquad \Rightarrow \quad -\sqrt{6} - \sqrt{2} < 4\sin x < \sqrt{6} - \sqrt{2}$

$\Rightarrow \quad \frac{-(\sqrt{6}+\sqrt{2})}{4} < \sin x < \frac{\sqrt{6}-\sqrt{2}}{4} \qquad \Rightarrow \quad -\frac{5\pi}{12} < x < \frac{\pi}{12}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, $a = -10,\ b = 2$

$\therefore \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$ **Ans.**

Illustration 22 : Find the values of x in the interval $[0,2\pi]$ which satisfy the inequality :
$3|2\sin x - 1| \geq 3 + 4\cos^2 x$.

Solution : The given inequality can be written as :

$3|2\sin x - 1| \geq 3 + 4(1 - \sin^2 x) \Rightarrow 3|2\sin x - 1| \geq 7 - 4\sin^2 x$

Let $\sin x = t \Rightarrow 3|2t - 1| \geq 7 - 4t^2$

Case I : For $2t - 1 \geq 0$ i.e. $t \geq 1/2$ we have, $|2t - 1| = (2t - 1)$

$\Rightarrow 3(2t - 1) \geq 7 - 4t^2 \Rightarrow 6t - 3 \geq 7 - 4t^2$

$\Rightarrow 4t^2 + 6t - 10 \geq 0 \Rightarrow 2t^2 + 3t - 5 \geq 0$

$\Rightarrow (t-1)(2t+5) \geq 0 \Rightarrow t \leq -\frac{5}{2}$ and $t \geq 1$

Now for $t \geq \frac{1}{2}$, we get $t \geq 1$ from above conditions i.e. $\sin x \geq 1$

The inequality holds true only for x satisfying the equation $\sin x = 1 \therefore x = \frac{\pi}{2}$

(for $x \in [0,2\pi]$)

Case II : For $2t - 1 < 0 \Rightarrow t < \frac{1}{2}$

we have, $|2t - 1| = -(2t - 1)$

$\Rightarrow -3(2t - 1) \geq 7 - 4t^2 \Rightarrow -6t + 3 \geq 7 - 4t^2$

$\Rightarrow 4t^2 - 6t - 4 \geq 0 \Rightarrow 2t^2 - 3t - 2 \geq 0$

$\Rightarrow (t - 2)(2t + 1) \geq 0 \Rightarrow t \leq -\frac{1}{2}$ and $t \geq 2$

Again, for $t < \frac{1}{2}$ we get $t \leq -\frac{1}{2}$ from above conditions

i.e. $\sin x \leq -\frac{1}{2} \Rightarrow \frac{7\pi}{6} \leq x \leq \frac{11}{6}\pi$ (for $x \in [0,2\pi]$)

Thus, $x \in \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right] \cup \left\{\frac{\pi}{2}\right\}$ **Ans.**

Illustration 23 : Find the values of θ, for which $\cos 3\theta + \sin 3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$ is always positive.

Solution : Given expression can be written as :

$4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$

Applying given condition, we get

$\Rightarrow -4(\sin^3\theta - \cos^3\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow -4(\sin\theta - \cos\theta)(\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow -4(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(4\sin\theta\cos\theta - 3) > 0$

$\Rightarrow (\sin\theta - \cos\theta)\{-4 - 4\sin\theta\cos\theta + 3 + 4\sin\theta\cos\theta - 3\} > 0$

$\Rightarrow -4(\sin\theta - \cos\theta) > 0$

$\Rightarrow -4\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) > 0 \Rightarrow \sin\left(\theta - \frac{\pi}{4}\right) < 0 \Rightarrow 2n\pi - \pi < \theta - \frac{\pi}{4} < 2n\pi, n \in I$

$\Rightarrow 2n\pi - \frac{3\pi}{4} < \theta < 2n\pi + \frac{\pi}{4} \Rightarrow \theta \in \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right), n \in I$ **Ans.**

Illustration 24 : The number of values of x in the interval $[0,\ 5\pi]$ satisfying the equation

$3\sin^2 x - 7\sin x + 2 = 0$ is - **[JEE 98]**

(A) 0 (B) 5 (C) 6 (D) 10

Solution : $3\sin^2 x - 7\sin x + 2 = 0$

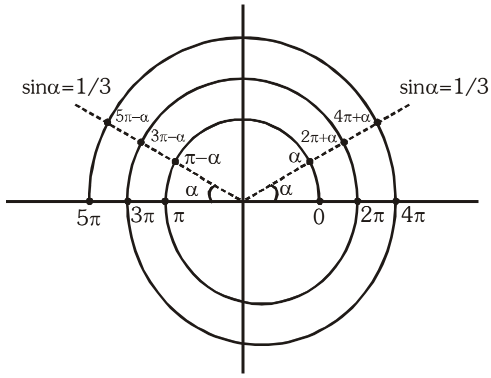
$\Rightarrow\ (3\sin x - 1)(\sin x - 2) = 0$

$\because\ \sin x \neq 2$

$\Rightarrow\ \sin x = \frac{1}{3} = \sin\alpha$ (say)

where α is the least positive value of x

such that $\sin\alpha = \frac{1}{3}$.



Clearly $0 < \alpha < \frac{\pi}{2}$. We get the solution,

$x = \alpha,\ \pi - \alpha,\ 2\pi + \alpha,\ 3\pi - \alpha,\ 4\pi + \alpha$ and $5\pi - \alpha$.

Hence total six values in $[0,\ 5\pi]$ **Ans. (C)**

ANSWERS FOR DO YOURSELF

1 : (i) (a) $\theta = n\pi + (-1)^n \frac{\pi}{6},\ n \in I$ **(b)** $\theta = (2n+1)\frac{\pi}{3},\ n \in I$ **(c)** $\theta = \frac{4n\pi}{3},\ n \in I$

(d) $\theta = \frac{n\pi}{2},\ n \in I$ **(e)** $\theta = n\pi \pm \frac{\pi}{12},\ n \in I$ **(f)** $\theta = 2n\pi + (-1)^{n+1}\pi,\ n \in I$

2 : (i) (a) $x = n\pi + (-1)^{n+1}\frac{\pi}{6},\ n \in I$ **(b)** $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8},\ n, k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3},\ n \in I$ **(d)** $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\frac{\sqrt{17}-1}{8}\right)$ or $\sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right),\ n \in I$

(ii) $\theta = \left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}\right\}$

3 : (i) (a) $x = 2n\pi - \frac{\pi}{4},\ n \in I$ **(b)** $2m\pi + \frac{\pi}{2},\ m \in I$

4 : (i) $\theta = n\pi$ or $\theta = \frac{m\pi}{3} \pm \frac{\pi}{9};\ n, m \in I$ **(ii)** $x = \frac{n\pi}{3},\ n \in I$ and $k\pi \pm \frac{\pi}{3},\ k \in I$

5 : (i) D **(ii)** $x = \frac{\pi}{4}$

6 : (i) $\bigcup_{n \in I}\left[2n\pi - \frac{2\pi}{3},\ 2n\pi + \frac{2\pi}{3}\right]$ **(ii)** $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

EXERCISE (O-1)

1. The general solution of $\tan 3x = 1$ is -

(A) $n\pi + \frac{\pi}{4} (n \in I)$ (B) $\frac{n\pi}{3} + \frac{\pi}{12} (n \in I)$ (C) $n\pi\ (n \in I)$ (D) $n\pi \pm \frac{\pi}{4} (n \in I)$ **TE0092**

2. If $2\tan^2\theta = \sec^2\theta$, then the general solution of θ -

(A) $n\pi + \frac{\pi}{4} (n \in I)$ (B) $n\pi - \frac{\pi}{4} (n \in I)$ (C) $n\pi \pm \frac{\pi}{4} (n \in I)$ (D) $2n\pi \pm \frac{\pi}{4} (n \in I)$ **TE0001**

3. Number of principal solution(s) of the equation $4\cdot 16^{\sin^2 x} = 2^{6\sin x}$ is

(A) 1 (B) 2 (C) 3 (D) 4 **TE0015**

4. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general solution of θ is -

(A) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$ (B) $n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{4}$ (C) $n\pi + (-1)^n \frac{\pi}{3} + \frac{\pi}{4}$ (D) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}$

where $n \in I$ **TE0014**

5. The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in (0, 10) is -

(A) 3 (B) 6 (C) 10 (D) 11 **TE0020**

6. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is :

(A) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3}, \pi\right\}$ (C) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (D) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$ **TE0005**

7. The equation $\sin x \cos x = 2$ has :

(A) one solution (B) two solutions (C) infinite solutions (D) no solution **TE0019**

8. If $0 \le x \le 3\pi$, $0 \le y \le 3\pi$ and $\cos x . \sin y = 1$, then the possible number of values of the ordered pair (x, y) is-

(A) 6 (B) 12 (C) 8 (D) 15 **TE0026**

9. If $\frac{\tan 2\theta + \tan\theta}{1 - \tan\theta\tan 2\theta} = 0$, then the general value of θ is -

(A) $n\pi$; $n \in I$ (B) $\frac{n\pi}{3}$; $n \in I$ (C) $\frac{n\pi}{4}$ (D) $\frac{n\pi}{6}$; $n \in I$ **TE0012**

where $n \in I$

10. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$, where $\alpha, \beta \in [-\pi, \pi]$, numbers of pairs of α, β which satisfy both the equations is

(A) 0 (B) 1 (C) 2 (D) 4 **TE0093**

11. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ (C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{48}, \pi\right)$ **TE0094**

12. The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is

(A) zero (B) one (C) two (D) four **TE0095**

PASSAGE :

Whenever the terms on the two sides of the equation are of different nature, then equations are known as Non standard form, some of them are in the form of an ordinary equation but can not be solved by standard procedures. Non standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, inequalities.

On the basis of above information, answer the following questions :

13. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2},\ 0 < x \le \frac{\pi}{2}$ has

(A) one real solutions (B) more than one real solutions
(C) no real solution (D) none of the above **TE0025**

14. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is-

(A) 0 (B) 1 (C) 2 (D) infinitely many **TE0096**

15. The number of solutions of the equation $\sin x = x^2 + x + 1$ is-

(A) 0 (B) 1 (C) 2 (D) None **TE0022**

EXERCISE (S-1)

1. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$ **TE0058**

2. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$. **TE0051**

3. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$ **TE0059**

4. Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \cdot \cos 2\theta \cdot \cos 3\theta = 1/4$. **TE0054**

5. Find the general solution of the equation, $2 + \tan x \cdot \cot\frac{x}{2} + \cot x \cdot \tan\frac{x}{2} = 0$ **TE0097**

6. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$. **TE0068**

7. Find the general solution of the trigonometric equation $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2}$. **TE0065**

8. Find the value of θ, which satisfy $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$. **TE0098**

9. Find the range of y such that the equation, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$. **TE0056**

10. Find the general values of θ for which the quadratic function $(\sin\theta)x^2 + (2\cos\theta)x + \frac{\cos\theta+\sin\theta}{2}$ is the square of a linear function. **TE0057**

11. Prove that the equations

(a) $\sin x \cdot \sin 2x \cdot \sin 3x = 1$ (b) $\sin x \cdot \cos 4x \cdot \sin 5x = -1/2$ **TE0066**

have no solution.

12. If α and β are the roots of the equation, $a\cos\theta + b\sin\theta = c$ then match the entries of **column-I** with the entries of **column-II**.

	Column-I		**Column-II**
(A)	$\sin\alpha + \sin\beta$	(P)	$\frac{2b}{a+c}$
(B)	$\sin\alpha \cdot \sin\beta$	(Q)	$\frac{c-a}{c+a}$
(C)	$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}$	(R)	$\frac{2bc}{a^2+b^2}$
(D)	$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}$	(S)	$\frac{c^2-a^2}{a^2+b^2}$

TE0099

EXERCISE (JM)

1. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation
$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is :- **[JEE(Main) 2016]**

(1) 9 (2) 3 (3) 5 (4) 7 **TE0074**

2. If sum of all the solutions of the equation $8\cos x\cdot\left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$ **TE0075**

3. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is
[JEE(Main)-Jan 19]

(1) 2 (2) 1 (3) 3 (4) 4 **TE0076**

4. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is : **[JEE(Main)-Jan 19]**

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{8}$ (4) $\frac{5\pi}{4}$ **TE0077**

5. Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is
[JEE(Main)-Apr 19]

(1) $\frac{13\pi}{6}$ (2) π (3) 2π (4) $\frac{5\pi}{3}$ **TE0078**

6. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the eauation.
[JEE(Main)-Apr 19]

(1) $\sin x = |\sin y|$ (2) $\sin x = 2 \sin y$ (3) $2|\sin x| = 3\sin y$ (4) $2\sin x = \sin y$ **TE0079**

7. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x,\ x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **[JEE(Main)-Apr 19]**

(1) 5 (2) 4 (3) 7 (4) 3 **TE0080**

8. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha \sin x = 2\alpha - 7$ has a solution. Then S is equal to : **[JEE(Main)-Apr 19]**

(1) [2, 6] (2) [3,7] (3) R (4) [1,4] **TE0081**

EXERCISE (JA)

1. The number of values of θ in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \ne \frac{n\pi}{5}$ for n = 0, ±1,±2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is **[JEE 2010, 3]**
TE0084

2. The positive integer value of n > 3 satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **TE0085**
[JEE 2011, 4]

3. The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$ in the interval $[0, 2\pi]$ is **[JEE 2015, 4M, –0M]**

TE0088

4. Let $S=\left\{x\in(-\pi,\pi): x\neq 0,\pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x+\operatorname{cosec}x+2(\tan x-\cot x)=0$ in the set S is equal to - **[JEE(Advanced)-2016, 3(–1)]**

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$ **TE0089**

5. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a\cos x+2b\sin x=c,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ has two distinct real roots α and β with $\alpha+\beta=\frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is ______ **TE0090**

[JEE(Advanced)-2018, 3(0)]

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x)=\sin(\pi\cos x)$ and $g(x)=\cos(2\pi\sin x)$ be two functions defined for $x>0$. Define the following sets whose elements are written in the increasing order :

$X=\{x: f(x)=0\}$, $Y=\{x: f'(x)=0\}$, $Z=\{x: g(x)=0\}$, $W=\{x: g'(x)=0\}$.

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

List-I	List-II
(I) X	(P) $\supseteq\left\{\frac{\pi}{2},\frac{3\pi}{2},4\pi,7\pi\right\}$
(II) Y	(Q) an arithmetic progression
(III) Z	(R) NOT an arithmetic progression
(IV) W	(S) $\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6}\right\}$
	(T) $\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$
	(U) $\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

6. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (R), (S) (2) (I), (P), (R) (3) (II), (Q), (T) (4) (I), (Q), (U) **TE0091**

7. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (Q), (T) (2) (IV), (P), (R), (S) (3) (III), (R), (U) (4) (III), (P), (Q), (U)

TE0091

ANSWER KEY

EXERCISE (O-1)

1. B	**2.** C	**3.** C	**4.** D	**5.** A	**6.** C	**7.** D	**8.** A
9. B	**10.** D	**11.** A	**12.** C	**13.** C	**14.** A	**15.** A	

EXERCISE (S-1)

1. $x = 2n\pi + \frac{\pi}{6}$, $n \in I$

2. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ & π

3. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}$, $n \in I$

4. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

5. $x = 2n\pi \pm \frac{2\pi}{3}$, $n \in I$

6. $x = \pi/16$

7. $x = 2n\pi + \frac{\pi}{12}$

8. $\theta = 2n\pi$ **or** $2n\pi + \frac{\pi}{2}$; $n \in I$

9. $-\sqrt{2} \le y \le \sqrt{2}$; $\frac{\pi}{2}, \pi$

10. $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2$, $n \in I$

12. (A) R; (B) S; (C) P; (D) Q

EXERCISE (JM)

1. 4	**2.** 1	**3.** 1	**4.** 1	**5.** 3	**6.** 1	**7.** 1
8. 1						

EXERCISE (JA)

1. 3	**2.** 7	**3.** 8	**4.** C	**5.** 0.5	**6.** 3	**7.** 2