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APPLICATION OF DERIVATIVE

TANGENT & NORMAL

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JEE (Main/Advanced) Syllabus

JEE(Main) Syllabus:

Applications of derivatives: Rate of change of quantities, monotonic increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

EXERCISE (O-2)

EXERCISE (S-1)

EXERCISE (S-2)

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ANSWER KEY

JEE(Advanced) Syllabus:

Geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, Rolle's Theorem and Lagrange's Mean Value Theorem.

RATE MEASURE, TANGENT & NORMAL

1. RATE MEASUREMENT:

Whenever one quantity y varies with another quantity x, satisfying some rule y = f(x), then $\frac{dy}{dx}$ (or f'(x)) represents the rate of change of y with respect to x and $\frac{dy}{dx}\Big]_{x=a}$ (or f'(a)) represents the

rate of change of y with respect to x at x = a.

- **Illustration 1:** The volume of a cube is increasing at a rate of 9cm³/s. How fast is the surface area increasing when the length of an edge is 10cm?
- Solution: Let x be the length of side, V be the volume and S be the surface area of the cube. Then $V=x^3$ and $S=6x^2$, where x is a function of time t.

$$\frac{dV}{dt} = 9cm^3 / s = \frac{d}{dt}(x^3) = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{3}{\mathrm{x}^2}$$

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}$$

$$\frac{\mathrm{dS}}{\mathrm{dt}}\bigg]_{\mathrm{x=10cm}} = 3.6 \; \mathrm{cm^2/s}.$$

- **Illustration 2:** x and y are the sides of two squares such that $y = x x^2$. Find the rate of change of the area of the second square with respect to the first square.
- **Solution:** Given x and y are sides of two squares. Thus the area of two squares are x^2 and y^2

We have to obtain
$$\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x} \cdot \frac{dy}{dx}$$
(i)

where the given curve is,
$$y = x - x^2 \implies \frac{dy}{dx} = 1 - 2x$$
 (ii)

Thus,
$$\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$$
 [from (i) and (ii)]

or
$$\frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x}$$
 $\Rightarrow \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$

Do yourself - 1:

- (i) What is the rate of change of the area of a circle with respect to its radius r at r = 6cm.
- (ii) A stone is dropped into a quiet lake and waves move in circles at the speed of 5cm/s. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area increasing?

2. APPROXIMATION USING DIFFERENTIALS:

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variable. In general dy = f(x)dx or df(x) = f(x)dx

Note:

- (i) For the independent variable 'x', increment Δ x and differential dx are equal but this is not the case with the dependent variable 'y' i.e. Δ y \neq dy.
 - \therefore Approximate value of y when increment Δx is given to independent variable x in y = f(x) is

$$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx} \cdot \Delta x$$

(ii) The relation dy = f'(x) dx can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

Illustration 3: Find the approximate value of square root of 25.2.

Solution: Let $f(x) = \sqrt{x}$

Now,
$$f(x + \Delta x) - f(x) = f'(x)$$
. $\Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, 25.2 = 25 + 0.2

Taking x = 25 and $\Delta x = 0.2$, we have

$$f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$$

or
$$f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \implies f(25.2) = 5.02$$

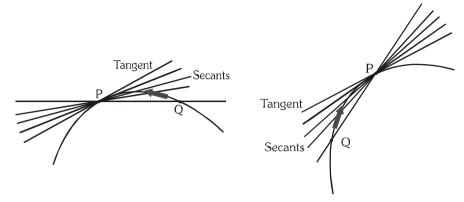
or
$$\sqrt{(25.2)} = 5.02$$

Do yourself - 2:

(i) Find the approximate value of $(0.009)^{1/3}$.

3. TANGENT TO THE CURVE AT A POINT:

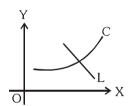
The tangent to the curve at 'P' is the line through P whose slope is limit of the secant slopes as $Q \rightarrow P$ from either side.



4. MYTHS ABOUT TANGENT:

(a) Myth: A line meeting the curve only at one point is a tangent to the curve.

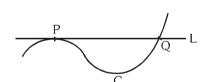
Explanation: A line meeting the curve in one point is not necessarily tangent to it.



Here L is not tangent to C

(b) Myth: A line meeting the curve at more than one point is not a tangent to the curve.

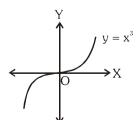
Explanation : A line may meet the curve at several points and may still be tangent to it at some point



Here L is tangent to C at P, and cutting it again at Q.

(c) Myth: Tangent at a point to the curve can not cross it at the same point.

Explanation: A line may be tangent to the curve and also cross it.



Here X-axis is tangent to $y = x^3$ at origin.

5. NORMAL TO THE CURVE AT A POINT :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

6. POINTS TO REMEMBER:

(a) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \frac{dy}{dx}\Big|_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(b) Equation of tangent at
$$(x_1, y_1)$$
 is; $y - y_1 = \frac{dy}{dx} \Big]_{(x_1, y_1)} (x - x_1)$

(c) Equation of normal at
$$(x_1, y_1)$$
 is; $y - y_1 = -\frac{1}{\frac{dy}{dx}}\Big|_{(x_1, y_1)}$

Note:

- (i) The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.
- (ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.
- (iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or dx/dy = 0.
- (iv) If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.
- (v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 y^2 + x^3 + 3x^2y y^3 = 0$, the tangents at the origin are given by $x^2 y^2 = 0$ i.e. x + y = 0 and x y = 0

Illustration 4: Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

Solution: The equation of the curve is $y = (x^3 - 1)(x - 2)$ (i)

It cuts x-axis at y = 0. So, putting y = 0 in (i), we get $(x^3 - 1)(x - 2) = 0$

$$\Rightarrow (x-1)(x-2)(x^2+x+1) = 0 \Rightarrow x-1 = 0, x-2 = 0 \qquad \left[\because x^2+x+1 \neq 0\right]$$
$$\Rightarrow x = 1, 2.$$

Thus, the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0). Now,

$$y = \left(x^3 - 1\right)\left(x - 2\right) \implies \frac{dy}{dx} = 3x^2\left(x - 2\right) + \left(x^3 - 1\right) \implies \left(\frac{dy}{dx}\right)_{(1,0)} = -3 \text{ and } \left(\frac{dy}{dx}\right)_{(2,0)} = 7$$

The equations of the tangents at (1, 0) and (2, 0) are respectively

$$y-0=-3(x-1)$$
 and $y-0=7(x-2) \implies y+3x-3=0$ and $7x-y-14=0$ **Ans**.

The equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$ is -Illustration 5:

(A)
$$x = 2$$

(B)
$$x = \pi$$

(C)
$$x + \pi = 0$$

(D)
$$2x = \pi$$

Solution:

$$\therefore$$
 $x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$, so the given point $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now from the given equation $\frac{dy}{dx} = 1 + \cos^2 x - \sin^2 x \implies \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},\frac{\pi}{2}\right)} = 1 + 0 - 1 = 0$

The curve has vertical normal at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The equation to this normal is $x = \frac{\pi}{2}$

$$\Rightarrow x - \frac{\pi}{2} = 0 \Rightarrow 2x = \pi$$

Ans. (D)

Illustration 6: The equation of normal to the curve $x + y = x^y$, where it cuts x-axis is -

(A)
$$y = x + 1$$

(B)
$$y = -x + 1$$

(C)
$$y = x - 1$$

(D)
$$y = -x - 1$$

Solution:

Given curve is $x + y = x^y$

(D)
$$y = -x -$$

at x-axis y=0,

$$\therefore x + 0 = x^0$$

$$x = 1$$

$$\therefore$$
 Point is A(1, 0)

Now to differentiate $x + y = x^y$ take log on both sides

$$\Rightarrow \log(x+y) = y \log x$$

$$\Rightarrow \log(x+y) = y \log x \qquad \qquad \therefore \quad \frac{1}{x+y} \left\{ 1 + \frac{dy}{dx} \right\} = y \cdot \frac{1}{x} + (\log x) \frac{dy}{dx}$$

Putting
$$x = 1, y = 0$$

Putting
$$x = 1, y = 0$$

$$\left\{1 + \frac{dy}{dx}\right\} = 0 \implies \left(\frac{dy}{dx}\right)_{(1,0)} = -1$$

 \therefore slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1} = 1 \implies y = x-1$

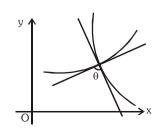
Ans. (C)

Do yourself - 3:

- Find the distance between the point (1,1) and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point where the curve cuts y-axis.
- Find the equation of a line passing through (-2,3) and parallel to tangent at origin for the (ii) circle $x^2 + y^2 + x - y = 0$.

7. ANGLE OF INTERSECTION BETWEEN TWO CURVES:

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection.



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Orthogonal curves:

If the angle between two curves at each point of intersection is 90° then they are called **orthogonal**

For example, the curves $x^2 + y^2 = r^2 & y = mx$ are orthogonal curves.

The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16, 8) is-Illustration 7:

(A)
$$60^{\circ}$$

(C)
$$\tan^{-1}\left(\frac{3}{5}\right)$$

(C)
$$\tan^{-1}\left(\frac{3}{5}\right)$$
 (D) $\tan^{-1}\left(\frac{4}{3}\right)$

Solution:

$$x^2 = 32y \implies \frac{dy}{dx} = \frac{x}{16} \implies y^2 = 4x \implies \frac{dy}{dx} = \frac{2}{y}$$

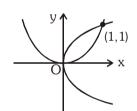
$$\therefore \text{ at } (16, 8), \left(\frac{dy}{dx}\right)_1 = 1, \left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$$

So required angle =
$$\tan^{-1} \left(\frac{1 - \frac{1}{4}}{1 + 1\left(\frac{1}{4}\right)} \right) = \tan^{-1} \left(\frac{3}{5}\right)$$
 Ans. (C)

Check the orthogonality of the curves $y^2 = x \& x^2 = y$. Illustration 8:

Solving the curves simultaneously we get points of intersection as (1, 1) and (0, 0). **Solution:**

At (1,1) for first curve
$$2y\left(\frac{dy}{dx}\right)_1 = 1 \Rightarrow m_1 = \frac{1}{2}$$



& for second curve
$$2x = \left(\frac{dy}{dx}\right)_2 \Rightarrow m_2 = 2$$

 $m_1 m_2 \neq -1$ at (1,1).

But at (0,0) clearly x-axis & y-axis are their respective tangents hence they are orthogonal at (0,0) but not at (1,1). Hence these curves are not said to be orthogonal.

If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is -Illustration 9:

(A)
$$\frac{1}{2}$$

(B)
$$\frac{1}{3}$$

Solution:

$$y = 1 - ax^2 \implies \frac{dy}{dx} = -2ax$$
 $y = x^2 \implies \frac{dy}{dx} = 2x$

$$y = x^2 \implies \frac{dy}{dx} = 2x$$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = -1$

$$\Rightarrow (-2ax)(2x) = -1$$

$$\Rightarrow 4ax^2 = 1$$

Now eliminating y from the given equations we have $1 - ax^2 = x^2$

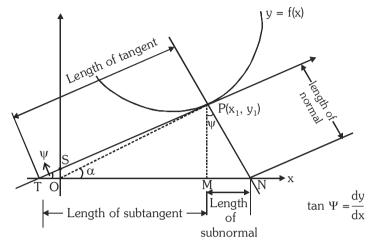
$$\Rightarrow (1+a)x^2 = 1$$

Eliminating
$$x^2$$
 from (i) and (ii) we get $\frac{4a}{1+a} = 1$ $\Rightarrow a = \frac{1}{3}$ Ans. (B)

Do yourself -4:

- (i) If two curves $y = a^x$ and $y = b^x$ intersect at an angle α , then find the value of $\tan \alpha$.
- (ii) Find the angle of intersection of curves $y = 4 x^2$ and $y = x^2$.

8. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL:



- (a) Length of the tangent (PT) = $\frac{y_1 \sqrt{1 + [f'(x_1)]^2}}{f'(x_1)}$
- **(b)** Length of Subtangent (MT) = $\frac{y_1}{f'(x_1)}$
- (c) Length of Normal (PN) = $\left| y_1 \sqrt{1 + \left[f'(x_1) \right]^2} \right|$
- (d) Length of Subnormal (MN) = $|y_1f(x_1)|$
- (e) Initial ordinate: Y intercept of tangent at point P(x, y) = OS
- (f) Radius vector (Polar radius): Line segment joining origin to point P(x, y) = OP
- (g) Vectorial angle: Angle made by radius vector with positive direction of x-axis in anticlock wise direction is called vectorial angle. In given figure α is vectorial angle.

Illustration 10: The length of the normal to the curve $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ at $\theta = \frac{\pi}{2}$ is -

(B)
$$\frac{a}{2}$$

(C)
$$\sqrt{2}a$$

(D)
$$\frac{a}{\sqrt{2}}$$

Solution:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a\left(1+\cos\theta\right)} = \tan\frac{\theta}{2} \implies \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \tan\left(\frac{\pi}{4}\right) = 1$$

Also at
$$\theta = \frac{\pi}{2}$$
, $y = a \left(1 - \cos \frac{\pi}{2} \right) = a$

∴ required length of normal =
$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$$
 Ans. (C)

Illustration 11: The length of the tangent to the curve $x = a \left(\cos t + \log \tan \frac{t}{2} \right)$, $y = a \sin t$ is

Solution:

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) / \left(\frac{dx}{dt}\right) = \frac{a \cos t}{a \left(-\sin t + \frac{1}{\sin t}\right)} = \tan t$$

$$\therefore \text{ length of the tangent} = y \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} = a \sin t \frac{\sqrt{1 + \tan^2 t}}{\tan t} = a \sin t \left(\frac{\sec t}{\tan t}\right) = a \text{ Ans. (C)}$$

Do yourself - 5:

- Prove that at any point of a curve, the product of the length of sub tangent and the length of sub normal is equal to square of the ordinates of point of contact.
- Find the length of subtangent to the curve $x^2 + y^2 + xy = 7$ at the point (1, -3).

Miscellaneous Illustrations:

Find the slope of normal at the point with abscissa x = -2 of the graph of the function Illustration 12:

$$f(\mathbf{x}) = |\mathbf{x}^2 - |\mathbf{x}||$$

 $f(x) = |x^2 - |x||$ At x = -2, f(x) becomes $f(x) = x^2 + x$ Solution:

$$f(\mathbf{x}) = \mathbf{x}^2 + \mathbf{x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1 = -3$$

Slope of normal $=\frac{1}{3}$

Illustration 13: If y = 4x - 5 is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then (A) p = 2, q = -7 (B) p = -2, q = 7 (C) p = -2, q = -7 (D) p = 2, q = 7

(A)
$$p = 2$$
, $q = -7$

(B)
$$p = -2$$
, $q = 7$

(C)
$$p = -2$$
, $q = -3$

(D)
$$p = 2$$
, $q = 7$

Solution:

$$\frac{dy}{dx} = 4 \quad \& \quad 9 = 8p + q$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3px^2$$

$$6\frac{dy}{dx} = 3p(4)$$
 \Rightarrow $\frac{dy}{dx} = 2p = 4$ \Rightarrow $p = 2 & q = -7$

ANSWERS FOR DO YOURSELF

- (i) 12π cm
- (ii) $80 \text{ } \pi\text{cm}^2/\text{s}$
- (i) 0.208

- $\left| \frac{\ell \text{na} \ell \text{nb}}{1 + \ell \text{na} \, \ell \text{nb}} \right| \qquad \text{(ii)} \quad \tan^{-1} \left(\frac{4\sqrt{2}}{7} \right)$
- **(ii)** 15

 $(A) x = \frac{\pi}{4}$

 $(A) 3tx - 2y = at^3$

1.

2.

(D) No where

(D) None of these

TN0004

EXERCISE (O-1)

(C) $x = \pi$

 $(C) 3tx + 2y = at^3$

The slope of the curve $y = \sin x + \cos^2 x$ is zero at the point, where-

The equation of tangent at the point (at², at³) on the curve $ay^2 = x^3$ is-

(B) $tx - 3y = at^3$

 $(B) x = \frac{\pi}{2}$

3.	The equation of normal to the curve $y = x^3 - 2x^2 + 4$ at the point $x = 2$ is-						
J•	(A) x + 4y = 0		(C) $x + 4y = 18$	(D) $4x - y = 18$	TN0006		
4.	The slope of the normal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at point $\theta = \pi/2$ is-						
	(A) 0	(B) 1	(C) –1	(D) $1/\sqrt{2}$	TN0007		
5.	Consider the curve repr	resented parametrically by	the equation		11(0007		
			$y = 2t^2 + 3t - 5$ where t				
		er of point on the curve wh	nere the tangent is horizon	ntal and V the numb	er of point		
	where the tangent is ver $(A) H = 2$ and $V = 1$	tical then	(B) $H = 1$ and $V = 2$				
	(C) $H = 2$ and $V = 1$		(D) $H = 1$ and $V = 2$				
	•		•		TN0009		
6.		ouches the curve $y = be^{-x}$		(D) (L 0)			
	(A)(0,a)	(B)(0,0)	(C)(0,b)	(D)(b,0)	TN0010		
		2 2			1110010		
7.	The curve $x^2 - y^2 = 5$ ar	$and \frac{x^2}{18} + \frac{y^2}{8} = 1 \text{ cut each } c$	other at any common poir	nt at an angle-			
	(A) $\pi/4$	(B) $\pi/3$	(C) $\pi/2$	(D) None of these			
8.	The lines tangent to the	e curve $y^3 - x^2y + 5y - 2x$	$= 0$ and $v^4 - v^3v^2 + 5v + 7$	2y = 0 at the origin i	TN0013		
0.	an angle θ equal to-	$\frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{i=1}^{n} \frac{1}$	-0 and X - X y + 3X + 2	zy – o at the origin i	intersect at		
	$(A) \frac{\pi}{6}$	(B) $\frac{\pi}{4}$	(C) $\frac{\pi}{3}$	(D) $\frac{\pi}{2}$			
	TI 1 0:	1	0 1 2 4 10 4	• • •	TN0014		
9.	The angle of intersection (A) 90°	on between the curves y ² (B) 60°	$= 8x$ and $x^2 = 4y - 12$ at ((C) 45°	(D) 0°			
	(A) 90	(B) 00	(C) 43	(D) 0	TN0016		
10.		ent to the curve $x^2 + xy + y$		S-			
	(A) 3	(B) 5	(C) 15	(D) $3/5$	TINIO 0 1 F		
					TN0017		
11.	Let S be a square with sid	des of length x. If we appro	eximate the change in size	of the area of S by h	$\left. \frac{\mathrm{dA}}{\mathrm{dx}} \right _{\mathrm{x}=\mathrm{x}_0}$		
	when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is						
	$(A) h^2$	$(B) 2hx_0$	(C) x_0^2	(D) h	•		
	(1.1) 11	(D) 21111 ₍₎	(C) A ₀	(D) II	TN0003		
					21,000		

- A Spherical balloon is being inflated at the rate of 35cc/min. The rate of increase in the surface area (in **12.** cm2/min.) of the balloon when its diameter is 14 cm, is: [JEE-MAIN Online 2013]
 - (A) $\sqrt{10}$
- (B) $10\sqrt{10}$
- (C) 100
- (D) 10

TN0002

- **13.** If the surface area of a sphere of radius r is increasing uniformly at the rate 8cm²/s, then the rate of change of its volume is: [JEE-MAIN Online 2013]
 - (A) proportional to r²
- (B) constant
- (C) proportional to r
- (D) proportional to \sqrt{r}

TN0001

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

- A line L is perpendicular to the curve $y = \frac{x^2}{4} 2$ at its point P and passes through (10, -1). The coordinates 1. of the point P are
 - (A) (2,-1)
- (B) (6,7)
- (C)(0,-2)
- (D)(4,2)

TN0023

- The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to: 2.
 - (A) square of the abscissa of the point of tangency
 - (B) square root of the abscissa of the point of tangency
 - (C) cube of the abscissa of the point of tangency
 - (D) cube root of the abscissa of the point of tangency.

TN0025

- **3.** At any two points of the curve represented parametrically by $x = a(2 \cos t - \cos 2t)$; $y = a (2 \sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by:
 - (A) $2\pi/3$
- (B) $3\pi/4$
- (C) $\pi/2$
- (D) $\pi/3$

TN0028

- The angle between the tangent lines to the graph of the function $f(x) = \int_{0}^{x} (2t 5) dt$ 4. points where the graph cuts the x-axis is
 - (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{2}$

- 5. The coordinates of the points on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, where tangent is inclined an angle $\pi/4$ to the x-axis are-
 - (A)(a, a)

- (B) $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ (C) $\left(a\left(\frac{\pi}{2}+1\right), a\right)$ (D) $\left(a, a\left(\frac{\pi}{2}+1\right)\right)$

- If the tangent to the curve $2y^3 = ax^2 + x^3$ at a point (a, a) cuts off intercepts p and q on the coordinates axes, 6. where $p^2 + q^2 = 61$, then a equals-
 - (A)30
- (B) 30
- (C) 0

(D) ± 30

- 7. The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at point (4, 4) on coordinate axes is-
 - (A) $4\sqrt{2}$
- (B) $6\sqrt{3}$
- (C) $8\sqrt{2}$
- (D) $\sqrt{256}$

TN0012

- 8. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are : [On line 2016]
 - (A) $(t^2 + 3, t^3 1)$

(B) $(t^2 + 3, -t^3 - 1)$

(C) $(16t^2 + 3, -64t^3 - 1)$

(D) $(4t^2 + 3, -8t^3 - 1)$

TN0015

- 9. For a curve $\frac{(\text{length of normal})^2}{(\text{length of tangent})^2}$ is equal to -
 - (A) (subnormal)/(subtangent)

(B) (subtangent)/(subnormal)

(C) (subtangent × subnormal)

(D) constant

TN0018

- 10. At any point of a curve (subtangent) × (subnormal) is equal to the square of the-
 - (A) slope of the tangent at that point
- (B) slope of the normal at that point

(C) abscissa of that point

(D) ordinate of that point

TN0019

EXERCISE (S-1)

1. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0.

TN0047

Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

TN0048

3. The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal at the point (-2, 2) on the curve $y = x^2 + 6x + 10$. Find the value of a and b.

TN0049

- 4. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line x + 2y = 0.
- The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at P (-2,0) & cuts the y-axis at a point Q where its gradient is 3. Find a, b, c.

TN0052

Show that the angle between the tangent at any point 'A' of the curve $\ln(x^2 + y^2) = C \tan^{-1} \frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

TN0058

7. Water is being poured on to a cylindrical vessel at the rate of 1 m³/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

TN0034

- **8.** A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.
 - (i) how fast is the farther end of the shadow moving on the pavement?
 - (ii) how fast is his shadow lengthening?

TN0035

EXERCISE (S-2)

RATE MEASURE AND APPROXIMATIONS

Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve xy = 1 - y.

TN0061

2. A function is defined parametrically by the equations

$$f(t) = x = \begin{bmatrix} 2t + t^2 \sin \frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{bmatrix} \quad \text{and } g(t) = y = \begin{bmatrix} \frac{1}{t} \sin t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{bmatrix}$$

Find the equation of the tangent and normal at the point for t = 0 if exist.

TN0062

Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3} & (x^2/a^2) + (y^2/b^2) = 1$ may touch if c = a + b.

TN0066

4. Prove that the segment of the normal to the curve $x = 2a \sin t + a \sin t \cos^2 t$; $y = -a \cos^3 t$ contained between the co-ordinate axes is equal to 2a.

TN0067

5. (a) Show that the curves $\frac{x^2}{a^2 + K_1} + \frac{y^2}{b^2 + K_1} = 1 \& \frac{x^2}{a^2 + K_2} + \frac{y^2}{b^2 + K_2} = 1$ intersect orthogonally.

TN0056

(b) If the two curves C_1 : $x = y^2$ and C_2 : xy = k cut at right angles find the value of k.

TN0057

A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

TN0036

7. An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of 1.5 cm³/min. Find the rate at which level of water in the cone is rising, when the depth of water is 4 cm.

TN0037

8. Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm³/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

TN0038

A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1 cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

TN0041

10. Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

TN0042

11. A circular ink blot grows at the rate of 2 cm² per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$.

EXERCISE (JM)

1.	The eq	uation of the tan	gent to t	he curve $y = x +$	$\frac{4}{x^2}$, tha	t is parallel to the	e x-axis	, is :	-
	(4)	•	(0)		(2)			•	[AIEEE-2010]

(1) y = 0

TN0068

The intercepts on x-axis made by tangents to the curve, $y = \int_{0}^{x} |t| dt$, $x \in R$, which are parallel to the line 2.

y = 2x, are equal to

[JEE-MAIN 2013]

 $(1) \pm 1$

 $(2) \pm 2$

 $(3) \pm 3$

 $(4) \pm 4$

TN0082

The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1): **3.**

[JEE-MAIN 2015]

(1) meets the curve again in the third quadrant

(3) does not meet the curve again

(2) meets the curve again in the fourth quadrant

(4) meets the curve again in the second quadrant

Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$. A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the 4.

point:

[JEE-MAIN 2016]

 $(1)\left(\frac{\pi}{4},0\right)$

(2)(0,0)

 $(3)\left(0,\frac{2\pi}{3}\right)$

 $(4)\left(\frac{\pi}{6},0\right)$

TN0070

The normal to the curve y(x-2)(x-3) = x+6 at the point where the curve intersects the y-axis passes **5.** through the point: [JEE-MAIN 2017]

 $(1)\left(\frac{1}{2},\frac{1}{3}\right) \qquad (2)\left(-\frac{1}{2},-\frac{1}{2}\right) \qquad (3)\left(\frac{1}{2},\frac{1}{2}\right)$

(4) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

TN0071

If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is: 6.

 $(1) \frac{7}{2}$

(2) 4

 $(3) \frac{9}{2}$

(4) 6 [JEE-MAIN 2018]

TN0072

The tangent to the curve, $y = xe^{x^2}$ passing through the point (1,e) also passes through the point : 7.

[JEE-MAIN 2019]

 $(1)\left(\frac{4}{3},2e\right)$

(2)(2.3e)

(3) $\left(\frac{5}{3}, 2e\right)$ (4) (3,6e)

TN0073

EXERCISE (JA)

If the normal to the curve, y = f(x) at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis. Then 1. f'(3) =[JEE 2000 (Scr.) 1 out of 35]

(A) - 1

(B) $-\frac{3}{4}$

(C) $\frac{4}{3}$

(D) 1

The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are) [JEE 2002 (Scr.), 3] 2.

(A)
$$\left(\pm \frac{4}{\sqrt{3}}, -2\right)$$
 (B) $\left(\pm \sqrt{\frac{11}{3}}, 1\right)$ (C) $(0, 0)$

$$(B)\left(\pm\sqrt{\frac{11}{3}},1\right)$$

(D)
$$\left(\pm \frac{4}{\sqrt{3}}, 2\right)$$

TN0078

Tangent to the curve $y = x^2 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point **3.** Q. Then the coordinates of Q are [JEE 2005 (Scr.), 3]

$$(A)(-6,-11)$$

$$(B)(-9,-13)$$

$$(C)(-10, -15)$$

(D)
$$(-6, -7)$$

ANSWER KEY

(TANGENT & NORMAL)

EXERCISE (0-1)

EXERCISE (O-2)

EXERCISE (S-1)

1.
$$x + y - 1 = 0$$

EXERCISE (S-1) 1.
$$x + y - 1 = 0$$
2. $x = 1$ when $t = 1$, $m \to \infty$; $5x - 4y = 1$ if $t \ne 1$, $t = 1/3$
3. $a = 1$, $b = \frac{-5}{2}$
4. $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$
5. $a = -1/2$; $b = -3/4$; $c = 3$

3.
$$a = 1, b = \frac{-5}{2}$$

4.
$$x + 2y = \pi/2$$
 & $x + 2y = -3\pi/2$

5.
$$a = -1/2$$
; $b = -3/4$; $c = 3$

$$\theta = \tan^{-1}\frac{2}{C}$$

7.
$$1/9 \pi$$
 m/min

6.
$$\theta = \tan^{-1} \frac{2}{C}$$
 7. $1/9 \pi$ m/min **8.** (i) 6 km/h; (ii) 2 km/hr

EXERCISE (S-2)

1.
$$(0,1)$$
 2. $T: x-2y=0$; $N: 2x+y=0$ **5. (b)** $\pm \frac{1}{2\sqrt{2}}$ **6.** $(4,11)$ & $(-4,-31/3)$

5. **(b)**
$$\pm \frac{1}{2\sqrt{2}}$$

7.
$$3/8 \pi$$
 cm/min

8.
$$\frac{\sqrt{2}}{4\pi}$$
 cm/s

7.
$$3/8 \pi$$
 cm/min 8. $\frac{\sqrt{2}}{4 \pi}$ cm/s 9. $1 + 36 \pi$ cu. cm/sec 10. $1/48 \pi$ cm/s

10.
$$1/48 \pi \text{ cm/s}$$

11.
$$\frac{1}{4}$$
 cm/sec.

EXERCISE (JM)

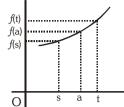
EXERCISE (JA)

MONOTONICITY

1. INCREASING / DECREASING / STRICTLY INCREASING / STRICTLY DECREASING NATURE OF A FUNCTION AT A POINT:



If $f(s) \le f(a) \le f(t)$ when ever s < a < t, where $s, t \in (a - h, a + h) \cap D_f$ for some h > 0, then f is said to be increasing at x = a.



(1) When 'a' be left end of the interval

$$f(a) \le f(x) \ \forall \ x \in (a, a+h) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is increasing at x = a.

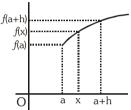
(2) When 'a' be right end of the interval

$$f(x) \le f(a) \ \forall \ x \in (a-h, a) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is increasing at x = a.



If f(s) < f(a) < f(t) when ever s < a < t, where $s, t \in (a - h, a + h) \cap D_f$ for some h > 0, then f(s) = 0 is said to be strictly increasing at x = a.



(1) When 'a' be left end of the interval

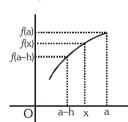
$$f(a) < f(x) \forall x \in (a, a+h) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is strictly increasing at x = a.

(2) When 'a' be right end of the interval

$$f(x) \le f(a) \ \forall \ x \in (a - h, a) \cap D_f$$
 for some $h \ge 0$

 \Rightarrow f is strictly increasing at x = a.



III. Decreasing at x = a:

If $f(s) \ge f(a) \ge f(t)$ when ever $s \le a \le t$, where $s, t \in (a - h, a + h) \cap D_f$ for some h > 0, then f is said to be decreasing at x = a.

(1) When 'a' be left end of the interval

$$f(a) \ge f(x) \ \forall \ x \in (a, a+h) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is decreasing at x = a.

(2) When 'a' be right end of the interval

$$f(x) \ge f(a) \ \forall \ x \in (a-h, a) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is decreasing at x = a.

IV. Strictly decreasing at x = a:

If f(s) > f(a) > f(t) when ever s < a < t, where $s, t \in (a - h, a + h) \cap D_f$ for some h > 0, then f is said to be strictly decreasing at x = a.

(1) When 'a' be left end of the interval

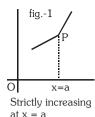
$$f(a) > f(x) \forall x \in (a, a+h) \cap D_f$$
 for some $h > 0$

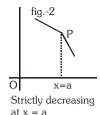
 \Rightarrow f is strictly decreasing at x = a.

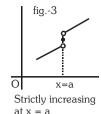
(2) When 'a' be right end of the interval

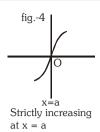
$$f(x) > f(a) \ \forall \ x \in (a - h, a) \cap D_f$$
 for some $h > 0$

 \Rightarrow f is strictly decreasing at x = a.









2. INCREASING & DECREASING NATURE OF A FUNCTION OVER AN INTERVAL:

Consider an interval $I \subseteq D_{\epsilon}$

I. Increasing Over an Interval I:

 $\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \le f(x_2)$, then f is increasing over interval I.

II. Decreasing Over an Interval I:

 $\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \ge f(x_2)$, then f is decreasing over interval I.

III. Strictly increasing over an Interval I:

 $\forall x_1, x_2 \in I, x_1 \le x_2 \Leftrightarrow f(x_1) \le f(x_2), \text{ then } f \text{ is strictly increasing over interval } I.$

IV. Strictly decreasing over an Interval I:

 $\forall x_1, x_2 \in I, x_1 \le x_2 \Leftrightarrow f(x_1) \ge f(x_2)$, then f is strictly decreasing over interval I.

3. MONOTONIC FUNCTION:

If a function is either increasing or decreasing over an interval then it is said to be monotonic function over the interval.

If a function is either strictly increasing or strictly decreasing over an interval then it is said to be strictly monotonic function over the interval.

4. FOR DIFFERENTIABLE FUNCTIONS:

Consider an interval $I \subseteq D_f$ that can be [a, b] or (a, b) or [a, b) or (a, b].

- (1) $f'(x) > 0 \forall x \in I \Rightarrow f$ is strictly increasing function over the interval I.
- (2) $f'(x) \ge 0 \ \forall \ x \in I \implies f$ is increasing function over the interval I.
- (3) $f'(x) \ge 0 \ \forall \ x \in I$ and f'(x) = 0 do not form any interval (that means f'(x) = 0 at discrete points) $\Rightarrow f$ is strictly increasing function over the interval I.
- (4) $f'(x) < 0 \forall x \in I \implies f$ is strictly decreasing function over the interval I.
- (5) $f'(x) \le 0 \ \forall \ x \in I \implies f$ is decreasing function over the interval I.
- (6) $f'(x) \le 0 \ \forall \ x \in I \text{ and } f'(x) = 0 \text{ do not form any interval (that means } f'(x) = 0 \text{ at discrete points)}$ $\Rightarrow f \text{ is strictly decreasing function over the interval I.}$

Illustration 1:

Let $f(x) = x^3 - 3x + 2$. Examine the monotonocity of function at points x = 0, 1 & 2.

Solution:

at
$$x = 0$$
,

$$f(0) = 2$$
, $f(0 + h) = h(h - 3) + 2 < 2$

$$f(0-h) = h(3-h) + 2 > 2$$

$$\Rightarrow f(0-h) > f(0) > f(0+h)$$

f is increasing or strictly increasing at x = 0.

at x = 1, f(1 - h) > f(1) < f(1 + h) neither increasing nor decreasing

Similarly at
$$x = 2$$
, $f(2 - h) < f(2) < f(2 + h)$ increasing at $x = 2$.

Е

Illustration 2: Prove that the function $f(x) = \log(x^3 + \sqrt{x^6 + 1})$ is strictly increasing.

Solution: Now, $f(x) = log(x^3 + \sqrt{x^6 + 1})$

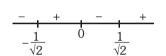
$$f'(x) = \frac{1}{x^3 + \sqrt{x^6 + 1}} \left(3x^2 + \frac{6x^5}{2\sqrt{x^6 + 1}} \right) = \frac{3x^2}{\sqrt{x^6 + 1}} \ge 0$$

 \Rightarrow f(x) is strictly increasing.

Illustration 3: Find the intervals of monotonocity of the function $y = x^2 - \log_e |x|$, $(x \neq 0)$.

Solution: Let $y = f(x) = x^2 - \log_e |x|$

$$f'(x) = 2x - \frac{1}{x}$$
; for all $x(x \neq 0)$



$$f'(x) = \frac{2x^2 - 1}{x}$$
 \Rightarrow $f'(x) = \frac{\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right)}{x}$

So $f'(x) \ge 0$ when $x \in \left[-\frac{1}{\sqrt{2}}, 0 \right] \cup \left[\frac{1}{\sqrt{2}}, \infty \right]$ and $f'(x) \le 0$

when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left(0, \frac{1}{\sqrt{2}}\right]$

 $\therefore f(x) \text{ is strictly increasing when } x \in \left[-\frac{1}{\sqrt{2}}, 0 \right]; \left[\frac{1}{\sqrt{2}}, \infty \right]$

and strictly decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right]; \left(0, \frac{1}{\sqrt{2}}\right]$

Do yourself - 1:

- (i) If function $f(x) = x^3 + \lambda x^2 \lambda x + 1$ is increasing at x = 0 & decreasing at x = 1, then find the greatest integral value of λ .
- (ii) If $f(x) = \sin x + \ln|\sec x + \tan x| 2x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then check the monotonicity of f(x)
- (iii) Prove that $y = e^x + \sin x$ is strictly increasing in $x \in R^+$

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ALLEN

5. **GREATEST AND LEAST VALUE OF A FUNCTION:**

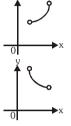
If a continuous function y = f(x) is increasing in the closed interval [a, b], then f(a) is the least value and f(b) is the greatest value of f(x) in [a, b] (figure-1)



(b) If a continuous function y = f(x) is decreasing in [a, b], then f(b) is the least and f(a) is the greatest value of f(x) in [a, b]. (figure-2)



(c) If a continuous function y = f(x) is increasing/decreasing in the (a,b), then no greatest and least value exist.



Show that $f(x) = \sin^{-1} \frac{x}{\sqrt{1+x^2}} - \ell nx$ is strictly decreasing in $x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find Illustration 4: its range.

Solution:
$$f(x) = \sin^{-1} \frac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1} x - \ell nx \implies f'(x) = \frac{1}{1+x^2} - \frac{1}{x} = \frac{-(1+x^2-x)}{x(1+x^2)}$$

$$\therefore f'(x) \le 0 \ \forall \ x \in \left[\frac{1}{\sqrt{3}}, \ \sqrt{3}\right]$$

f(x) is strictly decreasing.

$$f(x)|_{max} = f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{2}\ln 3$$
 & $f(x)|_{min} = f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{2}\ln 3$

$$\therefore \quad \text{Range of f(x)} = \left[\frac{\pi}{3} - \frac{1}{2} \ln 3, \ \frac{\pi}{6} + \frac{1}{2} \ln 3 \right]$$

Ans.

Find the greatest and least value of $f(x) = x^3 + 5x + e^x$ in [1, 3] *Illustration 5*:

 $f'(x) = 3x^2 + 5 + e^x \implies f(x)$ is strictly increasing. **Solution:**

Least value = f(1) = 6 + e

greatest value = $f(3) = (42 + e^3)$

6. PROVING INEQUALITIES USING MONOTONICITY:

Comparison of two functions f(x) and g(x) can be done by analysing their monotonic behaviour.

Let f (x) and g (x) are two function which are defined and differentiable for all $x \ge x_0$. Illustration 6: If $f(x_0) = g(x_0)$ and f'(x) > g'(x) for all $x > x_0$ then

(A)
$$f(x) \le g(x)$$
 for some $x \ge x_0$

(B)
$$f(x) = g(x)$$
 for some $x > x_0$

$$(A) f(x) < g(x) \text{ for some } x > x_0$$

$$(B) f(x) = g(x) \text{ for some } x > x_0$$

$$(C) f(x) > g(x) \text{ only for some } x > x_0$$

$$(D) f(x) > g(x) \text{ for all } x > x_0$$

(D)
$$f(x) > g(x)$$
 for all $x > x_0$

Let
$$h(x) = f(x) - g(x)$$

$$h'(x) = f'(x) - g'(x) > 0 \ \forall \ x > x_0$$

$$h(x_0) = 0$$

$$\Rightarrow$$
 h(x) > 0 \forall x > x₀

Ans. (D)

Do yourself - 2:

- (i) Let $f(x) = \frac{x^3}{3} \frac{x^2}{2} + 2$ in [-2, 2]. Find the greatest and least value of f(x) in [-2, 2]
- (ii) Prove that $x > \sin x > x \frac{x^3}{6}$ for all x > 0.

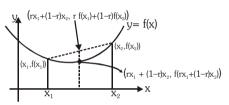
7. CONCAVITY OR CONVEXITY OF FUNCTION:

Convex function (concave upward):

If
$$f(rx_1 + (1-r)x_2) \le rf(x_1) + (1-r)f(x_2)$$

$$\forall x_1, x_2 \in I \text{ and } \forall r \in [0, 1]$$

then f is said to be convex function over I (or concave upward).



Strictly Convex function (strictly concave upward):

If
$$f(rx_1 + (1 - r)x_2) < rf(x_1) + (1 - r)f(x_2) \ \forall \ x_1, x_2 \in I \text{ where } x_1 \neq x_2 \text{ and } \forall \ r \in (0, 1)$$

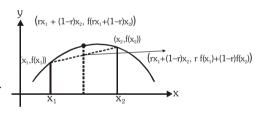
then f is said to be strictly convex function over I (or strictly concave upward).

Concave function (convex upward):

If
$$f(rx_1 + (1 - r)x_2) \ge rf(x_1) + (1 - r)f(x_2)$$

$$\forall x_1, x_2 \in I \text{ and } \forall r \in [0, 1]$$

then f is said to be concave function over I (or convex upward).



Strictly Concave function (Strictly convex upward):

If
$$f(rx_1 + (1 - r)x_2) > rf(x_1) + (1 - r)f(x_2) \ \forall x_1, x_2 \in I \text{ where } x_1 \neq x_2 \text{ and } \forall r \in (0, 1)$$

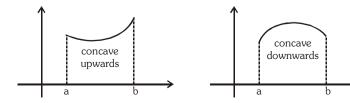
then f is said to be Strictly concave function over I (or Strictly convex upward).

Note:

- 1. Straight line or linear function is said to be concave up as well as concave down.
- 2. The sign of the 2^{nd} order derivative determines the concavity of the curve.

i.e. If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) \le 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).



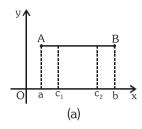
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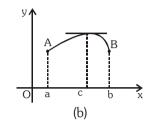
8. ROLLE'S THEOREM:

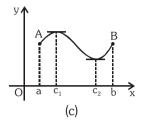
Let f be a function that satisfies the following three conditions:

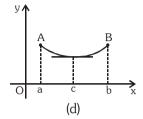
- (a) f is continuous on the closed interval [a, b].
- **(b)** f is differentiable on the open interval (a, b)
- (c) f(a) = f(b)

Then there exist at least one number c in (a, b) such that f(c) = 0.









Note: If f is differentiable function then between any two consecutive roots of f(x) = 0, there is at least one root of the equation f'(x) = 0.

(d) Geometrical Interpretation:

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 7: Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Solution : Here we observe that

(a) f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence f(x) is continuous in the [0,2] and differentiable in the (0,2).

(b)
$$f(0) = 0$$
, $f(2) = 2^3 - 3$. $(2)^2 + 2(2) = 0$
 $f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that f'(c) = 0

$$\Rightarrow f(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$$

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Illustration 8: Let Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$, when $1 \le x \le 2$ at the point $c = \frac{4}{3}$, then find a + b.

$$f(1) = f(2) \Rightarrow 1 + b + a = 8 + 4b + 2a$$

$$a + 3b + 7 = 0$$
(1

$$f'(c) = 3x^2 + 2bx + a = 0$$

$$\frac{16}{3} + \frac{8b}{3} + a = 0 \implies 3a + 8b + 16 = 0$$
(2

By solving
$$a = 8$$
, $b = -5$

Do yourself - 3:

- (i) Verify Rolle's theorem for $y = 1 (x^4)^{1/3}$ on the interval [-1,1]
- (ii) (a) Let $f(x) = 1 (x^2)^{1/3}$. Show that f(-1) = f(1) but there is no number c in (-1,1) such that f'(c) = 0.

Why does this not contradict Rolle's Theorem?

(b) Let $f(x) = (x-1)^{-2}$. Show that f(0) = f(2) but there is no number c in (0,2) such that f'(c) = 0. Why does this not contradict Rolle's Theorem?

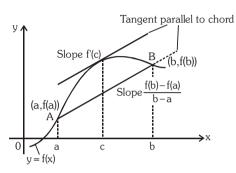
9. LAGRANGE'S MEAN VALUE THEOREM (LMVT):

Let f be a function that satisfies the following conditions:



(ii) f is differentiable in (a, b).

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



(a) Geometrical Interpretation:

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations:

If we think of the number (f(b) - f(a))/(b - a) as the average change in f over [a, b] and f'(c) as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration 9: Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Solution: Given
$$f(x) = 3x^2 + 5x + 7$$
 (i)
 $\Rightarrow f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again
$$f(x) = 6x + 5$$

Here
$$a = 1, b = 3$$

Now from Lagrange's mean value theorem

$$f(c) = \frac{f(b) - f(a)}{b - a}$$
 \Rightarrow $6c + 5 = \frac{f(3) - f(1)}{3 - 1} = \frac{49 - 15}{2} = 17 \text{ or } c = 2.$

Illustration 10: If f(x) is continuous and differentiable over [-2, 5] and $-4 \le f'(x) \le 3$ for all x in (-2, 5), then the greatest possible value of f(5) - f(-2) is -

Solution:

Apply LMVT

$$f'(x) = \frac{f(5) - f(-2)}{5 - (-2)}$$
 for some x in (-2, 5)

$$-28 \le f(5) - f(-2) \le 21$$

 \therefore Greatest possible value of f(5) - f(-2) is 21.

Do yourself - 4:

- (i) If $f(x) = x^2$ in [a, b], then show that there exist at least one c in (a, b) such that a, c, b are in A.P.
- (ii) Find C of LMVT for $f(x) = |x|^3$ in [2, 5].

10. SPECIAL NOTE:

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) Let f(x) is differentiable & either strictly increasing or strictly decreasing for $a \le x \le b$.

&

(b) f(a) and f(b) have opposite signs.

Then there is one & only one root of the equation f(x) = 0 in (a, b).

Miscellaneous Illustrations:

Illustration 11: If g(x) = f(x) + f(1 - x) and f''(x) < 0; $0 \le x \le 1$, show that g(x) strictly increasing in $x \in (0, 1/2)$ and strictly decreasing in $x \in (1/2, 1)$

Solution: $f'(x) < 0 \Rightarrow f(x)$ is strictly decreasing function.

$$Now, g(x) = f(x) + f(1 - x)$$

$$g'(x) = f'(x) - f(1-x)$$
(i)

Case I: If $x > (1 - x) \Rightarrow x > 1/2$

$$\therefore \ f(x) < f(1-x) \ \Rightarrow \ f(x) - f(1-x) < 0$$

$$\Rightarrow$$
 g'(x) < 0

$$\therefore g(x) \text{ strictly decreases in } x \in \left(\frac{1}{2}, 1\right)$$

Case II: If $x < (1 - x) \Rightarrow x < 1/2$

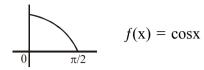
$$\therefore \quad f(x) > f(1-x) \quad \Rightarrow \quad f(x) - f(1-x) > 0$$

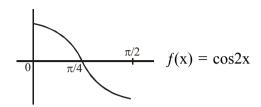
$$\Rightarrow$$
 g'(x) > 0

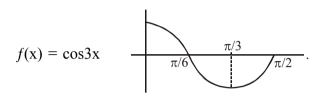
 \therefore g(x) strictly increases in x \in (0, 1/2)

- **Illustration 12:** Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$
 - (A) cosx
- $(B) \cos 2x$
- $(C) \cos 3x$
- (D) tanx

Solution:







Non monotonic

$$f(x) = \tan x$$
 is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

Option A & B are correct.

Illustration 13: Prove that the equation $e^{(x-1)} + x = 2$ has one solution.

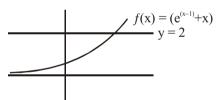
Solution:

Let
$$f(x) = e^{(x-1)} + x$$

$$f'(x) = e^{(x-1)} + 1$$

f(x) is strictly increasing function

$$\lim_{x \to \infty} f(x) = \infty \quad \& \quad \lim_{x \to -\infty} f(x) = 0$$



f(x) = 2 has exactly one solution.

ANSWERS FOR DO YOURSELF

- 1: (i) -4
 - (ii) Strictly increasing
- 2: (i) Greatest is $\frac{8}{3}$ and least value is $-\frac{8}{3}$.
- 3: (i) Rolle's theorem is valid
 - (ii) (a) f(x) is non-differentiable at x = 0 in (-1,1) (b) f(x) is discontinuous at x = 1 in (0,2)
- **4:** (ii) $C = \sqrt{\frac{117}{9}}$

 $(A) (-\infty, 4)$

(A)(0, e)

Function $f(x) = x^2(x-2)^2$ is-

(C) decreasing function

(A) increasing in [0, 1]; $[2, \infty)$

1.

2.

3.

MN0006

MN0007

Ε

(D) $(8, \infty)$

(D) None of these

EXERCISE (O-1) If the function $f(x) = 2x^2 - kx + 5$ is strictly increasing in [1, 2], then 'k' lies in the interval

(C) $(-\infty, 8]$

(C) $\left(0, \frac{1}{e}\right)$

(B) decreasing in [0, 1]; $[2, \infty]$

(D) increasing function

(B) $(4, \infty)$

(B)(0,1)

The function x^x strictly decreases on the interval-

4.		In $(-\infty, 10]$ and strictly increase $(-\infty, 10]$ and strictly decrease or every value of x		MN0004			
5.	Which one of the follow (A) f is not differentiab (C) f is even		old good for the fu (B) f is monoton (D) f has an extr	remum			
6.	by-			MN0010 decreases for all real values is given			
	(A) K < 1	$(B) K \ge 1$	(C) $K \ge \sqrt{2}$	(D) K < $\sqrt{2}$ MN0003			
7.	When $0 \le x \le 1$, $f(x) = (A)$ strictly increasing	x + x - 1 is- (B) strictly decreasing	(C) constant	(D) None of these MN0002			
8.		two continuous functions x_2 , then solution set of f (B) ϕ		\rightarrow R, such that $f(x_1) > f(x_2)$ and $f(g(3\alpha - 4))$ is (D) $R - [1, 4]$			
9.	If $2a + 3b + 6c = 0$ then	n at least one root of the eq	$\frac{1}{1}$	$\mathbf{MN0001}$ c = 0 lies in the interval-			
,,	(A)(0,1)	(B) $(1, 2)$	(C)(2,3)	(D) none			
10.	MN00. A value of c for which the conclusion of Mean values theorem holds for the function $f(x) = \log_e x$ the interval [1, 3] is-						
	$(A) 2\log_3 e$	(B) $\frac{1}{2}\log_{e}3$	$(C)\log_3 e$	(D) log _e 3 MN0015			
11.	The value of c in Lag [-1, 1] is-	grange's theorem for the	function $f(x) =$	$\begin{cases} x \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval			
	(A) 0	$(B) \frac{1}{2}$	(C) $-\frac{1}{2}$	(D) Non existent in the interval MN0016			
•							

MN0017

MN0012

The function $f(x) = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is -

(A) strictly increasing in its domain

(B) strictly decreasing in its domain

(C) strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$

(D) strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$

MN0011

If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at x = 2 on the closed interval [1, a], then the value of **15.** 'a' is equal to

(A)3

(B) 4

(C)6

(D) 1

MN0020

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. The equation $\sin x + x \cos x = 0$ has at least one root in

 $(A)\left(-\frac{\pi}{2},0\right)$

(B) $(0, \pi)$

(C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(0, \frac{\pi}{2}\right)$

MN0022

If $f: \mathbb{R} \to \mathbb{R}^+$ be a differentiable function and $g(x) = e^{2x}(2f(x) - 3(f(x))^2 + 2(f(x))^3) \ \forall \ x \in \mathbb{R}$, then which 2. of the following is/are always correct -

(A) g(x) is strictly increasing wherever f(x) is strictly increasing

(B) g(x) is strictly increasing wherever f(x) is strictly decreasing

(C) g(x) is strictly decreasing wherever f(x) is strictly decreasing

(D) g(x) is strictly decreasing wherever f(x) is strictly increasing

MN0021

3. Let a, b, c, d are non-zero real numbers such that 6a + 4b + 3c + 3d = 0, then the equation $ax^3 + bx^2 + cx + d = 0 \text{ has}$

(A) at least one root in [-2, 0]

(B) at least one root in [0, 2]

(C) at least two roots in [-2, 2]

(D) no root in [-2, 2]

MN0023

[MULTIPLE CORRECT CHOICE TYPE]

Let $F(x) = \int_{\sin x}^{\cos x} e^{(1+\arcsin t)^2} dt$ on $\left[0, \frac{\pi}{2}\right]$, then

(A) F''(c) = 0 for all $c \in \left(0, \frac{\pi}{2}\right)$

(B) F''(c) = 0 for some $c \in \left(0, \frac{\pi}{2}\right)$

(C) F'(c) $\neq 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$

(D) F (c) = 0 for some $c \in \left(0, \frac{\pi}{2}\right)$

MN0080



- For the equation $\frac{e^{-x}}{x+1} = a$; which of the following statement(s) is/are correct? **5.**
 - (A) If $a \in (0,\infty)$, then equation has 2 real and distinct roots
 - (B) If $a \in (-\infty, -e^2)$, then equation has 2 real & distinct roots.
 - (C) If $a \in (0,\infty)$, then equation has 1 real root
 - (D) If $a \in (-e,0)$, then equation has no real root.

MN0026

- Let $f: \left[0, \frac{\pi}{2}\right] \to \left[0, 1\right]$ be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right) = 1$, then
 - (A) $f'(\alpha) = \sqrt{1 f^2(\alpha)}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.
 - (B) $f'(\alpha) = \frac{2}{\pi}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.
 - (C) $f(\alpha) f'(\alpha) = \frac{1}{\pi}$ for at least one $\alpha \in \left(0, \frac{\pi}{2}\right)$
 - (D) $f'(\alpha) = \frac{8\alpha}{\pi^2}$ for at least one $\alpha \in \left(0, \frac{\pi}{2}\right)$.

MN0081

EXERCISE (S-1)

1. Find the intervals of monotonocity for the following functions & represent your solution set on the number line.

(a)
$$f(x) = 2$$
. e^{x^2-4x}

MN0028

(b)
$$f(x) = e^{x}/x$$

MN0029

(c)
$$f(x) = x^2 e^{-x}$$

MN0030 MN0031

(d)
$$f(x) = 2x^2 - ln |x|$$

- Also plot the graphs in each case & state their range.
- Let $f(x) = 1 x x^3$. Find all real values of x satisfying the inequality, $1 f(x) f^3(x) > f(1 5x)$ 2.

MN0035

Find the intervals of monotonocity of the functions in $[0, 2\pi]$ **3.**

(a)
$$f(x) = \sin x - \cos x \text{ in } x \in [0, 2\pi]$$

MN0032

(b)
$$g(x) = 2 \sin x + \cos 2x \text{ in } (0 \le x \le 2\pi)$$

MN0033

(b)
$$g(x) = 2 \sin x + \cos 2x \text{ in } (0 \le x \le 2\pi)$$

(c) $f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$

MN0034

Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{bmatrix} \max\{f(t): 0 \le t \le x\} & 0 \le x \le 1 \\ 0 \ge x \le 1 & 0 \le x \le 1 \end{bmatrix}$ 4.

Discuss the continuity & differentiability of g(x) in the interval (0,2).

MN0082

5. Find the greatest & the least values of the following functions in the given interval if they exist.

(a)
$$f(x) = \sin^{-1} \frac{x}{\sqrt{x^2 + 1}} - \ln x \ln \left[\frac{1}{\sqrt{3}}, \sqrt{3} \right]$$

MN0036

(b)
$$f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$$

(c) $y = x^5 - 5x^4 + 5x^3 + 1 \text{ in } [-1, 2]$

MN0037

(c)
$$y = x^5 - 5x^4 + 5x^3 + 1$$
 in $[-1, 2]$

MN0038

6. If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ increases for every $x \in R$ then find the range of values of 'a'.

MN0039

7. Find the set of values of x for which the inequality ln(1+x) > x/(1+x) is valid.

MN0041

8. (a) Let f, g be differentiable on R and suppose that f(0) = g(0) and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

MN0042

(b) Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

MN0043

- 9. Let $f(x) = 4x^3 3x^2 2x + 1$, use Rolle's theorem to prove that there exist c, 0 < c < 1 such that f(c) = 0.

 MN0046
- Assume that f is continuous on [a, b], a > 0 and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

MN0047

11. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that f(0) = 5, g(0) = 0, f(2) = 8, g(2) = 1. Show that there exists a number c satisfying 0 < c < 2 and f'(c) = 3 g'(c).

MN0048

12. For what value of a, m and b does the function $f(x) = \begin{bmatrix} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{bmatrix}$ satisfy the hypothesis of the mean value theorem for the interval [0, 2].

MN0049

EXERCISE (S-2)

1. Let f(x) be a strictly increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a.

MN0053

2. Find the values of 'a' for which the function $f(x) = \sin x - a \sin 2x - \frac{1}{3} \sin 3x + 2ax$ strictly increases throughout the number line.

MN0051

3. Find the minimum value of the function $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x.

MN0054

4. Find the value of x > 1 for which the function $F(x) = \int_{x}^{x^2} \frac{1}{t} ln\left(\frac{t-1}{32}\right) dt$ is strictly increasing and strictly decreasing.

MN0083

5. Prove that, $x^2 - 1 > 2x \ln x > 4(x - 1) - 2 \ln x$ for x > 1.

MN0055

Let a > 0 and f be continuous in [-a, a]. Suppose that f'(x) exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If f(a) = a and f(-a) = -a, show that f(0) = 0.

MN0058

EXERCISE (JM)

1. A function is matched below against an interval where it is supposed to be increasing, which of the following pairs is incorrectly matched? [AIEEE-2005]

interval

function

- $(1)(-\infty,\infty)$
- $(2)[2,\infty)$
- $x^3 3x^2 + 3x + 3$ $2x^3 3x^2 12x + 6$
- $(3)\left(-\infty,\frac{1}{3}\right]$
- $3x^2 2x + 1$
- $(4)(-\infty, -4)$
- $x^3 + 6x^2 + 6$

MN0061

[AIEEE-2007]

2. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in-(1) $(\pi/4, \pi/2)$

(2) $(-\pi/2, \pi/4)$

- $(3)(0, \pi/2)$
- (4) $(-\pi/2, \pi/2)$

MN0062

Statement 1 : The function $x^2(e^x + e^{-x})$ is increasing for all x > 0. **[JEE-MAIN Online 2013] Statement 2 :** The functions x^2e^x and x^2e^{-x} are increasing for all x > 0 and the sum of two increasing **3.**

functions in any interval (a, b) is an increasing function in (a, b).

- (1) Statement 1 is false; Statement 2 is true.
- (2) Statement 1 is true; Statement 2 is false.
- (3) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.
- (4) Statement 1 is true; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

MN0063

- If f and g are differentiable functions in [0, 1] satisfying f(0) = 2 = g(1), g(0) = 0 and 4. f(1) = 6, then for some $c \in [0, 1]$: [JEE-MAIN 2014]
 - (1) 2f'(c) = g'(c)

(2) 2f'(c) = 3g'(c)(4) f'(c) = 2g'(c)

(3) f'(c) = g'(c)

MN0064

- Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} \frac{d x}{\sqrt{b^2 + (d x)^2}}$, $x \in \mathbb{R}$, where a, b and d are non-zero real constants. 5. [JEE-MAIN 2019]
 - (1) f is a decreasing function of x
 - (2) f is neither increasing nor decreasing function of x
 - (3) f' is not a continuous function of x
 - (4) f is an increasing function of x

MN0084

- If the function f given by $f(x) = x^3 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing 6.
 - in [1, 5), then a root of the equation, $\frac{f(x)-14}{(x-1)^2} = 0 (x \ne 1)$ is: (1) 6 (2) 5 (3) 7

[JEE-MAIN 2019]

(4) -7

MN0085

EXERCISE (JA)

- For the function $f(x) = x \cos \frac{1}{x}, x \ge 1$, 1.
 - (A) for at least one x in the interval $[1, \infty)$, f(x+2) f(x) < 2
 - (B) $\lim_{x\to\infty} f'(x) = 1$
 - (C) for all x in the interval $[1, \infty)$, f(x+2) f(x) > 2
 - (D) f'(x) is strictly decreasing in the interval $[1, \infty)$

[JEE 2009, 4M] **MN0086**

Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell nx + \hat{\int} \sqrt{1 + \sin t} dt$. Then which 2.

of the following statement(s) is/(are) true?

[JEE 10, 3M]

- (A) f "(x) exists for all $x \in (0, \infty)$
- (B) f'(x) exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$
- (C) there exists $\alpha > 1$ such that |f'(x)| < |f(x)| for all $x \in (\alpha, \infty)$
- (D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| < \beta$ for all $x \in (0, \infty)$

MN0087

- Let $f:(0,1) \to \mathbb{R}$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that 0 < b < 1. Then **3.**
 - (A) f is not invertible on (0,1)

(B)
$$f \neq f^{-1}$$
 on (0,1) and $f'(b) = \frac{1}{f'(0)}$

- (C) $f = f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$ (D) f^{-1} is differentiable on (0,1) **[JEE 2011, 4M]**

MN0072

4. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is

[JEE 2011, 4M] MN0073

Paragraph for Question 5 and 6

Let
$$f(x) = (1-x)^2 \sin^2 x + x^2$$
 for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right) f(t) dt$ for all $x \in (1,\infty)$.

- **5.** Consider the statements:
 - **P**: There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$
 - **Q**: There exists some $x \in \mathbb{R}$ such that 2f(x) + 1 = 2x(1+x)

Then

[JEE 2012, 3M, -1M]

[JEE 2012, 3M, -1M]

(A) both **P** and **Q** are true

(B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true

(D) both **P** and **Q** are false

MN0088

- 6. Which of the following is true?
 - (A) g is increasing on $(1,\infty)$
 - (B) g is decreasing on $(1, \infty)$
 - (C) g is increasing on (1,2) and decreasing on $(2,\infty)$
 - (D) g is decreasing on (1,2) and increasing on $(2,\infty)$

MN0088

If $f(x) = \int_{0}^{x} e^{t^2} (t-2)(t-3) dt$ for all $x \in (0,\infty)$, then-7.

[JEE 2012, 4M]

- (A) f has a local maximum at x = 2
- (B) f is decreasing on (2,3)
- (C) there exists some $c \in (0, \infty)$ such that f''(c) = 0
- (D) f has a local minimum at x = 3

MN0089

The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is 8.

[JEE 2013, 2M]

- (A) 6
- (B) 4

- (C) 2
- (D) 0

MN0074

- Let $f(x) = x \sin \pi x$, x > 0. Then for all natural numbers n, f'(x) vanishes at [JEE 2013, 4M, -1M] 9.
 - (A) a unique point in the interval $\left(n, n + \frac{1}{2}\right)$
 - (B) a unique point in the interval $\left(n + \frac{1}{2}, n + 1\right)$
 - (C) a unique point in the interval (n, n + 1)
 - (D) two points in the interval (n, n + 1)

MN0075

- Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 5x + a$. Then 10.
 - (A) f(x) has three real roots if a > 4
 - (B) f(x) has only one real roots if a > 4
 - (C) f(x) has three real roots if a < -4
 - (D) f(x) has three real roots if -4 < a < 4

[JEE(Advanced)-2014, 3] **MN0076**

The total number of distinct $x \in [0, 1]$ for which $\int_{0}^{x} \frac{t^2}{1+t^4} dt = 2x - 1$ is 11.

> [JEE(Advanced)-2016, 3(0)] MN0090

- Let $f: \mathbb{R} \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value **12.** zero at some point in the interval (0, 1)? [JEE(Advanced)-2017]
 - (A) $e^x \int_0^x f(t) \sin t dt$

(B) $x^9 - f(x)$

(C) $f(x) + \int_{0}^{\frac{\pi}{2}} f(t) \sin t dt$

(D) $x - \int_0^{\frac{\pi}{2} - x} f(t) \cos t dt$

MN0091

- If $f: \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that f''(x) > 0 for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, f(1) = 1, **13.** [JEE(Advanced)-2017] then
 - (A) $0 < f'(1) \le \frac{1}{2}$ (B) $f'(1) \le 0$ (C) f'(1) > 1
- (D) $\frac{1}{2} < f'(1) \le 1$

MN0079

ANSWER KEY

(MONOTONOCITY)

EXERCISE (O-1)

1. Α 2. \mathbf{C} 3. Α **4.** C

5. B

6. C **7.** C

C 8.

9. A **10.** A

11. D

12. C

13. A

14. D

15. A

EXERCISE (O-2)

В 1.

2. A

3. В **4.** B,C,D **5.** B,C,D **6.** C,D

EXERCISE (S-1)

(a) Strictly increasing in $[2, \infty)$ & Strictly decreasing in $(-\infty, 2]$ 1.

(b) Strictly increasing in $[1, \infty)$ & Strictly decreasing in $(-\infty, 0)$; (0, 1]

(c) Strictly increasing in [0, 2] & Strictly decreasing in $(-\infty, 0]$; $[2, \infty)$

(d) Strictly increasing for $x \ge \frac{1}{2}$ or $-\frac{1}{2} \le x < 0$ & Strictly decreasing for $x \le -\frac{1}{2}$ or $0 < x \le \frac{1}{2}$

2. $(-2, 0) \cup (2, \infty)$

3. (a) Strictly increasing in $[0, 3\pi/4]$; $[7\pi/4, 2\pi]$ & Strictly decreasing in $[3\pi/4, 7\pi/4]$

(b) Strictly increasing in $[0, \pi/6]$; $[\pi/2, 5\pi/6]$; $[3\pi/2, 2\pi]$ & Strictly decreasing in $[\pi/6, \pi/2]$; $[5\pi/6, 3\pi/2]$

(c) Strictly increasing in $[0, \pi/2] \cup [3\pi/2, 2\pi]$ and Strictly decreasing in $[\pi/2, 3\pi/2]$

Continuous but not diff. at x = 14.

5. (a) $(\pi/6)+(1/2)\ln 3, (\pi/3)-(1/2)\ln 3$

(b) Maximum at x = 1 and f(-1) = 18; Minimum at x = 1/8 and f(1/8) = -9/4

(c) 2 & -10

 $a \ge 0$ 6.

7. $(-1,0) \cup (0,\infty)$

12. a = 3, b = 4 and m = 1

EXERCISE (S-2)

1.

 $(0, 1/3) \cup (1, 5)$

 $[1,\infty)$ 2.

3. -10

Strictly increasing in $[3, \infty)$ and Strictly decreasing in (1, 3]4.

EXERCISE (JM)

1. 3 2. 2

2 3.

4.

5.

3 6.

EXERCISE (JA)

B, C, D

B, C 2.

Α **3.**

4.

5. C

6. В 7. A,B,C,D

8. \mathbf{C} 9. B,C **10.** B,D

11. 1

12. B,D

13. C

MAXIMA-MINIMA

1. INTRODUCTION:

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

- What is the shape of a vessel that can with-stand maximum pressure?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?

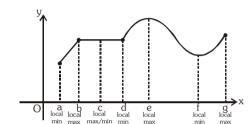
These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

MAXIMA & MINIMA:

(a) Local Maxima/Relative maxima:

A function f(x) is said to have a local maxima at x = a if $f(a) \ge f(x) \ \forall \ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.



(b) Local Minima/Relative minima:

A function f(x) is said to have a local minima at x = a if $f(a) \le f(x) \ \forall \ x \in (a - h, a + h) \cap D_{f(x)}$ Where h is some positive real number.

(c) Absolute maxima (Global maxima):

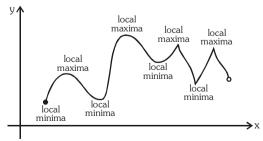
A function f has an absolute maxima (or global maxima) at c if $f(c) \ge f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minima (Global minima):

A function f has an absolute minima at c if $f(c) \le f(x)$ for all x in D and the number f(c) is called the minimum value of f on D.

Note:

- (i) The term 'extrema' is used for both maxima or minima.
- (ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
- (iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.
- (iv) It is not necessary that f(x) always has local maxima/minima at end points of the given interval when they are included.

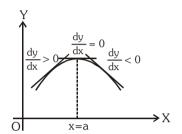


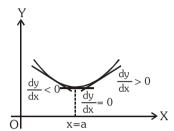
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA:

(a) First derivative test:

If f'(x) = 0 at a point (say x = a) and

- (i) If f(x) changes sign from positive to negative in the neighbourhood of x = a then x = a is said to be a point **local maxima.**
- (ii) If f'(x) changes sign from negative to positive in the neighbourhood of x = a then x = a is said to be a point **local minima.**





Note: If f'(x) does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either increasing or decreasing throughout this neighbourhood implying that x=a is not a point of extremum of f.

Illustration 1: Let $f(x) = x + \frac{1}{x}$; $x \ne 0$. Discuss the local maximum and local minimum values of f(x).

Solution :

Here,
$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x - 1)(x + 1)}{x^2}$$

Using number line rule, f(x) will have local maximum at x = -1 and local minimum at x = 1 \therefore local maximum value of f(x) = -2 at x = -1and local minimum value of f(x) = 2 at x = 1

Illustration 2:

If
$$f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3, \end{cases}$$
, then

- (A) f(x) is increasing on [-1, 2)
- (B) f(x) is continuous on [-1, 3]
- (C) f'(x) does not exist at x = 2
- (D) f(x) has the maximum value at x = 2

Solution :

Given,
$$f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 6x+12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$$

(A) which shows f'(x) > 0 for $x \in [-1, 2)$ So, f(x) is increasing on [-1, 2)Hence, (A) is correct. (B) for continuity of f(x). (check at x = 2) RHL = 35, LHL = 35 and f(2) = 35

So, (B) is correct

(C) Rf'(2) = -1 and Lf'(2) = 24

so, not differentiable at x = 2.

Hence, (C) is correct.

(D) we know f(x) is increasing on [-1, 2) and decreasing on (2, 3],

Thus maximum at x = 2,

Hence, (D) is correct.

 \therefore (A), (B), (C), (D) all are correct.

Ans.

Do yourself - 1:

- (i) Find local maxima and local minima for the function $f(x) = x^3 3x$.
- (ii) If function $f(x) = x^3 62x^2 + ax + 9$ has local maxima at x = 1, then find the value of a.
- (b) Second derivative test:

If f(x) is continuous and differentiable at x = a where f'(a) = 0 and f''(a) also exists then for ascertaining maxima/minima at x = a, 2^{nd} derivative test can be used -

- (i) If $f''(a) > 0 \implies x = a$ is a point of local minima
- (ii) If $f'(a) < 0 \implies x = a$ is a point of local maxima
- (iii) If f'(a) = 0 \Rightarrow second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

Illustration 3: If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at x = a and x = b respectively, then ordered pair (a, b) is -

$$(A)(3,-2)$$

(B)
$$(2, -3)$$

$$(C)(-2,3)$$

$$(D)(-3,2)$$

Solution:

$$f(x) = 2x^3 - 3x^2 - 36x + 6$$

$$f'(x) = 6x^2 - 6x - 36$$
 & $f''(x) = 12x - 6$

$$\alpha$$
 1 (λ) 12 λ 0

$$100 \text{ M} \text{ I (A)} \quad 0 \quad \Rightarrow \quad$$

Now
$$f(x) = 0 \implies 6(x^2 - x - 6) = 0 \implies (x - 3)(x + 2) = 0 \implies x = -2, 3$$

$$f''(-2) = -30$$

 \therefore x = -2 is a point of local maximum

$$f''(3) = 30$$

 \therefore x = 3 is a point of local minimum

Hence, (-2, 3) is the required ordered pair.

Ans. (C)

Illustration 4: Find the point of local maxima of $f(x) = \sin x$ (1+cosx) in $x \in (0, \pi/2)$.

Solution:

Let
$$f(x) = \sin x (1 + \cos x) = \sin x + \frac{1}{2} \sin 2x$$

$$\Rightarrow$$
 f'(x) = cos x + cos 2x

$$f''(x) = -\sin x - 2\sin 2x$$

Now
$$f'(x) = 0 \implies \cos x + \cos 2x = 0$$

$$\Rightarrow$$
 cos 2x = cos $(\pi - x)$ \Rightarrow x = $\pi/3$

Also
$$f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$$

f(x) has a maxima at $x = \pi/3$

Ans.

Illustration 5: Find the global maximum and global minimum of $f(x) = \frac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution:

$$f(x) = \frac{e^x + e^{-x}}{2}$$
 is differentiable at all x in its domain.

Then
$$f'(x) = \frac{e^x - e^{-x}}{2}, f''(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = 0 \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$$

f''(0) = 1 : x = 0 is a point of local minimum

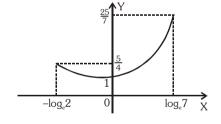
Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of f(x) at all these three points $x = -\log_e 2$, 0, $\log_e 7$

$$\Rightarrow f(-\log_e 2) = \frac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \frac{5}{4}$$

$$f(0) = \frac{e^0 + e^{-0}}{2} = 1$$

$$f(\log_e 7) = \frac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \frac{25}{7}$$



 \therefore x = 0 is absolute minima & x = $\log_e 7$ is absolute maxima Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\frac{25}{7}$ at $x = \log_e 7$

Ans.

Do yourself - 2:

- (i) Find local maximum value of function $f(x) = \frac{\ell nx}{x}$
- (ii) If $f(x) = x^2 e^{-2x}$ (x > 0), then find the local maximum value of f(x).

Illustration 6: Identify a point of maxima/minima in $f(x) = (x + 1)^4$.

Solution:
$$f(x) = (x + 1)^4$$

$$f(x) = 4(x+1)^3$$

$$\frac{-\text{ve}}{-1}$$
 $\frac{-\text{dy}}{\text{dx}}$

at x = -1 f(x) is having local minima

$$\therefore$$
 at $x = -1$ f(x) has point of minima.

Illustration 7: Find point of local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution:
$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

$$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2 (x^2 - 4x + 3)$$

= 5x² (x - 1)(x - 3)

$$f(x) = 0 \implies x = 0, 1, 3$$

$$f''(x) = 10x(2x^2 - 6x + 3)$$

But at x = 0, derivative sign is positive in its neighbourhood.

Now
$$f'(1) < 0 \implies Maxima at x = 1$$

$$f''(3) > 0 \implies Minima at x = 3$$

$$\Rightarrow$$
 Neither maxima nor minima at $x = 0$.

Do yourself - 3:

(i) Identify the point of local maxima/minima in $f(x) = (x-3)^{10}$.

3. USEFUL FORMULAE OF MENSURATION TO REMEMBER:

- (a) Volume of a cuboid = ℓ bh.
- **(b)** Surface area of a cuboid = $2(\ell b + bh + h\ell)$.
- (c) Volume of a prism = area of the base \times height.
- (d) Lateral surface area of prism = perimeter of the base \times height.
- (e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).
- (f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.
- (g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height. (Note that slant surfaces of a pyramid are triangles).
- **(h)** Volume of a cone = $\frac{1}{3} \pi r^2 h$.
- (i) Curved surface area of a cylinder = $2 \pi rh$.
- (j) Total surface area of a cylinder = $2 \pi rh + 2 \pi r^2$.
- (k) Volume of a sphere $=\frac{4}{3} \pi r^3$.
- (I) Surface area of a sphere = $4 \pi r^2$.
- (m) Area of a circular sector $=\frac{1}{2} r^2 \theta$, when θ is in radians.

SUMMARY OF WORKING RULES FOR SOLVING REAL LIFE OPTIMIZATION **PROBLEM:**

First: When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second: Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

If y = f(x) is a quantity to be maximum or minimum, find those values of x for which Third: dy/dx = f'(x) = 0.

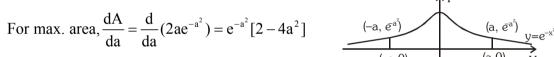
Using derivative test, test each value of x for which f'(x) = 0 to determine whether it Fourth: provides a maximum or minimum or neither.

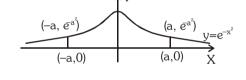
If the derivative fails to exist at some point, examine this point as possible maximum or Fifth:

If the function y = f(x) is defined only for $x \in [a,b]$ then examine x = a & x = b for Sixth: possible extreme values.

Illustration 8: Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Area of the rectangle will be A = 2a. e^{-a^2} Solution:





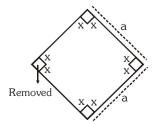
$$\frac{dA}{da} = 0 \implies a = \pm \frac{1}{\sqrt{2}}$$

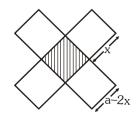
& sign of $\frac{dA}{da}$ changes from positive to negative at $a = +\frac{1}{\sqrt{2}}$

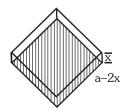
$$\Rightarrow$$
 $x = \frac{1}{\sqrt{2}}$ are points of maxima \Rightarrow $A_{max} = \frac{2}{\sqrt{2}} \cdot e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units. **Ans.**

A box of maximum volume with top open is to be made by cutting out four equal squares Illustration 9: from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get **Solution:** a box with a square base of side (a - 2x) and height x.







$$\therefore \frac{dV}{dx} = (a - 2x)^2 + x \cdot 2(a - 2x)(-2)$$

For V to be extremum $\frac{dV}{dx} = 0$ \Rightarrow x = a/2, a/6

But when x = a/2; V = 0 (minimum) and we know minimum and maximum occurs alternately in a continuous function.

Hence, V is maximum when x = a/6.

Ans.

Illustration 10: If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

Solution : Let x be the radius of cylinder and y be its height

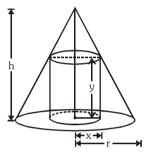
$$V = \pi x^2 y$$

x, y can be related by using similar triangles

$$\frac{y}{r-x} = \frac{h}{r} \implies y = \frac{h}{r}(r-x)$$

$$\Rightarrow V(x) = \pi x^2 \frac{h}{r}(r-x) \implies V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2)$$



$$V'(x) = 0 \implies x = 0, \frac{2r}{3}$$

$$V''(x) = \frac{\pi h}{r} (2r - 6x)$$

 $V''(0) = 2\pi h \implies x = 0$ is point of minima

$$V''\left(\frac{2r}{3}\right) = -2\pi h \implies x = \frac{2r}{3} \text{ is point of maxima}$$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Do yourself - 4:

- (i) Find the two positive numbers x & y such that their sum is 60 and xy^3 is maximum.
- (ii) If from a wire of length 36 metre, a rectangle of greatest area is made, then find its two adjacent sides in metre.
- (iii) If ab = 2a + 3b where a > 0, b > 0, then find the minimum value of ab.
- (iv) Of all closed right circular cylinders of a given volume of 100 cubic centimetres, find the dimensions of cylinder which has minimum surface area.

Important note:

(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

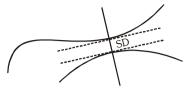
i.e.
$$xy = \frac{1}{4}[(x+y)^2 - (x-y)^2]$$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e.
$$(x + y)^2 = (x - y)^2 + 4xy$$

5. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES:

Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)



Note: Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then

AP will be maximum or minimum if it is normal to the curve at P.

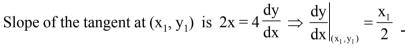
Proof:
$$F(x) = (x-a)^2 + (f(x)-b)^2 \implies F'(x) = 2(x-a) + 2(f(x)-b) \cdot f'(x)$$

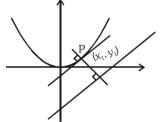


$$\therefore f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x). m_{AP} = 1.$$

Illustration 11: Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line y = x - 4.

Solution: Let $P(x_1y_1)$ be a point on the curve $x^2 = 4y$ at which normal is also a perpendicular to the line y = x - 4.





$$\therefore \quad \frac{\mathbf{x}_1}{2} = 1 \quad \Rightarrow \quad \mathbf{x}_1 = 2$$

$$\therefore \quad x_1^2 = 4y_1 \implies y_1 = 1$$

Hence required point is (2, 1)

Do yourself - 5:

(i) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line x + y = -2.

6. SOME SPECIAL POINTS ON A CURVE:

- (a) Stationary points: The stationary points are the points of domain where f(x) = 0.
- **(b) Critical points :** There are three kinds of critical points as follows :
 - (i) The point at which f'(x) = 0
 - (ii) The point at which f'(x) does not exists
 - (iii) The end points of interval (if included)

These points belongs to domain of the function.

Note: Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/local minima.

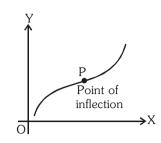
(c) **Point of inflection :** A point where the graph of a function has a tangent line and where the strict concavity changes is called a

point of inflection. For finding point of inflection of any

function, compute the points (x-coordinate) where $\frac{d^2y}{dx^2} = 0$

or $\frac{d^2y}{dx^2}$ does not exist. Let the solution is x = a, if $\frac{d^2y}{dx^2} = 0$ at

x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.



if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

Illustration 12: Find the critical point(s) & stationary point(s) of the function $f(x) = (x-2)^{2/3}(2x+1)$

Solution: f(x) = (x)

$$f(x) = (x-2)^{2/3}(2x+1)$$

 $f(x) = (x-2)^{2/3} \cdot 2 + (2x+1)\frac{2}{3}(x-2)^{-1/3} = 2(x-2)^{2/3} + \frac{2}{3}(2x+1)\frac{1}{(x-2)^{1/3}}$

$$= \left[2(x-2) + \frac{2}{3}(2x+1)\right] \frac{1}{(x-2)^{1/3}} = \frac{2(5x-5)}{3(x-2)^{1/3}}$$

f(x) does not exist at x = 2 and f(x) = 0 at x = 1

 \therefore x = 1, 2 are critical points and x = 1 is stationary point.

Illustration 13: The point of inflection for the curve $y = x^{\frac{5}{3}}$ is -

(A)(1,1)

(B)(0,0)

(C)(1,0)

(D)(0,1)

Solution:

Here
$$\frac{d^2y}{dx^2} = \frac{10}{9x^{1/3}}$$

From the given points we find that (0, 0) is the point of the curve where

 $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point.

 \therefore (0, 0) is the required point

Ans. (B)

Illustration 14: Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

Solution:

$$f(x) = 3x^4 - 4x^3$$

$$f(x) = 12x^3 - 12x^2$$

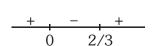
$$f'(x) = 12x^2(x-1)$$

$$f(x) = 0 \implies x = 0, 1$$

examining sign change of f(x)

thus x = 1 is a point of local minima

$$f''(x) = 12(3x^2 - 2x)$$



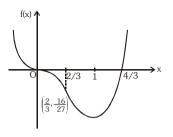
$$f''(x) = 12x(3x - 2)$$

$$f'(x) = 0 \implies x = 0, \frac{2}{3}$$

Again examining sign of f'(x)

thus x = 0, $\frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Do yourself - 6:

- Find the critical points and stationary point of the function $f(x) = \frac{e^x}{x}$ (i)
- Find the point of inflection for the curve $y = x^3 6x^2 + 12x + 5$ (ii)
- For the function $f(x) = \frac{x^4}{12} \frac{5x^3}{6} + 3x^2 + 7$
 - (a) Find the interval in which f''(x) > 0
 - (b) Find the interval in which f''(x) < 0
 - (c) Find the points of inflection of f(x).

Miscellaneous Illustrations:

Illustration 15: Let a cuboid having square base has area 6. Then find its maximum volume.

Total area = $2a^2 + 4ah = 6$ **Solution:**

$$h = \frac{(6-2a^2)}{4a}$$
 \Rightarrow $V = a^2h = \frac{a^2(6-2a^2)}{4a}$

$$\frac{\mathrm{dv}}{\mathrm{da}} = 6 - 6a^2 = 0$$

$$a = 1 \& \frac{d^2v}{da^2} = -ve$$

Minimum V = 1

ANSWERS FOR DO YOURSELF

- local max. at x = -1, local min. at x = 11: (i)
- (ii) 121

- 2: (i) 1/e
- (ii) $1/e^2$
- (i) local minima at x = 33:

- (i) x = 15 & y = 45 (ii) 9 & 9 (iii) 24 (iv) $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm. $k = 2\left(\frac{50}{\pi}\right)^{1/3}$ cm.
- 5: (i) (2,-4)
- (i) x = 1 is a critical point as well as stationary point (Note x = 0 is not in the domain of f(x))
 - (ii) x = 2
- (iii) (a) $(-\infty, 2) \cup (3, \infty)$
- **(b)** (2, 3)
- (c) x = 2 & x = 3

EXERCISE (O-1)

1.	If f(x)	= X	+ x - 1	+ x - 2	, then-
----	---------	------	-----------	-----------	---------

(A)
$$f(x)$$
 has minima at $x = 1$

(B) f(x) has maxima at x = 0

(C)
$$f(x)$$
 has neither maxima nor minima at $x = 3$

(D) None of these

MM0009

2. The maximum area of a right angled triangle with hypotenuse h is:- [JEE-MAIN Online 2013]

(A)
$$\frac{h^2}{\sqrt{2}}$$

(B)
$$\frac{h^2}{2}$$

(C)
$$\frac{h^2}{4}$$

(D)
$$\frac{h^2}{2\sqrt{2}}$$

MM0008

3. The cost of running a bus from A to B, is Rs. $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed (in km/h) of the bus is : [JEE-MAIN Online 2013]

- (A) 40
- (B) 60
- (C) 45
- (D) 50

MM0015

4. The greatest value of $x^3 - 18x^2 + 96x$ in the interval (0, 9) is-

- (A) 128
- (B) 60
- (C) 160
- (D) 120

MM0006

5. Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

(A) 2

(B) e

 $(C) e^2$

(D) 1

MM0007

6. The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum if the angle between them is:

- (A) $\pi/6$
- (B) $\pi/4$
- (C) $\pi/3$
- (D) $5\pi/12$

MM0016

7. Consider the function $f(x) = x \cos x - \sin x$, then identify the statement which is correct.

(A) f is neither odd nor even

(B) f is monotonic decreasing at x = 0

(C) f has a maxima at $x = \pi$

(D) f has a minima at $x = -\pi$

MM0010

8. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at x = 3 and maximum at x = -1, then -

(A) a = -3, b = -9, c = 0

(B) a = 3, b = 9, c = 0

(C) a = -3, b = -9, $c \in R$

(D) none of these

MM0005

9. If $f(x) = \int_{x}^{x^2} (t-1) dt$, $1 \le x \le 2$, then global maximum value of f(x) is

(A) 1

(B) 2

(C) 4

(D) 5

MM0095

10. Range of the function $f(x) = \frac{\ell nx}{\sqrt{x}}$ is

- $(A)(-\infty,e)$
- (B) $(-\infty, e^2)$
- $(C)\left(-\infty,\frac{2}{e}\right]$
- $(D)\left(-\infty,\frac{1}{e}\right)$

- (D) None of these

MM0017

Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(C)
$$\sqrt{\frac{a^2 + b^2}{2}}$$

$$(D) \frac{a+2b}{3}$$

MM0018

13. $f(x) = 1 + [\cos x]x$, in $0 \le x \le \frac{\pi}{2}$ (where [.] denotes greatest integer function)

(B) has a maximum value 2

(D) is not differentiable at $x = \frac{\pi}{2}$

MM0003

The minimum value of asecx + bcosecx, 0 < a < b, $0 < x < \pi/2$ is-

(C) $(a^{2/3} + b^{2/3})^{3/2}$

(D) None of these

MM0013

P is a point on positive x-axis, Q is a point on the positive y-axis and 'O' is the origin. If the line passing 15. through P and Q is tangent to the curve $y = 3 - x^2$ then the minimum area of the triangle OPQ, is

(A)2

(B)4

(D)9

MM0020

16. A minimum value of sinx cos2x is-

(A) 1

(B) - 1

(C) $-2/3\sqrt{6}$

(D) None of these

MM0002

The rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is minimum when **17.**

(A) x = 0

(B) $x = 1/\sqrt{2}$

(C) $x = -1/\sqrt{2}$

(D) $x = \pm 1/\sqrt{2}$

MM0014

For the function $f(x) = \int_{0}^{x} \frac{\sin t}{t} dt$, where x > 0, **18.**

(A) maximum occurs at $x = n\pi$, n is even

(B) minimum occurs at $x = n\pi$, n is odd

(C) maximum occurs at $x = n\pi$, n is odd

(D) None of these

MM0096

The set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ posses a negative **19.** point of inflection.

(A) $(-\infty, -2) \cup (0, \infty)$ (B) $\{-4/5\}$

(C)(-2,0)

(D) empty set

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

- If (a, b) be the point on the curve $y = |x^2 4x + 3|$ which is nearest to the circle $x^2 + y^2 4x 4y + 7 = 0$, then 1. (a + b) is equal to -
 - (A) $\frac{7}{4}$
- (B)0

(C)3

(D)2

MM0026

- If $f(x) = \begin{cases} e^{x+1} e^x & x \le 0 \\ e^{1-x} 1 & 0 < x < 1 \\ x + \ell nx & x \ge 1 \end{cases}$, then -2
 - (A) x = 0 is point of local maxima, x = 1 is neither local maxima nor local minima.
 - (B) x = 1 is point of local minima, x = 0 is point of local maxima
 - (C) x = 0 and x = 1 both are points of local maxima
 - (D) x = 0 and x = 1 both are points of local minima

MM0022

- Let $f(x) = \begin{bmatrix} x^{3/5} & \text{if } x \le 1 \\ & & \text{, then the number of critical points on the graph of the function is} \\ -(x-2)^3 & \text{if } x > 1 \end{bmatrix}$ **3.**
 - (A) 1

(C)3

(D)4

MM0027

- 4. The set of all values of 'a' for which the function,
 - $f(x) = (a^2 3a + 2) \left(\cos^2 \frac{x}{4} \sin^2 \frac{x}{4}\right) + (a 1)x + \sin 1$ does not possess critical points is:
 - (A) $[1, \infty)$
- (B) $(0, 1) \cup (1, 4)$ (C) (-2, 4)
- (D) $(1,3) \cup (3,5)$

MM0028

- Let $f(x) = ax^2 b|x|$, where a and b are constants. Then at x = 0, f(x) has **5.**
 - (A) a maxima whenever a > 0, b > 0
 - (B) a maxima whenever a > 0, b < 0
 - (C) minima whenever a > 0, b > 0
 - (D) neither a maxima nor minima whenever a > 0, b < 0

MM0023

- 6. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per additional tree drops by 10 apples. Number of trees that should be added to the existing orchard for maximising the output of the trees, is
 - (A)5

(B) 10

(C) 15

(D) 20

MM0029

[MULTIPLE CORRECT CHOICE TYPE]

- Let f(x) be a cubic polynomial such that it has point of inflection at x = 2 and local minima at x = 4, then-7. (A) f(x) has local minima at x = 0(B) f(x) has local maxima at x = 0
 - (C) $\lim_{x\to\infty} f(x) \to \infty$

MM0034

- The function $f(x) = \int_{-2015}^{x} t(e^t e^2)(e^t 1)(t + 2014)^{2015}(t 2015)^{2016}(t 2016)^{2017} dt$ has-8.
 - (A) local minima at x = -2014

(B) local minima at x = 2

(C) local maxima at x = 2

(D) local minima at x = 2016

MM0097

- Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha) \cdot f(\beta) < 0$, then which 9. of the following can be true for the equation f(x) = 0?
 - (A) three equal roots

- (B) three distinct real roots
- (C) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$ (D) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$.

MM0031

- Let A(p,q) and B(h,k) are points on the curve $4x^2 + 9y^2 = 1$, which are nearest and farthest from the line 10. 72 + 8x = 9y respectively, then -
 - (A) $p+q=-\frac{1}{5}$ (B) $p+q=\frac{1}{5}$ (C) $h+k=-\frac{1}{5}$

MM0036

- The function $f(x) = \int_{1}^{x} \sqrt{1 t^4} dt$ is such that : 11.
 - (A) it is defined on the interval [-1, 1]
- (B) it is an strictly increasing function

(C) it is an odd function

(D) the point (0,0) is the point of inflection

MM0098

EXERCISE (S-1)

A cubic f(x) vanishes at x = -2 & has relative minimum/maximum at x = -1 and x = 1/3. 1. If $\int_{-1}^{1} f(x) dx = \frac{14}{3}$, find the cubic f(x).

MM0099

- Investigate for maxima & minima for the function, $f(x) = \int_{1}^{x} [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$ 2. **MM0100**
- **3.** Find the greatest & least value for the function;
 - $y = x + \sin 2x$, $0 \le x \le 2\pi$ (a)

MM0039

(b) $y = 2\cos 2x - \cos 4x$, $0 \le x \le \pi$

MM0040

- 4. Suppose f(x) is a function satisfying the following conditions:
 - (i) f(0) = 2, f(1) = 1

- f has a minimum value at $x = \frac{5}{2}$ and
- for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b & the function f(x).

Let P(x) be a polynomial of degree 5 having extremum at x = -1, 1 and $\lim_{x \to 0} \left(\frac{P(x)}{x^3} - 2 \right) = 4$. If M and m are the maximum and minimum value of the function y = P'(x) on the set $A = \{x | x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

MM0042

6. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

MM0047

7. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2 + 3}$, find the equations of the tangent lines of minimum and maximum slope.

MM0045

- A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.
- 9. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 x^2$.

MM0044

10. If $y = \frac{ax + b}{(x-1)(x-4)}$ has a turning value at (2,-1) find a & b and show that the turning value is a maximum.

MM0043

11. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft? Assume that one side of the rectangle lies on the base of the triangle.

MM0051

12. Let f(x) be a cubic polynomial which has local maximum at x = -1 and f(x) has a local minimum at x = 1. If f(-1) = 10 and f(3) = -22, then find the distance between its two horizontal tangents.

MM0038

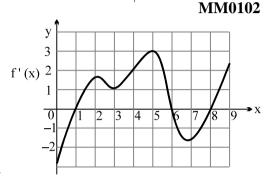
EXERCISE (S-2)

1. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum, the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.

MM0063

- 2. Consider the function $f(x) = \begin{bmatrix} \sqrt{x} \ln x & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{bmatrix}$
 - (a) Find whether f is continuous at x = 0 or not.
 - (b) Find the minima and maxima if they exist.
 - (c) Does f'(0)? Find $\lim_{x\to 0} f'(x)$.
 - (d) Find the inflection points of the graph of y = f(x)...

- 3. Consider the function $y = f(x) = \ln(1 + \sin x)$ with $-2\pi \le x \le 2\pi$, $x \ne \frac{-\pi}{2}$, $\frac{3\pi}{2}$. Find
 - (a) the zeroes of f(x)
 - (b) inflection points if any on the graph
 - (c) local maxima and minima of f(x)
 - (d) asymptotes of the graph
 - (e) sketch the graph of f(x) and compute the value of the definite integral $\int_{-\pi/2}^{\pi/2} f(x) dx$.



- 4. The graph of the derivative f' of a continuous function f is shown with f(0) = 0
 - (i) On what intervals *f* is strictly increasing or strictly decreasing?
 - (ii) At what values of x does f have a local maximum or minimum?
 - (iii) On what intervals is f'' > 0 or f'' < 0
 - (iv) State the x-coordinate(s) of the point(s) of inflection.
 - (v) Assuming that f(0) = 0, sketch a graph of f.

MM0052

5. Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

MM0064

6. The value of 'a' for which $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

MM0053

7. The function f(x) defined for all real numbers x has the following properties

f(0) = 0, f(2) = 2 and $f'(x) = k(2x - x^2)e^{-x}$ for some constant k > 0. Find

- (a) the intervals on which f is strictly increasing and strictly decreasing and any local maximum or minimum values.
- (b) the intervals on which the graph f is concave down and concave up.
- (c) the function f(x) and plot its graph.

MM0054

8. The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

MM0066

- 9. Find all the values of the parameter 'a' for which the function; f(x) = 0 and x in (x 7x) in 5x in area sing 0 has an arrivial 0
 - $f(x) = 8ax a \sin 6x 7x \sin 5x$ increasing & has no critical points for all $x \in R$.

MM0055

48

EXERCISE (JM)

Let $f: R \to R$ be defined by $f(x) = \begin{cases} k-2x, & \text{if } x \le -1 \\ 2x+3, & \text{if } x > -1 \end{cases}$ If f has a local minimum at x = -1, then a possible value of k is: 1. [AIEEE-2010]

(1) 1

(2) 0

(4) -1

MM0067

For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_{0}^{x} \sqrt{t} \sin t dt$. Then f has:-2.

[AIEEE-2011]

- (1) local minimum at π and local maximum at 2π
- (2) local maximum at π and local minimum at 2π
- (3) local maximum at π and 2π
- (4) local minimum at π and 2π

MM0103

The shortest distance between line y - x = 1 and curve $x = y^2$ is :-3.

[AIEEE-2011]

 $(1) \frac{8}{3\sqrt{2}}$

(2) $\frac{4}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{4}$

(4) $\frac{3\sqrt{2}}{8}$

MM0068

Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ 4.

Statement - 1: x = 0 is point of minima of f.

Statement - 2: f(0) = 0.

[AIEEE-2011]

- (1) Statement-1 is false, statement-2 is true.
- (2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
- (3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.
- (4) Statement-1 is true, statement-2 is false.

- **5.** A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is: [AIEEE-2012]
 - (1) 9/2
- (2) 9/7
- (4) 2/9

MM0070

Let a, b \in R be such that the function f given by $f(x) = \ln |x| + bx^2 + ax$, $x \ne 0$ has extreme values at 6. x = -1 and x = 2.

Statement–1: f has local maximum at x = -1 and at x = 2.

Statement-2: $a = \frac{1}{2}$ and $b = \frac{-1}{4}$.

[AIEEE-2012]

- (1) Statement–1 is true, Statement–2 is false.
- (2) Statement–1 is false, Statement–2 is true.
- (3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.
- (4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1.

MM0071

[JEE-MAIN 2019]

 $(1)\frac{1}{2}$

 $(2) \frac{1}{3} \sqrt{\frac{7}{3}}$

 $(3) \frac{1}{6} \sqrt{\frac{7}{3}}$

 $(4) \frac{\sqrt{5}}{6}$

MM0078

EXERCISE (JA)

- Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, 1. $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then
 - (A) a = b and $c \neq b$

(B) a = c and $a \neq b$

(C) $a \neq b$ and $c \neq b$

(D) a = b = c

MM0083

Let f be a function defined on **R** (the set of all real numbers) such that **(b)** $f'(x) = 2010 (x-2009) (x-2010)^2 (x-2011)^3 (x-2012)^4$, for all $x \in \mathbb{R}$. If g is a function defined on R with values in the interval $(0,\infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$,

then the number of points in R at which g has a local maximum is [JEE 2010, 3 + 3]

MM0084

2. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is [JEE 2012, 4M]

MM0085

3. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is [JEE 2012, 4M]

MM0086

- 4. The function f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x|| has a local minimum or a local maximum at [JEE 2013, 3M,-1M] x =
 - (A) -2
- (B) $\frac{-2}{3}$
- (C) 2
- (D) $\frac{2}{2}$

MM0088

The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \ge 1$, for all x > 0, is -**5.**

[JEE(Advanced)-2016, 3(-1)]

- (A) $\frac{1}{64}$
- (B) $\frac{1}{32}$ (C) $\frac{1}{27}$
- (D) $\frac{1}{25}$

MM0090

Let $f: \mathbb{R} \to (0, \infty)$ and $g: \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous 6.

functions on \mathbb{R} . Suppose f'(2) = g(2) = 0, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced)-2016, 4(-2)]

- (A) f has a local minimum at x = 2
- (B) f has a local maximum at x = 2

(C) f''(2) > f(2)

(D) f(x) - f''(x) = 0 for at least one $x \in \mathbb{R}$

7. If
$$f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$
, then

[JEE(Advanced)-2017]

- (A) f'(x) = 0 at exactly three points in $(-\pi, \pi)$
- (B) f(x) attains its maximum at x = 0
- (C) f(x) attains its minimum at x = 0
- (D) f'(x) = 0 at more than three points in $(-\pi, \pi)$

MM0092

8. Let $f: \mathbb{R} \to \mathbb{R}$ be given by f(x) = (x-1)(x-2)(x-5). Define $F(x) = \int_0^x f(t) dt$, x > 0. Then which

of the following options is/are correct?

[JEE(Advanced)-2019, 4(-1)]

- (1) F has a local minimum at x = 1
- (2) F has a local maximum at x = 2
- (3) $F(x) \neq 0$ for all $x \in (0, 5)$
- (4) F has two local maxima and one local minimum in $(0, \infty)$

MM0104

9. Let
$$f(x) = \frac{\sin \pi x}{x^2}, x > 0$$

Let $x_1 < x_2 < x_3 < ... < x_n < ...$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < ... < y_n < ...$ be all the points of local minimum of f.

Then which of the following options is/are correct?

[JEE(Advanced)-2019, 4(-1)]

(1)
$$|\mathbf{x}_{n} - \mathbf{y}_{n}| > 1$$
 for every n

(2)
$$x_1 < y_1$$

(3)
$$x_n \in \left(2n, 2n + \frac{1}{2}\right)$$
 for every n

(4)
$$x_{n+1} - x_n > 2$$
 for every n

ANSWER KEY

(MAXIMA-MINIMA)

EXERCISE (O-1)

1. A.C **2.** C **3.** B

5. B

6. C

7. B **8.** C **9.** C

10. C

11. B

12. A

14. C

15. B

16. B

17. D

18. C

19. A

EXERCISE (O-2)

1. C

3. C

4.

4. C

13. C

5. A

6. C

7. B,C **8.** A,C,D **9.** B,C,D

11. A,B,C,D **10.** A,D

EXERCISE (S-1)

1.
$$f(x) = x^3 + x^2 - x + 2$$

2. Max. at
$$x = 1$$
; $f(1) = 0$, min. at $x = 7/5$; $f(7/5) = -108/3125$

3. (a) Max at
$$x = 2\pi$$
, Max value = 2π , Min. at $x = 0$, Min value = 0

(b) Max at
$$x = \pi/6$$
 & also at $x = 5\pi/6$ and Max value = $3/2$, Min at $x = \pi/2$, Min value = -3

4.
$$a = \frac{1}{4}$$
; $b = -\frac{5}{4}$; $f(x) = \frac{1}{4}(x^2 - 5x + 8)$

5. 6

6.
$$75\sqrt{3}$$
 sq. units

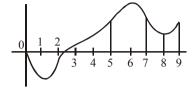
7.
$$3x + 4y - 9 = 0$$
; $3x - 4y + 9 = 0$.

10.
$$a = 1, b = 0$$

EXERCISE (S-2)

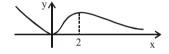
- (a) f is continuous at x = 0;
 - **(b)** -2/e;
 - (c) does not exist, does not exist;
 - (d) pt. of inflection x = 1
- 3. (a) $x = -2\pi, -\pi, 0, \pi, 2\pi,$
 - (b) no inflection point,
 - (c) local maxima at $x = -3\pi/2$, $\pi/2$, 2π and local minima at $x = -2\pi$
 - (d) $x = 3\pi/2$ and $x = -\pi/2$,
 - (e) $-\pi \ln 2$

- **4.** (i) I in [1, 6]; [8, 9] and D in [0, 1]; [6, 8];
 - (ii) L.Min. at x = 1 and x = 8; L.Max. at x = 0, 6, 9
 - (iii) $(0,2) \cup (3,5) \cup (7,9)$ and $(2,3) \cup (5,7)$;
 - (iv) x = 2, 3, 5, 7
 - (v) Graph is



5. $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$

- **6.** 320
- 7. (a) strictly increasing in [0,2] and strictly decreasing in $(-\infty, 0]$; $[2, \infty)$, local min. value = 0 and local max. value = 2
 - **(b)** concave up for $(-\infty, 2-\sqrt{2}]$; $[2+\sqrt{2}, \infty)$ and concave down in $[2-\sqrt{2}, 2+\sqrt{2}]$
 - (c) $f(x) = \frac{1}{2}e^{2-x} \cdot x^2$



- 8. $\frac{4}{3\sqrt{3}}$
- **9.** (6, ∞)

EXERCISE (JM)

- **1.** 4
- **2.** 2
- **3.** 4
- **4.** 3
- **5.** 4
- **6.** 3

- **8.** 3
- **9.** 1
- **10.** 4
- **11.** 4
- **12.** 3
- **13.** 3
- EXERCISE (JA)
- **1.** (a) D, (b) 1
- **2.** 5
- **3.** 9
- **4.** A,B
- **5.** C
- **6.** A,D
- **7.** B,D

7. 4

- **8.** 1,2,3
- **9.** 1,3,4

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Important Notes					

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