Inverse trigonometric functions

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important identities from trig

1. $1 + \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right), \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$
2. $1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right), \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right) + 1$
3. $\sqrt{1 + \sin(\theta)} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)$
4. Remember **to use the right triangle to convert one inverse trigonometric function to another.** This can sometimes make it a lot easier to solve an equation.
5. General values
    a. $\sin(\theta) = \sin(\alpha) \Rightarrow \theta = n\pi + (-1)^n\alpha$
    b. $\cos(\theta) = \cos(\alpha) \Rightarrow \theta = 2n\pi \pm \alpha$
    c. $\tan(\theta) = \tan(\alpha) \Rightarrow \theta = n\pi \pm \alpha$

basics

Standard ranges used from respective graphs. The inverse trigonometric relations are functions only when we take a certain range. Then they become one-one and onto functions.

$f(x)$	domain	range
$\arcsin(x)$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\csc^{-1}(x)$	**R** $-(-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - (0)$
$\arctan(x)$	$(-\infty, \infty)$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\sec^{-1}(x)$	**R** $-(-1, 1)$	$[0, \pi] - \left(\frac{\pi}{2}\right)$

$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$

6. $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$, and the same for the pairs arccot, arctan and arcsec, arccsc.
7. $\arcsin(-x) = -\arcsin(x)$, the same for arctan and arccsc.
8. $\arccos(-x) = \pi - \arccos(x)$, the same for arcsec and arccot.
9. $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ and the same for the pairs arccot, arctan and arcsec, arccsc.
10. $\cot^{-1}(x) = \arctan(\frac{1}{x})$ and $\cot^{-1}(x) = \frac{\pi}{2} - \arctan(x)$
11. $\arctan(1/3) + \arctan(1/2) = \frac{\pi}{4}$

identities

12. If $\sin(\theta) = x$, then $\theta = \arcsin(x)$, and $2\theta = 2\arcsin(x)$ and so on. Use these with the other functions as well.
13. Certain important identities grouped below.
    a. $\arcsin(x) + \arcsin(y) = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
    b. $\arcsin(x) - \arcsin(y) = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$
    c. $\arccos(x) + \arccos(y) = \arccos\left(xy - \left(\sqrt{1-x^2}\right)\left(\sqrt{1-y^2}\right)\right)$
    d. $\arccos(x) - \arccos(y) = \arccos\left(xy + \left(\sqrt{1-x^2}\right)\left(\sqrt{1-y^2}\right)\right)$
    e. $\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$
    f. $\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{1+xy}\right)$

    All of these are only true when $0 \leq x \leq 1\ \forall\, x \in$ **R**, $0 \leq y \leq 1\ \forall\, y \in$ **R**.
14. $\arctan\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \arctan(x)$ and such.
15. $2\arctan(x) = \arctan\left(\frac{2x}{1-x^2}\right)$
16. $2\arctan(x) = \arcsin\left(\frac{2x}{1+x^2}\right)$
17. $2\arctan(x) = \arccos\left(\frac{1-x^2}{1+x^2}\right)$
18. For **simplification**, if $\arctan(z)$ is given, where z is any of the below expressions,

a. if $\sqrt{a^2 - x^2}$ is asked, let $x = a\sin(\theta)$ or $x = a\cos(\theta)$.

b. if $\sqrt{x^2 - a^2}$ is asked, let $x = a\sec(\theta)$ or $x = a\csc(\theta)$.

c. if $\sqrt{a^2 + x^2}$ is asked, let $x = a\tan(\theta)$ or $x = a\cot(\theta)$.

d. if $\sqrt{\frac{a+x}{a-x}}$ is asked, let $x = a\cos(\theta)$.

After this, substitute the appropriate identities such as $\sin(2\theta)$ or $\cos(2\theta)$.

19. Few equalities using the $90^\circ - \theta$ conversion.

a. $\tan(\cot^{-1}(x)) = \cot(\tan^{-1}(x))$

b. $\cos(\sin^{-1}(x)) = \sin(\cos^{-1}(x))$

c. $\sec(\csc^{-1}(x)) = \csc(\sec^{-1}(x))$

20. If $2\arcsin(x) = 2\theta$, then $2\theta = \arcsin(2x\sqrt{1 - x^2})$. Then, $2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Similarly it follows for the other inverse trigonometric functions.

21. If $\sin\theta > \arcsin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, then it follows that

$2\arcsin(x) = \pi - \arcsin(2x\sqrt{1 - x^2})$.

other important properties

- If $-n \le x^2 \le n^2 \; \forall n \in \mathbf{Z}$, then $0 \le x^2 \le n^2$.
- $\arctan(1/2) + \arctan(1/3) = \pi/4$
- If $|x| > n$, then the linear graph for x would look like