The process of calculating derivative is called differentiation.

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=f'(x)=\frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

Note : $\frac{dy}{dx}$ can also be represented as y_1 or y' or Dy or $f'(x)$. $\frac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Illustration 1 : Differentiate each of following functions by first principle :

(i) $f(x)=\tan x$ (ii) $f(x)=e^{\sin x}$

Solution : (i) $f'(x)=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}=\lim_{h\to 0}\frac{\tan(x+h-x)\left[1+\tan x\tan(x+h)\right]}{h}$

$=\lim_{h\to 0}\frac{\tanh}{h}.(1+\tan^2 x)=\sec^2 x.$ **Ans.**

(ii) $f'(x)=\lim_{h\to 0}\frac{e^{\sin(x+h)}-e^{\sin x}}{h}=\lim_{h\to 0}e^{\sin x}\frac{\left[e^{\sin(x+h)-\sin x}-1\right]}{\sin(x+h)-\sin x}\left(\frac{\sin(x+h)-\sin x}{h}\right)$

$=e^{\sin x}\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=e^{\sin x}\cos x$ **Ans.**

Do yourself -1 :

(i) Differentiate each of following functions by first principle:

(a) $f(x)=\ell nx$ (b) $f(x)=\frac{1}{x}$

2. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x\ell na,\ a>0$	(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x)\log_a e,\ a>0,\ a\neq 1$	(vi)	sinx	cosx
(vii)	cosx	$-\sin x$	(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx	(x)	cosecx	$-\operatorname{cosec}x\cdot\cot x$
(xi)	cotx	$-\operatorname{cosec}^2 x$	(xii)	constant	0
(xiii)	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}},\ -1<x<1$	(xiv)	$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}},\ -1<x<1$
(xv)	$\tan^{-1}x$	$\frac{1}{1+x^2},\ x\in R$	(xvi)	$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$
(xvii)	$\operatorname{cosec}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$	(xviii)	$\cot^{-1}x$	$\frac{-1}{1+x^2},\ x\in R$

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3. FUNDAMENTAL THEOREMS :

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ (b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"PRODUCT RULE"**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as **"QUOTIENT RULE"**

(e) If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as **"CHAIN RULE"**

Note : In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

Illustration 2 : If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Solution : $y = e^x.\tan x + x \cdot \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \cdot \tan x + e^x \cdot \sec^2 x + 1 \cdot \log x + x \cdot \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$ **Ans.**

Illustration 3 : If $y = \frac{\log x}{x} + e^x \sin 2x + \log_5 x$, find $\frac{dy}{dx}$.

Solution : On differentiating we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin 2x) + \frac{d}{dx}(\log_5 x) = \frac{\frac{1}{x}\cdot x - \log x \,.\, 1}{x^2} + e^x \sin 2x + 2e^x \,.\, \cos 2x + \frac{1}{x \log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1 - \log x}{x^2}\right) + e^x(\sin 2x + 2\cos 2x) + \frac{1}{x \log_e 5}$ **Ans.**

Illustration 4 : If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, find $\frac{dy}{dx}$.

Solution : $y = \log_e(\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}} \cdot \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{1}{2\sqrt{1+x^2}} .2x$$

$$= \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left\{1+\left(\sqrt{1+x^2}\right)^2\right\}\sqrt{1+x^2}} = \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)(2+x^2)\sqrt{1+x^2}}$$ **Ans.**

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Do yourself -2 :

(i) Find $\frac{dy}{dx}$ if -

(a) $y = (x + 1)(x + 2)(x + 3)$ (b) $y = e^{5x} \tan(x^2 + 2)$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of a function :

(a) which is the product or quotient of a number of functions or

(b) of the form $[f(x)]^{g(x)}$ where f & g are both derivable functions.

It is convenient to take the logarithm of the function first & then differentiate.

Illustration 5 : If $y = (\sin x)^{\ell n\, x}$, find $\frac{dy}{dx}$

Solution : $\ell n\, y = \ell n\, x.\ \ell n\, (\sin x)$

On differentiating we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ \ell n\ (\sin x) + \ell n\ x.\ \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x} + \cot x\ \ell\, n\, x\right]$$ **Ans.**

Illustration 6 : If $x = \exp\left(\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x\,[1 + \tan(\log x) + \sec^2 x]$ (B) $2x\,[1 + \tan(\log x)] + \sec^2 x$

(C) $2x\,[1 + \tan(\log x)] + \sec x$ (D) $2x + x[1 + \tan(\log x)]^2$

Solution : Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right) \Rightarrow \quad \tan(\log x) = (y - x^2)/x^2$$

$\Rightarrow \quad y = x^2 + x^2 \tan(\log x)$

On differentiating, we get

$$\therefore \quad \frac{dy}{dx} = 2x + 2x \tan(\log x) + x \sec^2(\log x) \Rightarrow 2x\,[1 + \tan(\log x)] + x \sec^2(\log x)$$

$= 2x + x[1 + \tan(\log x)]^2$ **Ans. (D)**

Illustration 7 : If $y = \frac{x^{1/2}(1 - 2x)^{2/3}}{(2 - 3x)^{3/4}(3 - 4x)^{4/5}}$ find $\frac{dy}{dx}$

Solution : $\ell n\, y = \frac{1}{2}\ \ell n\, x + \frac{2}{3}\ \ell n\ (1 - 2x) - \frac{3}{4}\ \ell n\ (2 - 3x) - \frac{4}{5}\ell n\ (3 - 4x)$

On differentiating we get,

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1 - 2x)} + \frac{9}{4(2 - 3x)} + \frac{16}{5(3 - 4x)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} - \frac{4}{3(1 - 2x)} + \frac{9}{4(2 - 3x)} + \frac{16}{5(3 - 4x)}\right)$$ **Ans.**

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Do yourself -3 :

(i) Find $\frac{dy}{dx}$ if $y = x^x$ **(ii)** Find $\frac{dy}{dx}$ if $y = e^x.e^{x^2}.e^{x^3}.e^{x^4}$

5. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)}{g'(\theta)}$

Illustration 8: If $y = a\cos t$ and $x = a(t - \sin t)$ find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution : $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$ **Ans.**

Illustration 9 : Prove that the function represented parametrically by the equations. $x = \frac{1+t}{t^3}; y = \frac{3}{2t^2} + \frac{2}{t}$ satisfies the relationship : $x(y')^3 = 1 + y'$ (where $y' = \frac{dy}{dx}$)

Solution : Here $x = \frac{1+t}{t^3} = \frac{1}{t^3} + \frac{1}{t^2}$

Differentiating w.r. to t

$$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$y = \frac{3}{2t^2} + \frac{2}{t}$$

Differentiating w.r. to t

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t = y'$$

Since $x = \frac{1+t}{t^3} \Rightarrow x = \frac{1+y'}{(y')^3}$ or $x(y')^3 = 1 + y'$

Ans.

6. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

Illustration 10: Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Solution : $\frac{d(\log_e \tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}} = \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x\cos x}$ **Ans.**

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Do yourself-4 :

(i) Find $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ if $y = \cos^4 t$ & $x = \sin^4 t$.

(ii) Find the slope of the tangent at a point P(t) on the curve $x = at^2$, $y=2at$.

(iii) Differentiate $x^{\ell nx}$ with respect to ℓnx.

7. DIFFERENTIATION OF IMPLICIT FUNCTIONS : $\phi(x, y) = 0$

(a) To find dy /dx of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x & then collect terms with dy/dx together on one side.

(b) Also $\frac{dy}{dx} = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}$, where $\frac{\partial\phi}{\partial x}$ = partial derivative of $\phi(x, y)$ w.r.t. x taking y as a constant and

$\frac{\partial\phi}{\partial y}$ = partial derivative of $\phi(x, y)$ w.r.t. y taking x as a constant.

(c) In the case of implicit functions, generally, both x & y are present in answers of dy/dx.

Illustration 11 : If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Solution : Let $u = x^y$ and $v = y^x$

$u + v = 2 \quad \Rightarrow \quad \frac{du}{dx}+\frac{dv}{dx}=0$

Now $u = x^y$ and $v = y^x$

$\Rightarrow \ \ell n\, u = y\, \ell nx$ and $\ell n\, v = x\, \ell n\, y$

$\Rightarrow \ \frac{1}{u}\frac{du}{dx}=\frac{y}{x}+\ell nx\,\frac{dy}{dx}$ and $\frac{1}{v}\frac{dv}{dx}=\ell n\, y+\frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \ \frac{du}{dx}=x^y\left(\frac{y}{x}+\ell nx\frac{dy}{dx}\right)$ and $\frac{dv}{dx}=y^x\left(\ell n\, y+\frac{x}{y}\frac{dy}{dx}\right)$

$\Rightarrow \ x^y\left(\frac{y}{x}+\ell n\, x\frac{dy}{dx}\right)+y^x\left(\ell ny+\frac{x}{y}\frac{dy}{dx}\right)=0 \ \Rightarrow \ \frac{dy}{dx}=-\frac{\left(y^x\ell ny+x^y.\frac{y}{x}\right)}{\left(x^y\ell nx+y^x.\frac{x}{y}\right)}$ **Ans.**

Aliter :

$\phi(x,y) = x^y + y^x - 2 = 0$

$\frac{dy}{dx}=\frac{-\partial\phi/\partial x}{\partial\phi/\partial y}=\frac{yx^{y-1}+y^x\ell ny}{x^y\ell nx+xy^{x-1}}$

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Illustration 12 : If $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+\dfrac{\sin x}{1+\cos x.....}}}$, prove that $\dfrac{dy}{dx}=\dfrac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$.

Solution : Given function is $y=\dfrac{\sin x}{1+\dfrac{\cos x}{1+y}}=\dfrac{(1+y)\sin x}{1+y+\cos x}$

or $y + y^2 + y\cos x = (1 + y)\sin x$ .....(i)

Differentiate both sides with respect to x,

$$\frac{dy}{dx}+2y\frac{dy}{dx}+\frac{dy}{dx}\cos x-y\sin x = (1 + y)\cos x+\frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1 + y)\cos x + y\sin x$$

or $\dfrac{dy}{dx}=\dfrac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$ **Ans.**

Aliter :

From (i) $\phi(x,y) = (1 + y)\sin x - y - y^2 - y\cos x = 0$

$$\frac{dy}{dx}=-\frac{\partial\phi/\partial x}{\partial\phi/\partial y}=-\frac{(1+y)\cos x+y\sin x}{\sin x-1-2y-\cos x}=\frac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$$

Do yourself -5 :

(i) Find $\dfrac{dy}{dx}$, if $x + y = \sin(x - y)$

(ii) If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

8. DIFFERENTIATION BY TRIGONOMETRIC TRANSFORMATION :

Some Standard Substitutions :

Expression	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2+x^2}$	$x = \tan\theta$ or $a\cot\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\dfrac{a+x}{a-x}}$ or $\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos2\theta$
$\sqrt{2ax-x^2}$	$x = a(1 - \cos\theta)$

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Illustration 13: If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find

(i) $f(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Solution : $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & \frac{-\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1} x & x > 1 \\ 2\tan^{-1} x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1} x) & x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1 \Rightarrow f'(1)$ does not exist **Ans.**

Illustration 14 : $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) -1

Solution : Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$. **Ans (A)**

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Illustration 15 : Obtain differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Solution : Assume $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} \Rightarrow u = v$$

$$\therefore \quad \frac{du}{dv} = 1.$$ **Ans.**

Do yourself : 6

(i) If $y = \cos^{-1}(4x^3 - 3x)$, then find :

(a) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (b) $f'(0)$, (c) $f'\left(\frac{\sqrt{3}}{2}\right)$.

9. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If g is inverse of f, then

(a) $g\{f(x)\} = x$ $\quad g'\{f(x)\}f'(x)=1$ (b) $f\{g(x)\} = x$ $\quad f'\{g(x)\}g'(x) = 1$

Illustration 16 : If g is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then g'(x) equals :-

(A) $1 + x^n$ (B) $1 + [f(x)]^n$ (C) $1 + [g(x)]^n$ (D) none of these

Solution : Since g is the inverse of f. Therefore

$f(g(x)) = x$ for all x

$$\Rightarrow \frac{d}{dx}f(g(x)) = 1 \text{ for all } x$$

$$\Rightarrow f'(g(x))\, g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^n$$ **Ans. (C)**

Do yourself -7 :

(i) If g is inverse of f and $f(x) = 2x + \sin x$; then g'(x) equals:

(A) $-\frac{3}{x^2}+\frac{1}{\sqrt{1-x^2}}$ (B) $2 + \sin^{-1}x$ (C) $2 + \cos g(x)$ (D) $\frac{1}{2+\cos(g(x))}$

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10. HIGHER ORDER DERIVATIVES :

Let a function $y = f(x)$ be defined on an interval (a, b). If $f(x)$ is differentiable function, then its derivative $f'(x)$ [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If $f'(x)$ is again differentiable function on (a, b), then its derivative $f''(x)$ [or d^2y/dx^2 or y''] is called second derivative of y w.r.t. x.

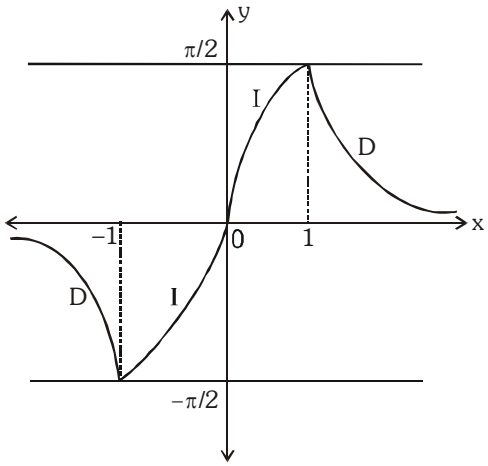
Similarly, the 3rd order derivative of y w.r.to x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and denoted by $f'''(x)$ or y''' and so on.

Note : If $x = f(\theta)$ and $y = g(\theta)$ where 'θ' is a parameter then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ & $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{d\theta}$

In general $\frac{d^ny}{dx^n} = \frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\Big/\frac{dx}{d\theta}$.

11. ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$



Important points :

(i) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) f is continuous for all x but not differentiable at $x = 1, -1$

(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad |x| < 1 \\ \text{non existent} & \text{for} \quad |x| = 1 \\ \frac{-2}{1+x^2} & \text{for} \quad |x| > 1 \end{cases}$

(iv) Increasing in $(-1, 1)$ & Decreasing in $(-\infty, -1) \cup (1, \infty)$

(b) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if} \quad x \ge 0 \\ -2\tan^{-1}x & \text{if} \quad x < 0 \end{cases}$

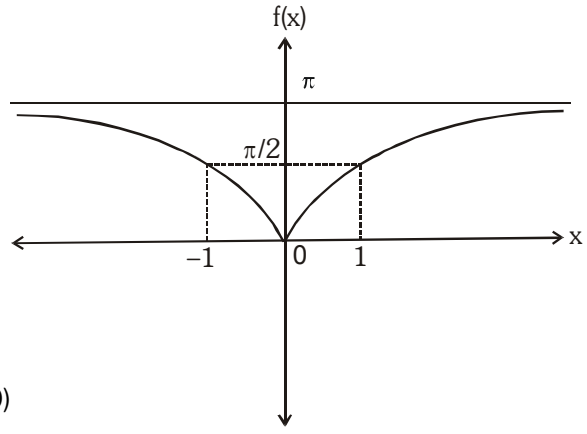
Important points :

(i) Domain is $x \in R$ & range is $[0, \pi)$

(ii) Continuous for all x but not differentiable at $x = 0$

(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad x > 0 \\ \text{non existent} & \text{for} \quad x = 0 \\ -\frac{2}{1+x^2} & \text{for} \quad x < 0 \end{cases}$

(iv) Increasing in $(0, \infty)$ & Decreasing in $(-\infty, 0)$



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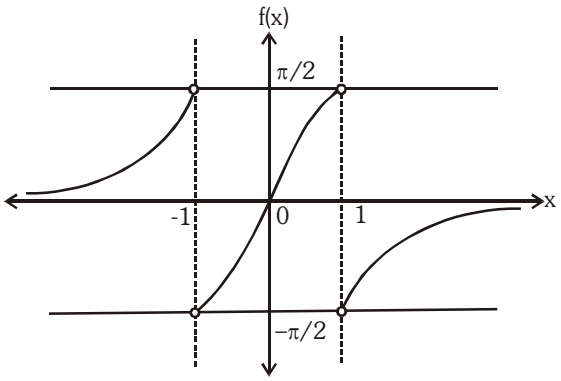
(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x|<1 \\ \pi + 2\tan^{-1}x & x<-1 \\ -(\pi - 2\tan^{-1}x) & x>1 \end{cases}$

Important points :



(i) Domain is $R - \{1, -1\}$ & range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(ii) It is neither continuous nor differentiable

at $x = 1, -1$

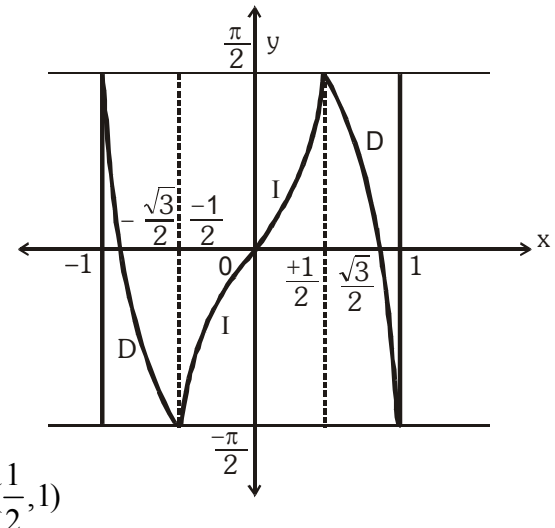
(iii) $\dfrac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(iv) Increasing $\forall$ x in its domain

(v) It is bounded for all x

(d) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if} \quad -1 \le x < -\dfrac{1}{2} \\ 3\sin^{-1}x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if} \quad \dfrac{1}{2} < x \le 1 \end{cases}$

Important points :



(i) Domain is $x \in [-1, 1]$ & range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\dfrac{1}{2}, \dfrac{1}{2}$

(iv) $\dfrac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in (-\dfrac{1}{2}, \dfrac{1}{2}) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in (-1, -\dfrac{1}{2}) \cup (\dfrac{1}{2}, 1) \end{cases}$

(v) Increasing in $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and Decreasing in $\left[-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]$

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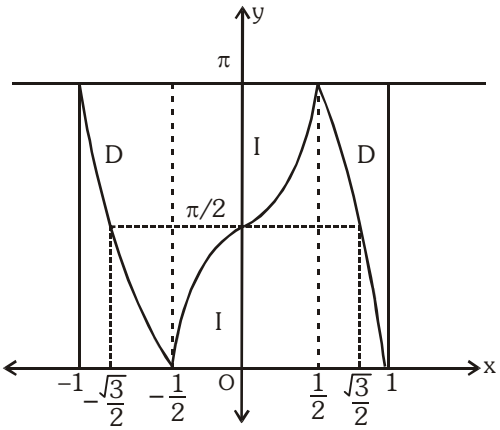
(e) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1} x - 2\pi & \text{if} \quad -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$

Important points :

(i) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}$



(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(v) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ &

Decreasing in $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

GENERAL NOTE :

Concavity is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} \ge 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} \le 0 \Rightarrow$ Concave downwards

---

Illustration 17: Find the interval for which $f(x) = x^3 + x + 1$ is

(i) concave upwards (ii) concave downwards.

Solution : $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x$

(i) $f''(x) = 6x \ge 0 \Rightarrow$ Concave upwards

$\Rightarrow \quad x \in [0, \infty)$

(ii) $f''(x) = 6x \le 0 \Rightarrow$ Concave downwards

$\Rightarrow \quad x \in (-\infty, 0]$ **Ans.**

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Illustration 18: If $x = a(t + \sin t)$ and $y = a(1 - \cos t)$, find $\frac{d^2y}{dx^2}$.

Solution : Here $x = a(t + \sin t)$ and $y = a(1-\cos t)$

Differentiating both sides w.r.t. t, we get :

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again differentiating both sides, we get,

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2(t/2)\cdot\frac{1}{a(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$ **Ans.**

Illustration 19: $y = f(x)$ and $x = g(y)$ are inverse functions of each other then express $g'(y)$ and $g''(y)$ in terms of derivative of $f(x)$.

Solution : $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \text{.....(i)}$$

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right).\frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2}\cdot\left(\frac{1}{f'(x)}\right)$$

$$\Rightarrow \quad g''(y) = -\frac{f''(x)}{(f'(x))^3} \qquad \text{.....(ii)}$$

Which can also be remembered as $\boldsymbol{\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}}$ **Ans.**

Do yourself : 8

(i) If $y = xe^{x^2}$ then find y''.

(ii) Find y'' at $x = \pi/4$, if $y = x\tan x$.

(iii) Prove that the function $y = e^x \sin x$ satisfies the relationship $y'' - 2y' + 2y = 0$.

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12. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note : Sometimes it is batter to expand the determinant first & then differentiate.

Illustration 20: If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find f'(x).

Solution : Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

$$\text{or} \quad f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know if any two rows or columns are equal, then value of determinant is zero.

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \qquad \therefore \quad f'(x) = 6\,(2x^2 - x^2)$$

Therefore, $f'(x) = 6x^2$ **Ans.**

Do yourself : 9

(i) If $f(x) = \begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find $f'(1)$. **(ii)** If $f(x) = \begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$, then find $f'(1)$.

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13. L' HÔPITAL'S RULE :

(a) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(b) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) Indeterminate forms of the type 1^∞, ∞^0 or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.

Illustration 21: Evaluate $\lim_{x\to 0} |x|^{\sin x}$

Solution : $\lim_{x\to 0} |x|^{\sin x} = \lim_{x\to 0} e^{\sin x \log_e |x|} = e^{\lim_{x\to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$

$= e^{\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}}$ (applying L'Hôpital's rule)

$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x\cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)} = e^{-(1)^2\cdot(0)} = e^0 = 1$ **Ans.**

Illustration 22: Solve $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$.

Solution : Here $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

$= \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x}$ $\left(\frac{-\infty}{-\infty} \text{form}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x}\cdot 2\cos 2x}{\frac{1}{\sin x}\cdot \cos x}$ {applying L'Hôpital's rule}

$= \lim_{x\to 0^+} \frac{\left(\frac{(2x)}{\sin(2x)}\right)\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} = \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$ **Ans.**

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Illustration 23: Evaluate $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution : Here, $A = \lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$ (∞^0 form)

$$\therefore \quad \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{e^n}{\pi}\right) = \lim_{n\to\infty}\frac{n\log e - \log\pi}{n}\left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{n\to\infty}\frac{\log e - 0}{1} \quad \{\text{applying L'Hôpital's rule}\}$$

$\log A = 1 \Rightarrow A = e^1$ or $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n} = e$ **Ans.**

Do yourself : 10

(i) Using L'Hôpital's rule find (a) $\lim_{x\to0}\frac{\tan x - x}{x^3}$ (b) $\lim_{x\to0}\frac{e^x - x - 1}{x^2}$

(ii) Using L'Hôpital's rule verify that : (a) $\lim_{x\to0}\frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim_{x\to0}\frac{\ell n(1+x)}{x} = 1$

Miscellaneous Illustrations :

Illustration 24: Find second order derivative of $y = \sin x$ with respect to $z = e^x$.

Solution :

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{e^x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{\cos x}{e^x}\right)\cdot\frac{dx}{dz} = \frac{-e^x\sin x - \cos x e^x}{\left(e^x\right)^2}\cdot\frac{1}{e^x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} = -\frac{(\sin x + \cos x)}{e^{2x}}$$ **Ans.**

Illustration 25: Let a function f satisfies $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}\ \forall\ x, y \in R$ and $f'(0) = a$, $f(0) = b$, then find $f(x)$ hence find $f''(x)$.

Solution : $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Diff. w.r.t. 'x'

$$f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}f'(x) \qquad \left(\because\ x \,\&\, y \text{ are independent to each other, } \therefore \frac{dy}{dx} = 0\right)$$

$$f'\left(\frac{x+y}{2}\right) = f'(x)$$

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Let x = 0 & y = x $\quad f'\left(\frac{x}{2}\right) = f'(0) = a$

$\Rightarrow \quad f'(x) = a$

On integrating, we get $f(x) = ax + b \quad (\because f(0) = b)$

$\Rightarrow \quad f''(x) = 0$

Illustration 26: Prove that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$

Solution : Let $\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.....\infty$

$= \lim_{n\to\infty} \cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.....\cos\frac{x}{2^n}$

$\Rightarrow \quad \lim_{n\to\infty} \frac{\sin x}{2^n \sin\frac{x}{2^n}} = \frac{\sin x}{x} \quad \Rightarrow \quad \cos\frac{x}{2}.\cot\frac{x}{2^2}\cos\frac{x}{2^3}.....\infty = \frac{\sin x}{x}$

$\Rightarrow \quad \ell n\left(\cos\frac{x}{2}\right) + \ell n\left(\cos\frac{x}{2^2}\right) + \ell n\left(\cos\frac{x}{2^3}\right) + ...\infty = \ell n \sin x - \ell n x$

Diff. w.r.t. x

$$-\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + ...\infty\right) = \cot x - \frac{1}{x}$$

Diff. w.r.t. x again

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$$

Hence proved

ANSWERS FOR DO YOURSELF

1 : **(i)** **(a)** $\frac{1}{x}$ **(b)** $-\frac{1}{x^2}$

2 : **(i)** **(a)** $3x^2 + 12x + 11$ **(b)** $5e^{5x}\tan(x^2+2) + 2xe^{5x}\sec^2(x^2+2)$

3 : **(i)** $x^x(\ell nx + 1)$ **(ii)** $y(1 + 2x + 3x^2 + 4x^3)$

4. **(i)** -1 **(ii)** $\frac{1}{t}$ **(iii)** $2(x^{\ell nx})(\ell nx)$

5 : **(i)** $\frac{\cos(x-y)-1}{\cos(x-y)+1}$ **(ii)** $y' = -\left(\frac{2x+e^y}{xe^y+1}\right)$, -1

6 : **(i)** **(a)** -6 **(b)** 3 **(c)** -6

7 : **(i)** D

8 : **(i)** $y'' = 4y + 2xy'$ **(ii)** $\pi + 4$

9 : **(i)** $e(\sin 1 + \cos 1) - 1$ **(ii)** 9

10. **(i)** **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{2}$

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EXERCISE (O-1)

1. If $y=\frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}}+\frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}}+\frac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$, then $\frac{dy}{dx}$ is equal to-

(A) 0 (B) 1

(C) $(\alpha+\beta+\gamma)x^{\alpha+\beta+\gamma-1}$ (D) $\alpha\beta\gamma$

DF0001

2. If $f(x)=|\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to -

(A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) 1 (D) –1

DF0002

3. $\frac{d}{dx}(e^x \sin\sqrt{3}\,x)$ equals-

(A) $e^x \sin(\sqrt{3}\,x+\pi/3)$ (B) $2e^x \sin(\sqrt{3}\,x+\pi/3)$

(C) $\frac{1}{2}e^x \sin(\sqrt{3}\,x+\pi/3)$ (D) $\frac{1}{2}e^x \sin(\sqrt{3}\,x-\pi/3)$

DF0003

4. $\frac{d}{dx}\left(\ell n \sin\sqrt{x}\right)$ is equal to-

(A) $\frac{\tan\sqrt{x}}{2\sqrt{x}}$ (B) $\frac{\cot\sqrt{x}}{\sqrt{x}}$ (C) $\frac{\cot\sqrt{x}}{2x}$ (D) $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

DF0004

5. If $y=\sqrt{\frac{1-x}{1+x}}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{y}{1-x^2}$ (B) $\frac{y}{x^2-1}$ (C) $\frac{y}{1+x^2}$ (D) $\frac{y}{y^2-1}$

DF0005

6. If $y=\ell n\left\{\frac{x+\sqrt{(a^2+x^2)}}{a}\right\}$, then the value of $\frac{dy}{dx}$ is-

(A) $\sqrt{a^2-x^2}$ (B) $a\sqrt{a^2+x^2}$ (C) $\frac{1}{\sqrt{a^2+x^2}}$ (D) $x\sqrt{a^2+x^2}$

DF0006

7. If $(\cos x)^y=(\sin y)^x$, then $\frac{dy}{dx}$ equals-

(A) $\frac{\log\sin y-y\tan x}{\log\cos x+x\cot y}$ (B) $\frac{\log\sin y+y\tan x}{\log\cos x-x\cot y}$

(C) $\frac{\log\sin y+y\tan x}{\log\cos x+x\cot y}$ (D) $\frac{\log\sin y+y\tan x}{\log\cos y-y\cot x}$

DF0008

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8. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2^x + 2^y}{2^x - 2^y}$ (B) $\frac{2^x + 2^y}{1 + 2^{x+y}}$ (C) $2^{x-y}\left(\frac{2^y - 1}{1 - 2^x}\right)$ (D) $\frac{2^x + y - 2^x}{2^y}$

DF0009

9. If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, then $\frac{dy}{dx}$ equals-

(A) $-\tan\frac{t}{2}$ (B) $\cot\frac{t}{2}$ (C) $-\cot\frac{t}{2}$ (D) $\tan\frac{t}{2}$

DF0010

10. The differential coefficient of $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$ w.r.t. $\sqrt{1 - x^2}$ is-

(A) $1/x^2$ (B) $2/x^3$ (C) $x/2$ (D) $2/x$

DF0011

11. If $x^3 - y^3 + 3xy^2 - 3x^2y + 1 = 0$, then at (0, 1) $\frac{dy}{dx}$ equals-

(A) 1 (B) –1 (C) 2 (D) 0

DF0012

12. $\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)\right)$ equals, if $-\pi < \theta < \pi$ -

(A) 1/2 (B) 1 (C) $\sec\theta$ (D) $\operatorname{cosec}\theta$

DF0013

13. If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, then $\frac{dy}{dx}$ is equal to-

(A) 1 (B) 0 (C) –1 (D) –2

DF0015

14. If g is the inverse of f and $f'(x) = \frac{1}{1 + x^3}$ then g'(x) is equal to-

(A) $1 + [g(x)]^3$ (B) $\frac{-1}{2(1 + x^2)}$ (C) $\frac{1}{2(1 + x^2)}$ (D) $\frac{1}{1 + \left[g(x)\right]^3}$

DF0017

15. If $x^2 + y^2 = 1$, then-

(A) $yy'' - 2(y')^2 + 1 = 0$ (B) $yy'' + (y')^2 + 1 = 0$

(C) $yy'' + (y')^2 - 1 = 0$ (D) $yy'' + 2(y')^2 + 1 = 0$

DF0018

16. Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ ($x \neq 0$), then the value of f'(27) is-

(A) 15 (B) 45 (C) 0 (D) 35

DF0019

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EXERCISE (O-2)

1. Let u(x) and v(x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to -

(A) 1 (B) 0 (C) 7 (D) –7

DF0021

2. Let f(x) be a polynomial function of second degree. If f(1) = f(–1) and a, b, c are in A.P., then f'(a), f'(b) and f'(c) are in

(A) G.P. (B) H.P. (C) A.G.P. (D) A.P.

DF0022

3. Suppose the function $f(x) - f(2x)$ has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function $f(x) - f(4x)$ at x = 1, has the value equal to

(A) 19 (B) 9 (C) 17 (D) 14

DF0023

4. Suppose that f(0) = 0 and f'(0) = 2, and let g(x) = $f\left(-x + f\left(f(x)\right)\right)$. The value of g'(0) is equal to

(A) 0 (B) 1 (C) 6 (D) 8

DF0028

5. If f is differentiable in (0, 6) & f'(4) = 5, then $\underset{x\to 2}{\text{Limit}} \frac{f(4) - f(x^2)}{2 - x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

DF0029

6. If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\frac{e^x}{(1+e^x)^3}$ (C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{-1}{(1+e^x)^3}$

DF0030

7. If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = K$ then the value of K is equal to

(A) 1 (B) –1 (C) 2 (D) 0

DF0033

8. Let $f(x)$ be differentiable at x = h, then $\underset{x\to h}{\text{Lim}} \frac{(x+h)f(x) - 2h f(h)}{x - h}$ is equal to -

(A) $f(h) + 2hf'(h)$ (B) $2f(h) + hf'(h)$ (C) $hf(h) + 2f'(h)$ (D) $hf(h) - 2f'(h)$

DF0034

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9. Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)$ has the value equal to

(A) $\sqrt{2}-1$ (B) $\frac{\sqrt{2}+1}{\sqrt{2}}$ (C) $2-\sqrt{2}$ (D) $\sqrt{2}+1$

DF0036

Multiple objective type :

10. If $f(x) = (2x-3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then

(A) $g'(2\pi) = \frac{7}{3}$ (B) $g'(2\pi) = \frac{3}{7}$ (C) $g''(2\pi) = \frac{7}{3}$ (D) $g''(2\pi) = 0$

DF0037

11. If $f(x) = x.|x|$, then its derivative is :

(A) 2x (B) –2x (C) 2|x| (D) 2xsgnx

DF0038

12. $\lim\limits_{x\to\pi}\frac{x^{\pi}-\pi^{x}}{x^{x}-\pi^{\pi}}$ is equal to -

(A) $\log_{\pi e}\left(\frac{e}{\pi}\right)$ (B) $\log_{\pi e}\left(\frac{\pi}{e}\right)$

(C) $\tan(\cot^{-1}(\ell n\pi) - \cot^{-1}(1))$ (D) $\tan(\tan^{-1}(1) - \tan^{-1}(\ell n\pi))$

DF0042

EXERCISE (S-1)

1. Let f, g and h are differentiable functions. If f(0) = 1 ; g(0) = 2 ; h(0) = 3 and the derivatives of their pair wise products at x = 0 are (f g)' (0) = 6 ; (g h)' (0) = 4 and (h f)' (0) = 5 then compute the value of (fgh)'(0).

DF0046

2. **(a)** If $y = (\cos x)^{\ell nx} + (\ell nx)^{x}$ find $\frac{dy}{dx}$

DF0047

(b) If $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$. Find $\frac{dy}{dx}$.

DF0048

3. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ell n\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ell n\, y'$. Where ' denotes the derivative.

DF0049

4. If $y = \ln\left(x^{e^x}.a^y\right)^{y^x}$ find $\frac{dy}{dx}$.

DF0050

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5. If $x = \text{cosec}\,\theta - \sin\theta$; $y = \text{cosec}^n\theta - \sin^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$.

DF0052

6. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w.r.t. $\sqrt{1-x^4}$.

DF0055

7. If $\sqrt{1-x^6}+\sqrt{1-y^6} = a^3.(x^3-y^3)$, prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

DF0057

8. Suppose $f(x) = \tan(\sin^{-1}(2x))$

(a) Find the domain and range of f.

(b) Express f(x) as an algebraic function of x.

(c) Find f' (1/4)

DF0061

9. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + \ldots..$ to n terms.

Find dy/dx, expressing your answer in 2 terms.

DF0065

10. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$, $u \in \left(0,\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}},1\right)$ prove that $2\frac{dy}{dx}+1 = 0$.

DF0066

11. **(a)** If $y = y(x)$ and it follows the relation $e^{xy} + y\cos x = 2$, then find **(i)** y'(0) and **(ii)** y''(0).

DF0069

(b) A twice differentiable function f(x) is defined for all real numbers and satisfies the following conditions

$f(0) = 2$; $f'(0) = -5$ and $f''(0) = 3$.

The function g(x) is defined by $g(x) = e^{ax} + f(x)\ \forall\ x \in R$, where 'a' is any constant. If $g'(0) + g''(0) = 0$. Find the value(s) of 'a'.

DF0070

12. If $x = 2\cos t - \cos 2t$ & $y = 2\sin t - \sin 2t$, find the value of (d^2y/dx^2) when $t = (\pi/2)$.

DF0071

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EXERCISE (S-2)

1. Find a polynomial function f(x) such that $f(2x) = f'(x)\, f''(x)$.

DF0075

2. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = ky$, then find the value of 'k'.

DF0076

3. Let $y = x \sin kx$. Find the possible value of k for which the differential equation $\frac{d^2y}{dx^2} + y = 2k\cos kx$ holds true for all $x \in R$.

DF0077

4. The function $f: R \to R$ satisfies $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

DF0079

5. Let $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$. Define the function $f'(x)$ for all x and find $f''(0)$ if it exist.

DF0081

6. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

DF0084

7. If $f: R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$, then prove that $f(2) = f(1) - f(0)$.

DF0085

8. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Find the continuous function f(x) satisfying $f'(1) = f(-1)$.

DF0056

9. (a) Let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. Find the value of g' where $f(x) = 2$.

DF0062

(b) Let $f: R \to R$ be defined as $f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$ and $g(x) = f^{-1}(x)$, then find $g'(-1)$.

DF0063

(c) Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \frac{1}{f^{-1}(x)}$.

If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

DF0064

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EXERCISE (S-3)

Evaluate the following limits using L'Hospital's Rule or otherwise (Q.No.1 to 5) :

1. $\lim\limits_{x\to 0}\left[\dfrac{1}{x\sin^{-1}x}-\dfrac{1-x^2}{x^2}\right]$ **DF0087**

2. $\lim\limits_{x\to 0}\dfrac{x+\ell n\left(\sqrt{x^2+1}-x\right)}{x^3}$ **DF0088**

3. $\lim\limits_{x\to 0}\left[\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right]$ **DF0089**

4. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ell n(1-x)}{x\cdot\tan^2 x}$ **DF0090**

5. $\lim\limits_{x\to 0}\dfrac{x^{6000}-(\sin x)^{6000}}{x^2.(\sin x)^{6000}}$ **DF0091**

6. Given a real valued function f(x) as follows :

$f(x)=\dfrac{x^2+2\cos x-2}{x^4}$ for $x<0$; $f(0)=\dfrac{1}{12}$ & $f(x)=\dfrac{\sin x-\ell n(e^x\cos x)}{6x^2}$ for $x>0$. Test the continuity and differentiability of f(x) at $x=0$.

DF0093

7. **Column - I** contains function defined on R and **Column-II** contains their properties. Match them :-

	Column - I		**Column - II**
(A)	$\lim\limits_{n\to\infty}\left(\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right)^n$ equal	(P)	e
(B)	$\lim\limits_{x\to 0^+}\dfrac{1}{(1+\text{cosec}\,x)^{\frac{1}{\ell n(\sin x)}}}$ equals	(Q)	e^2
(C)	$\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\cos^{-1}x\right)^{1/x}$ equals	(R)	$e^{-2/\pi}$
		(S)	$e^{\pi/6}$

DF0096

EXERCISE (JM)

1. Let $f:(-1, 1)\to R$ be a differentiable function with $f(0)=-1$ and $f'(0)=1$. Let $g(x)=[f(2f(x)+2)]^2$. Then g'(0) :- **[AIEEE-2010]**

(1) 4 (2) –4 (3) 0 (4) –2

DF0097

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2. $\frac{d^2x}{dy^2}$ equals :- **[AIEEE-2011]**

(1) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (2) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (3) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (4) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

DF0098

3. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to : **[JEE(Main)-2013]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) 1 (4) $\sqrt{2}$

DF0099

4. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then g'(x) is equal to :

[JEE(Main)-2014]

(1) $1 + x^5$ (2) $5x^4$ (3) $\frac{1}{1+\{g(x)\}^5}$ (4) $1+\{g(x)\}^5$

DF0100

5. If for $x\in\left(0,\frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then g(x) equals :-

[JEE(Main)-2017]

(1) $\frac{3}{1+9x^3}$ (2) $\frac{9}{1+9x^3}$ (3) $\frac{3x\sqrt{x}}{1-9x^3}$ (4) $\frac{3x}{1-9x^3}$

DF0101

6. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$, is: **[JEE(Main)-2019]**

(1) $\frac{3}{2\sqrt{2}}$ (2) $\frac{1}{3\sqrt{2}}$ (3) $\frac{1}{6}$ (4) $\frac{1}{6\sqrt{2}}$

DF0102

7. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval (–2, 2), g is :-

[JEE(Main)-2019]

(1) differentiable at all points
(2) not differentiable at two points
(3) not continuous
(4) not differentiable at one point

DF0103

8. For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then $(1+\log_e 2x)^2\frac{dy}{dx}$ is equal to : **[JEE(Main)-2019]**

(1) $\log_e 2x$ (2) $\frac{x\log_e 2x + \log_e 2}{x}$ (3) $x\log_e 2x$ (4) $\frac{x\log_e 2x - \log_e 2}{x}$

DF0104

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9. If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is : **[JEE(Main)-2019]**

(1) 12 (2) 33 (3) 9 (4) 15

DF0105

10. If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at $x = 0$ is equal to : **[JEE(Main)-2019]**

(1) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$ (2) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ (3) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (4) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

DF0106

11. The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x \in \left(0, \frac{\pi}{2}\right)\right)$ is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) 1 (4) 2

DF0107

EXERCISE (JA)

1. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[JEE 2009, 4]**

DF0110

2. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

DF0111

3. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1,3) is

[JEE(Advanced)-2014, 3]

DF0112

4. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in \mathbb{R}$. Then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

DF0113

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ANSWER KEY

EXERCISE (O-1)

1. A	**2.** B	**3.** B	**4.** D	**5.** B	**6.** C	**7.** B	**8.** C
9. C	**10.** D	**11.** A	**12.** A	**13.** D	**14.** A	**15.** B	**16.** A

EXERCISE (O-2)

1. A	**2.** D	**3.** A	**4.** C	**5.** D	**6.** B	**7.** D	**8.** A
9. C	**10.** B,D	**11.** C,D	**12.** A,C,D				

EXERCISE (S-1)

1. 16 **2.** (a) $Dy = (\cos x)^{\ell nx}\left[\frac{\ell n(\cos x)}{x} - \tan x\ \ell nx\right] + (\ell nx)^x\left[\frac{1}{1nx} + \ell n(\ell nx)\right]$;

(b) $\frac{dy}{dx} = e^{x^{e^x}} \cdot x^{e^x}\left[\frac{e^x}{x} + e^x \ell nx\right] + e^{x^{x^e}}\ x^{e-1}\ x^{x^e}[1 + e\ell nx] + x^{e^{e^x}}\ e^{e^x}\left[\frac{1}{x} + e^x \ell nx\right]$

4. $\frac{y}{x}\cdot\frac{x\ell nx + x\ell nx.\ell ny + 1}{\ell n\,x(1 - x - y\,\ell na)}$ **6.** $\frac{1+\sqrt{1-x^4}}{x^6}$ **8.** **(a)** $\left(-\frac{1}{2},\frac{1}{2}\right),(-\infty,\infty)$; **(b)** $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; **(c)** $\frac{16\sqrt{3}}{9}$

9. $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ **11.** (a) (i) $y'(0) = -1$; (ii) $y''(0) = 2$; (b) $a = 1, -2$ **12.** $\frac{-3}{2}$

EXERCISE (S-2)

1. $\frac{4x^3}{9}$ **2.** 25 **3.** k = 1, –1 or 0 **4.** 6

5. $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\frac{1}{3}$ **6.** $\frac{1-2x}{2\sqrt{1-x^2}}$

8. $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ell n\frac{3}{2}\right]x & \text{if } x \leq 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if } x > 0 \end{cases}$ **9.** (a) 1/6; (b) $\frac{1}{14}$; (c) –1

EXERCISE (S-3)

1. $\frac{5}{6}$ **2.** $\frac{1}{6}$ **3.** $-\frac{1}{3}$ **4.** $-\frac{1}{2}$ **5.** 1000

6. f is cont. but not derivable at x = 0 **7.** (A) S; (B) P ; (C) R

EXERCISE (JM)

1. 2	**2.** 2	**3.** 1	**4.** 4	**5.** 2	**6.** 4	**7.** 4	**8.** 4
9. 2	**10.** 1	**11.** 4					

EXERCISE (JA)

1. 2 **2.** 1 **3.** 8 **4.** B,C

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Important Notes

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